Representation of the exact solution for a kind of nonlinear partial differential equation

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Abstract

In this work, we give a new method for solving a nonlinear partial differential equation in the reproducing kernel space. Through numerical example, it is proved to be valid.

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1. Introduction

Giving exact solutions of a nonlinear PDE (Partial Differential Equation) is important in study of the theory of nonlinear problems and their applications. In Ref. [1], using the two-dimensional differential transform method, the author solved the following PDE: \( \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( D(u) \frac{\partial u}{\partial x} \right) \). For a nonlinear evolution equation with a nonlinear term in the form \( u_{tt} + au_{xx} + bu + cu^3 = 0 \), Yong Chen et al. [2] have also given an exact solution. In this work, we focus on a kind of nonlinear PDE with initial-boundary value conditions in the form

\[ \frac{\partial}{\partial x} \left( a(x) \frac{\partial u(x, t)}{\partial x} \right) - \frac{\partial^2 u(x, t)}{\partial t^2} + H(u(x, t)) = f(x, t) \tag{1.1} \]

where \( H(x) \) is a nonlinear term of any type. For the kind of PDEs of general type, we give a representation of the exact solution in the reproducing kernel space. A numerical example shows that this method is valid.

2. Several reproducing kernel spaces

Like in Ref. [3], we give several reproducing kernel spaces.

1. The reproducing kernel space

\[ W_2[0, 1] = \{ u(x) \mid u, u' \text{ are absolutely continuous functions,} \]
\[ u, u', u'' \in L^2[0, 1], \text{and} u(0) = u'(1) = 0 \}. \tag{2.1} \]
The inner product is
\[
(u, v)_{W_2} = \int_0^1 (4u(x)v(x) + 5u'(x)v'(x) + u''(x)v''(x))\,dx.
\] (2.2)

The reproducing kernel is
\[
R_{x}^{[2]}(x) = \frac{1}{6(1 + e^x)} \left( e^{x+\xi} - e^{2-x-\xi} + e^{2-|x-\xi|} - e^{\xi-|x-\xi|} \right)
+ \frac{1}{12(1 + e^x)} \left( e^{4-2x-2\xi} - e^{2x+2\xi} + e^{2|x-\xi|} - e^{4-2|x-\xi|} \right).
\] (2.3)

2. The reproducing kernel space
\[
W_3[0, \infty) = \{u(t) \mid u, u', u'' \text{ are absolutely continuous functions},
\quad u, u', u'' \in L^2[0, +\infty], \quad \text{and } u'(0) = 0\}.
\] (2.4)

The inner product and the reproducing kernel are of the following forms respectively:
\[
(u, v)_{W_3} = \int_{0}^{+\infty} \left( 36u(t)v(t) + 49u'(t)v'(t) + 14u''(t)v''(t) \right)\,dt
\] (2.5)
\[
R_{y}^{(3)}(t) = \frac{e^{-3(t+y)}}{240} \left( 1 + 5e^{2(t+y)} - 4e^{t+y} + 5e^{3(t+y)} - 4e^{3(t+y) - 2|t-y|} + e^{3(t+y) - 3|t-y|} \right).
\] (2.6)

3. The reproducing kernel space
\[
W_1[0, 1] = \{u(x) \mid u \text{ is an absolutely continuous function}, \quad \text{and } u, u' \in L^2[0, 1]\}.
\] (2.7)

The inner product and the reproducing kernel are respectively
\[
(u, v)_{W_1} = \int_{0}^{1} (u(x)v(x) + u'(x)v'(x))\,dx
\] (2.8)
\[
R_{x}^{[1]}(x) = \frac{1}{2(e^x - 1)} \left( e^{x+s} + e^{2-(x+s)} + e^{|x-s|} + e^{2-|x-s|} \right).
\] (2.9)

4. In this work, our problem needs that \(u(x, t)\) is of different smoothness for \(x\) and \(t\). So we construct the reproducing kernel space
\[
W_{2,3}(D) = \{u(x, t) \mid \frac{\partial^{n+m}}{\partial x^n \partial t^m} u(x, t) \text{ are two-variable complete continuous functions},
\quad n = 0, 1, m = 0, 1, 2, \quad \frac{\partial^{p+q}}{\partial x^p \partial t^q} u(x, t) \in L^2(D), \quad p = 0, 1, 2, \quad q = 0, 1, 2, 3,
\quad u(0, t) = 0, \quad \frac{\partial}{\partial x} u(1, t) = 0, \quad \frac{\partial}{\partial t} u(x, 0) = 0\}, \quad D = [0, 1] \times [0, +\infty).
\] (2.10)

The inner product is
\[
(u_1, u_2)_{W_{2,3}} = \int_D \left[ 144u_1(x, t)u_2(x, t) + 196 \frac{\partial}{\partial t} u_1(x, t) \frac{\partial}{\partial x} u_2(x, t) + 56 \frac{\partial^2}{\partial t^2} u_1(x, t) \frac{\partial^2}{\partial x^2} u_2(x, t)
\right.
\] (2.11)
\[
+ 4 \frac{\partial^3}{\partial x^3} u_1(x, t) \frac{\partial^3}{\partial x^3} u_2(x, t) + 180 \frac{\partial}{\partial t} u_1(x, t) \frac{\partial}{\partial x} u_2(x, t) + 245 \frac{\partial^2}{\partial x^2} u_1(x, t) \frac{\partial^2}{\partial t^2} u_2(x, t)
\] (2.11)
\[
+ 70 \frac{\partial^3}{\partial x^3} u_1(x, t) \frac{\partial^3}{\partial x^3} u_2(x, t) + 5 \frac{\partial^4}{\partial x^4} u_1(x, t) \frac{\partial^4}{\partial x^4} u_2(x, t)
\] (2.11)
\[
+ 36 \frac{\partial^4}{\partial x^4} u_1(x, t) \frac{\partial^2}{\partial x^2} u_2(x, t) + 49 \frac{\partial^3}{\partial x^2} u_1(x, t) \frac{\partial^3}{\partial x^2} u_2(x, t)
\] (2.11)
\[
+ 14 \frac{\partial^4}{\partial x^4} u_1(x, t) \frac{\partial^2}{\partial x^2} u_2(x, t) + \frac{\partial^5}{\partial x^5} u_1(x, t) \frac{\partial^5}{\partial x^5} u_2(x, t) \right] \,dx \,dt.
\] (2.11)
The reproducing kernel is
\[ K(\xi, \eta)(x, t) = R_{\xi}^{[2]}(x)R_{\eta}^{[3]}(t) \]  
(2.12)
where \( R_{\xi}^{[2]}(x) \) and \( R_{\eta}^{[3]}(t) \) are given by (2.3) and (2.6) respectively.

3. Transformation of the nonlinear partial differential equation

Given the nonlinear differential equation with initial-boundary value conditions
\[
\begin{align*}
\frac{\partial}{\partial x} \left[ a(x) \frac{\partial u(x, t)}{\partial x} \right] - \frac{\partial^2 u(x, t)}{\partial t^2} + H(u(x, t)) &= f(x, t) \\
\frac{\partial}{\partial t} u(x, 0) &= 0 \\
\frac{\partial}{\partial x} u(1, t) &= 0, \quad 0 \leq x \leq 1, \quad 0 \leq t < +\infty
\end{align*}
\]  
(3.1)
where \( u(x, t) \in W_{(2, 3)}(D) \) and \( a(x) \in C^1[0, 1] \), \( H(x) \in C^1(-\infty, +\infty) \), \( h(x) \in W_2[0, 1] \), \( f(x, t) \in W_{(1, 1)}(D) \) [3] are the given functions, Eq. (3.1) is equivalent to
\[
\begin{align*}
a(x) \frac{\partial}{\partial x} u(x, t) - a(0) \frac{\partial}{\partial x} u(x, t) \bigg|_{x=0} - \int_0^x u''(z, t) \, dz + \int_0^x H(u(z, t)) \, dz &= \int_0^x f(z, t) \, dz \\
u(x, 0) &= h(x).
\end{align*}
\]  
(3.2)

4. The definition of the operator \( L \)

The linear operator \( L : W_{(2, 3)}(D) \rightarrow W_1[0, 1] \) is defined by
\[
(Lu)(x) = \int_0^x u''(z, 0) \, dz.
\]  
(4.1)

Then Eq. (3.2) is transformed into the equation
\[
(Lu)(x) = F(x)
\]
\[
F(x) = a(x)h'(x) - a(0)h'(0) + \int_0^x (H(h(x)) - f(z, 0)) \, dz.
\]  
(4.2)

It is easy to prove that the linear operator \( L : W_{(2, 3)}(D) \rightarrow W_1[0, 1] \) is bounded.

**Lemma 4.1.** Let \( L^* \) denote the conjugate operator of \( L \); then \( L^* \) is a bounded operator from \( W_1[0, 1] \) to \( W_{(2, 3)}(D) \) and
\[
(L^* R_{\xi}^{[1]}(\cdot))(x, t) = \frac{d^2}{d\eta^2} R_{t}^{[3]}(\eta) \bigg|_{\eta=0} \int_0^x R_{x}^{[2]}(\xi) \, d\xi
\]  
(4.3)
(See Appendix A).

5. Decomposition into direct sum of \( W_{(2, 3)}(D) \)

Taking \( A = \{s_1, s_2, \ldots\} \) as a dense set of interval \([0, 1]\), put \( \varphi_i(x) = R_{s_i}^{[1]}(x) \), where \( R_{s_i}^{[1]}(x) \) is given by (2.9). And let
\[
\psi_i(x, t) = (L^* \varphi_i)(x, t), \quad i = 1, 2, \ldots
\]  
(5.1)
Now we orthonormalize the function system \( \{ \psi_i(x, t) \}_{i=1}^{\infty} \), and obtain an orthonormal system \( \{ \tilde{\psi}_i(x, t) \}_{i=1}^{\infty} \), which holds for all \( i \),
\[
\tilde{\psi}_i(x, t) = \sum_{k=1}^{i} \beta_{ik} \psi_k(x, t) \quad i = 1, 2, \ldots
\] (5.2)
where \( \beta_{ik} \) is the coefficient of orthonormalization. Then \( \text{span}(\tilde{\psi}_i(x, t))_{i=1}^{\infty} \) is a subspace of \( W_{(2,3)}(D) \):
\[
\text{span}(\tilde{\psi}_i(x, t))_{i=1}^{\infty} = \left\{ u(x, t) \mid u(x, t) = \sum_{i=1}^{n} c_i \tilde{\psi}_i(x, t), c_i \in R, n \in N \right\} .
\] (5.3)
Let \( S \) denote the closure of this subspace, that is
\[
S = \overline{\text{span}(\tilde{\psi}_i(x, t))_{i=1}^{\infty}}
\]
and \( S^\perp \) denotes the orthogonal complement space of \( S \) in \( W_{(2,3)}(D) \). Taking a set of points \( B = \{ p_1(\xi_1, \eta_1), p_2(\xi_2, \eta_2), \ldots \} \) as a dense set of region \( D = [0, 1] \times [0, +\infty) \), and putting
\[
\rho_j(x, t) = R_{\xi_j}^{[2]}(x)R_{\eta_j}^{[3]}(t), \quad j = 1, 2, \ldots
\]
where \( R_{\xi_j}^{[2]}(x)R_{\eta_j}^{[3]}(t) \) is the reproducing kernel of \( W_{(2,3)}(D) \), we proceed with the generalized Schmidt orthonormalization for the function system \( \{ \rho_j(x, t) \} \) about the orthonormal system \( \{ \tilde{\psi}_j(x, t) \} \), that is
\[
\tilde{\rho}_j(x, t) = \frac{\rho_j(x, t) - \sum_{k=1}^{\infty} (\rho_j(x, t), \tilde{\psi}_k(x, t)) \tilde{\psi}_k(x, t) - \sum_{m=1}^{j-1} (\rho_j(x, t), \tilde{\rho}_m(x, t)) \tilde{\rho}_m(x, t)}{\rho_j(x, t) - \sum_{k=1}^{\infty} (\rho_j(x, t), \tilde{\psi}_k(x, t)) \tilde{\psi}_k(x, t) - \sum_{m=1}^{j-1} (\rho_j(x, t), \tilde{\rho}_m(x, t)) \tilde{\rho}_m(x, t)}
\]
So we have
\[
\tilde{\rho}_j(x, t) = \sum_{k=1}^{\infty} \beta_{jk} \psi_k(x, t) + \sum_{m=1}^{j} \beta_{jm}^* \rho_m(x, t) \quad j = 1, 2, \ldots
\] (5.4)
However,
\[
W_{(2,3)}(D) = S \oplus S^\perp
\] (5.5)
and \( \{ \tilde{\psi}_i(x, t) \}_{i=1}^{\infty} \cup \{ \tilde{\rho}_j(x, t) \}_{j=1}^{\infty} \) is an orthonormal basis of \( W_{(2,3)}(D) \).

6. Solving the nonlinear partial differential equation

**Lemma 6.1.** Assume that \( u(x, t) \) is the solution of Eq. (4.2); then it follows that
\[
u(x, t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left( a(s_k)h'(s_k) - k(0)h'(0) + \int_{0}^{s_k} (H(h(z)) - f(z, 0))dz \right) \tilde{\psi}_i(x, t) + \sum_{j=1}^{\infty} \alpha_j \tilde{\rho}_j(x, t)
\] (6.1)
where \( a(x), h(x), f(x, t), H(x) \) are given functions in (3.1), and \( \alpha_j \) are coefficients to be determined.

**Proof.** Due to \( u(x, t) \in W_{(2,3)}(D) \), we have
\[
u(x, t) = \sum_{i=1}^{\infty} \left( u(x, t), \tilde{\psi}_i(x, t) \right)_{W_{(2,3)}} \tilde{\psi}_i(x, t) + \sum_{j=1}^{\infty} \left( u(x, t), \tilde{\rho}_j(x, t) \right)_{W_{(2,3)}} \tilde{\rho}_j(x, t)
\]
Let \( \alpha_j = \left( u(x, t), \tilde{\rho}_j(x, t) \right)_{W_{(2,3)}} \) and from (5.2)
\[
u(x, t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left( u(x, t), (L^* \phi_k)(x, t) \right)_{W_{(2,3)}} \tilde{\psi}_i(x, t) + \sum_{j=1}^{\infty} \left( u(x, t), \tilde{\rho}_j(x, t) \right)_{W_{(2,3)}} \tilde{\rho}_j(x, t)
\]
= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} ((Lu)(x), \phi_k(x))_{W_1} \tilde{\psi}_i(x,t) + \sum_{j=1}^{\infty} \left( u(x,t), \tilde{\rho}_j(x,t) \right)_{W_{23}} \tilde{\rho}_j(x,t)

= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} (F(x), \phi_k(x))_{W_1} \tilde{\psi}_i(x,t) + \sum_{j=1}^{\infty} \left( u(x,t), \tilde{\rho}_j(x,t) \right)_{W_{23}} \tilde{\rho}_j(x,t)

= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} F(s_k) \tilde{\psi}_i(x,t) + \sum_{j=1}^{\infty} \left( u(x,t), \tilde{\rho}_j(x,t) \right)_{W_{23}} \tilde{\rho}_j(x,t)

and from (4.2), we can obtain (6.1). □

Further we have

\[
\begin{align*}
\tilde{\psi}_i(x,0) = & \sum_{j=1}^{\infty} \alpha_j \tilde{\rho}_j(x,0) \\
= & \frac{1}{60} \sum_{r=1}^{\infty} \sum_{m=1}^{r} \beta_{lm} \beta_{ik} \int_0^{s_k} \int_0^{s_{n}} R_{x}^{[2]}(z)dz - \frac{1}{60} \sum_{k=1}^{i} \sum_{m=1}^{l} \beta_{lm} \beta_{ik} R_{y_n}^{[3]}(0) \int_0^{s_k} R_{x}^{[2]}(z)dz \\
& - \frac{1}{60} \sum_{r=1}^{\infty} \sum_{m=1}^{r} \beta_{lm} \beta_{ik} \int_0^{s_k} \int_0^{s_{n}} R_{x}^{[2]}(z)dz - \frac{1}{60} \sum_{k=1}^{i} \sum_{m=1}^{l} \beta_{lm} \beta_{jk} R_{y_n}^{[3]}(0) \int_0^{s_k} R_{x}^{[2]}(z)dz \\
& - \frac{1}{60} \sum_{r=1}^{\infty} \sum_{m=1}^{r} \beta_{lm} \beta_{ip} \int_0^{s_k} \int_0^{s_{n}} R_{x}^{[2]}(z)dz + \frac{1}{60} \sum_{k=1}^{i} \sum_{m=1}^{l} \beta_{lm} \beta_{ip} R_{y_n}^{[3]}(0) \int_0^{s_k} R_{x}^{[2]}(z)dz \\
& - \frac{1}{60} \sum_{k=1}^{i} \sum_{m=1}^{l} \beta_{lm} \int_0^{s_k} h(z)dz + \sum_{n=1}^{l} \beta_{jn} R_{y_n}^{[3]}(0) h(\xi_n), \quad l = 1, 2, \ldots.
\end{align*}
\]

Theorem 6.1. Assuming that \( A = \{s_1, s_2, \ldots\} \) is a dense set on interval \([0, 1]\). If \( \alpha_j, \quad j = 1, 2, \ldots, \) are the solutions of the infinite equations (6.3), then the solution of Eq. (3.1) can be given by (6.1).

7. Numerical example

Example. For the nonlinear partial differential equation

\[
\begin{align*}
\frac{\partial}{\partial x} \left[ \left( 4 - e^{-y} \right) \frac{\partial u(x,t)}{\partial x} \right] - \frac{\partial^2 u(x,t)}{\partial t^2} + (1 + u(x,t))^2 &= f(x,t) \\
u(x,0) = x^2 - 2x \\
\frac{\partial}{\partial t} u(x,0) = 0 \\
\frac{\partial}{\partial x} u(0,t) = 0 \\
\frac{\partial}{\partial x} u(1,t) = 0, \quad 0 \leq x \leq 1, \quad 0 \leq t \leq +\infty
\end{align*}
\]
where \( f(x, t) = 1 - e^{-\frac{t}{10}}(8 - 4e^{-x} - \frac{22}{5}x + 2xe^{-x} + \frac{2}{5}t^{2}x + \frac{11}{5}x^{2} - \frac{1}{25}t^{2}x^{2}) + x^{2}e^{-\frac{t}{10}}(4 - 4x + x^{2}) \), the true solution is \( u(x, t) = (x^{2} - 2x)e^{-\frac{t}{10}} \). By using Mathematica 4.2 and applying it to (6.3), we calculate the approximate solution \( u_{n}(x, t) \) truncating the series in (6.1). The numerical results are given by Table 1.

**Appendix A**

### A.1. Proof of Lemma 4.1

**Proof.** Because \( L \) is bounded, \( L^{*} \) is naturally bounded. And

\[
(L^{*}R_{3}^{[1]}(\cdot))(x, t) = \left( (L^{*}R_{3}^{[1]}(\cdot))(\cdot, \cdot), K_{(x,t)}(\cdot, \cdot) \right)_{W(2,3)}
\]

where \( \cdot, \cdot \) and \( \cdot \) denote the variables corresponding to the functions respectively. We have

\[
(L^{*}R_{3}^{[2]}(\cdot))(x, t) = \left( (L^{*}R_{3}^{[1]}(\cdot))(\cdot, \cdot), R_{3}^{[2]}(\cdot)R_{3}^{[3]}(\cdot) \right)_{W(2,3)}
\]

\[
= \left( R_{3}^{[1]}(x), L(R_{3}^{[2]}(\cdot)R_{3}^{[3]}(\cdot))(\cdot) \right)_{W_{1}}
\]

\[
= L(R_{3}^{[2]}(\cdot)R_{3}^{[3]}(\cdot))(s)
\]

\[
= \frac{d^{2}}{d\phi^{2}} R_{3}^{[3]}(\cdot) \int_{0}^{s} R_{3}^{[2]}(\cdot) d\phi \cdot \phi.
\]

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**References**

