On irreducible semigroups of matrices with traces in a subfield

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“bas do’aay-e saharat moones-e jaan khaahad bood to keh ischoon Hafez-e shakhhiz gholaami daari”

With my cordial regards, dedicated to Heydar Radjavi on his seventieth birthday

Abstract

In this paper we consider irreducible semigroups of matrices over a general field $K$ with traces in a subfield $F$. Motivated by a result of Omladic–Radjabalipour–Radjavi, we prove a block matrix representation theorem for the $F$-algebras generated by such semigroups. We use our main results to generalize certain classical triangularization results, e.g., those due to Guralnick and Radjavi. Some other consequences of our main results are presented.

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1. Introduction

Semigroups of matrices with traces or spectra in a subfield have been studied in [7] as well as [10]. In this note, we prove a key theorem (Theorem 2.2 below) which is further employed to prove, among other things, the main result of the section, namely Theorem 2.9. In the remaining part of this section, we use Theorem 2.9 to give a more precise description of irreducible semigroups of matrices with traces in a finite subfield $F$. 

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Throughout the paper, unless otherwise stated, \( K \) denotes a general field; and \( F \) stands for a subfield of \( K \). We view the members of \( Mn(K) \) as linear transformations acting on the left of \( Kn \) relative to a fixed basis where \( Kn \) is the vector space of \( n \times 1 \) column vectors. For a collection \( \mathcal{F} \) in \( Mn(K) \), a subspace \( \mathcal{M} \) is invariant for \( \mathcal{F} \) if \( T\mathcal{M} \subseteq \mathcal{M} \) for all \( T \in \mathcal{F} \); \( \mathcal{M} \) is hyperinvariant for \( \mathcal{F} \) if \( T\mathcal{M} \subseteq \mathcal{M} \) for all \( T \in \mathcal{F} \cup \mathcal{F}' \) where the symbol \( \mathcal{F}' \) denotes the commutant of \( \mathcal{F} \). A collection \( \mathcal{F} \) in \( Mn(K) \) is called reducible if \( \mathcal{F} = \{0\} \) or it has a nontrivial invariant subspace. This definition is slightly unconventional, but it simplifies some of the statements in what follows.

A collection \( \mathcal{F} \) of matrices in \( Mn(K) \) is called simultaneously triangularizable or simply triangularizable if there exists a basis for the vector space \( Kn \) relative to which all matrices in the family are upper triangular. Equivalently, there exists an invertible matrix \( S \) over \( K \) such that each member of \( S^{-1}\mathcal{F}S \) is upper triangular.

By an \( F \)-algebra \( A \) in \( Mn(K) \), we mean a subring of \( Mn(K) \) that is closed under scalar multiplication by the elements of the subfield \( F \). For a semigroup \( \mathcal{S} \) in \( Mn(K) \), we use \( \text{Alg}_F(\mathcal{S}) \) to denote the \( F \)-algebra generated by \( \mathcal{S} \). By \( \text{Alg}(\mathcal{S}) \) we simply mean \( \text{Alg}_K(\mathcal{S}) \).

For a given field \( F \) and \( k \in \mathbb{N} \) with \( k > 1 \), we say that \( F \) is \( k \)-closed if every polynomial of degree \( k \) over \( F \) is reducible over \( F \). It is plain that a field \( F \) is algebraically closed if and only if \( F \) is \( k \)-closed for all \( k \in \mathbb{N} \) with \( k > 1 \). It can be shown that finite extensions of prime fields, e.g., finite fields, are not \( k \)-closed for all \( k \in \mathbb{N} \) with \( k > 1 \).

A collection \( \mathcal{F} \) of matrices in \( Mn(F) \) is called absolutely irreducible if \( \mathcal{F} \) is irreducible over all extensions of \( F \). By Burnside’s Theorem, the collection \( \mathcal{F} \) is absolutely irreducible if and only if \( \text{Alg}(\mathcal{F}) = Mn(F) \). It is plain that an absolutely irreducible family of matrices in \( Mn(F) \) is irreducible and its commutant consists of scalars.

If \( \mathcal{J} \) is a multiplicative semigroup, a subset \( \mathcal{J} \) of \( \mathcal{F} \) is called a semigroup ideal of \( \mathcal{F} \) if \( JS, SJ \in \mathcal{J} \) whenever \( J \in \mathcal{J} \) and \( S \in \mathcal{F} \). In what follows, we make frequent use of the elementary well-known lemma below.

**Lemma 1.1.** Let \( F \) be a field, \( n \in \mathbb{N} \), and \( \mathcal{F} \) a semigroup in \( Mn(F) \). If \( \mathcal{F} \) is (absolutely) irreducible, then so is every nonzero semigroup ideal of \( \mathcal{F} \).

**Proof.** If \( \mathcal{F} \) is irreducible, see Lemma 2.1.10 of [13].

Next suppose that \( \mathcal{F} \) is absolutely irreducible. Then, the assertion is a quick consequence of the proof above in view of the definition of absolute irreducibility. \( \Box \)

Motivated by Lemma 2.1.12 of [13], we state the following useful lemma.

**Lemma 1.2.** Let \( F \) be a field, \( n \in \mathbb{N} \), \( \mathcal{J} \) a semigroup in \( Mn(F) \), and \( T \) a nonzero matrix in \( Mn(F) \). If \( \mathcal{F} \) is (absolutely) irreducible, then so is \( T\mathcal{F}|_{\mathcal{R}} \) where \( \mathcal{R} = TF^n \) is the range of \( T \).
Proof. The proof is easy (see [15, Lemma 2.1]). □

Remark. In [15], the counterpart of the preceding lemma over division rings is used to give a new proof of Levitzki’s Theorem [6, Theorem II.35] on division rings.

The following well-known result shows that triangularizability of a collection of triangularizable matrices does not depend on the ground field.

Corollary 1.3. Let $F$ be a field, $K$ a field extension of $F$, and $\mathcal{F}$ a family of triangularizable matrices in $M_n(F)$ where $n \in \mathbb{N}$. Then $\mathcal{F}$ is triangularizable over $F$ if and only if $\mathcal{F}$ is triangularizable over $K$.

Proof. The “only if” part is obvious. To see the “if” part, it suffices to show that $\hat{\mathcal{F}}$, the induced quotient transformations on $\mathcal{N}/\mathcal{M}$, is reducible whenever $\mathcal{M} \subset \mathcal{N}$ are two $F$-invariant subspaces for $\mathcal{F}$ with $\dim \mathcal{N}/\mathcal{M} > 1$. If $\hat{\mathcal{F}}$ is commutative, then reducibility easily follows from the hypothesis that each member of $\mathcal{F}$ is triangularizable over $F$. If there are $A, B \in \mathcal{F}$ such that $\hat{A}\hat{B} \neq \hat{B}\hat{A}$, then the ideal $\hat{I}$ generated by $\hat{A}\hat{B} - \hat{B}\hat{A}$ in $\text{Alg}(\hat{\mathcal{F}})$ is in particular an algebra of nilpotent transformations on $\mathcal{N}/\mathcal{M}$ for $\mathcal{F}$ is triangularizable over $K$. Hence, it follows from Levitzki’s Theorem [6, Theorem II.35] that $\hat{I}$ is triangularizable, and hence reducible, over $F$. Now reducibility of $\text{Alg}(\hat{\mathcal{F}})$, hence $\hat{\mathcal{F}}$, follows from that of the nonzero ideal $\hat{I}$ in light of Lemma 1.1, completing the proof. □

2. Main results

We start this section with the following key lemma which is very likely known to the experts.

Lemma 2.1. Let $F$ be a field, and $\mathcal{A}$ an irreducible algebra in $M_n(F)$. Then $\mathcal{A}$ is semisimple both as a subring and as a subalgebra of $M_n(F)$. Furthermore, the algebra $\mathcal{A}$ contains the identity matrix, and $\mathcal{A}$ is simple.

Proof. In light of Theorem IX.5.2(ii) of [5], it suffices to show that $\mathcal{A}$ is semisimple as a subalgebra of $M_n(F)$. To this end, note that $\mathcal{A}$ is an algebraic algebra. Therefore, it follows from Exercise IX.5.6 of [5] that the Jacobson radical of the algebra $\mathcal{A}$, denoted by $\text{Rad}(\mathcal{A})$, is nil (i.e., every element of $\text{Rad}(\mathcal{A})$ is nilpotent). Now Levitzki’s Theorem together with Lemma 1.1 yields $\text{Rad}(\mathcal{A}) = \{0\}$. That is, $\mathcal{A}$ is semisimple as a subalgebra, and hence as a subring, of $M_n(F)$. That $\mathcal{A}$ contains the identity matrix follows from Theorem 2.2 of [11]. To see that $\mathcal{A}$ is simple, suppose $\mathcal{J}$ is a nonzero ideal of $\mathcal{A}$. To show $\mathcal{J} = \mathcal{A}$, note that, in view of Lemma 1.1, the nonzero ideal $\mathcal{J}$ is an irreducible algebra in $M_n(F)$, and hence contains the identity matrix. Therefore, $\mathcal{J} = \mathcal{A}$, completing the proof. □
In what follows the following theorem is crucial.

**Theorem 2.2.** Let \( n \in \mathbb{N} \), \( F \) a field, \( \mathcal{S} \) an irreducible semigroup in \( M_n(F) \), and \( \mathcal{I} \) a nonzero semigroup ideal of \( \mathcal{S} \). Then either trace is identically zero on \( \mathcal{S} \) or

\[
\{ A \in \text{Alg}(\mathcal{S} \cup \{ I \}) : \text{tr}(A) = [0] \} = \{ 0 \}.
\]

**Proof.** Denote the left hand side of the asserted identity by \( J \). Set \( A := \text{Alg}(\mathcal{S}) \). We have \( A = \text{Alg}(\mathcal{I}) \), for the algebra \( A \) is irreducible, and hence simple by Lemma 2.1, and that \( \text{Alg}(\mathcal{I}) \) is a nonzero ideal of \( A \). Again from Lemma 2.1, we see that \( A = \text{Alg}(\mathcal{S} \cup \{ I \}) \). Now since \( A = \text{Alg}(\mathcal{S}) \) contains the identity matrix, it is not difficult to see that

\[
\mathcal{J} = \{ A \in \text{Alg}(\mathcal{S}) : \text{tr}(A) = [0] \} = \{ A \in \mathcal{J} : \text{tr}(A) = [0] \}
\]

is an ideal of the simple algebra \( \mathcal{S} \) consisting of matrices with traces zero. Thus either \( \mathcal{J} = A \) in which case \( \text{tr}(\mathcal{A}) = [0] \), or \( \mathcal{J} = \{ 0 \} \), as desired. \( \square \)

**Remarks**

1. In the preceding theorem, if \( n > 1 \) and the ground field \( F \) happens to be perfect; or the its characteristic is zero or does not divide \( n \), then, by Theorem 2.2.19 of [16], the hypothesis that trace is not identically zero on \( \mathcal{S} \) is redundant.

2. Under the hypotheses of the preceding theorem, if \( n > 1 \) and the ground field happens to be \( k \)-closed for each \( k \) dividing \( n \) with \( k > 1 \); or the semigroup \( \mathcal{S} \) happens to be absolutely irreducible, then using Theorem 2.2.21 of [16] (resp. the definition), in view of the proof above, we see that

\[
\{ A \in M_n(F) : \text{tr}(A) = [0] \} = \{ 0 \}.
\]

The following result characterizes all irreducible families of matrices over a field on which trace is permutable. See [3] for a description of matrix algebras with non-degenerate permutable trace in terms of their Jacobson radicals.

**Corollary 2.3.** Let \( n > 1 \), \( F \) a field, and \( \mathcal{F} \) an irreducible family of matrices in \( M_n(F) \) on which trace is permutable. Then either \( \text{tr}(\text{Alg}(\mathcal{F})) = [0] \) or \( \text{Alg}(\mathcal{F}) \) is an extension field of \( F \) in which case there exists an irreducible matrix \( A \in M_n(F) \) such that \( \text{Alg}(\mathcal{F}) = F[A] \).

**Proof.** Suppose that trace is not identically zero on \( \mathcal{A} := \text{Alg}(\mathcal{F}) \), we first show that \( \mathcal{F} \), hence \( \text{Alg}(\mathcal{F}) \), is commutative. To this end, let \( A, B \in \mathcal{F} \) be arbitrary. Since trace is permutable on \( \mathcal{F} \), it follows that \( \text{tr}((A - BA)\text{Alg}(\mathcal{F})) = [0] \), and hence \( AB = BA \) by Theorem 2.2. So we have proved that \( \text{Alg}(\mathcal{F}) \) is commutative. From this together with irreducibility of \( \text{Alg}(\mathcal{F}) \) we see that every nonzero element of \( \text{Alg}(\mathcal{F}) \) is invertible because the kernel of every element of \( \text{Alg}(\mathcal{F}) \) is invariant.
under \( F \). Therefore, \( \text{Alg}(F) \) is an extension field of \( F \). Hence, by the Primitive Element Theorem [5, Proposition 5.6.15], there exists a matrix \( A \in M_n(F) \) such that \( \text{Alg}(F) = F[A] \). □

We use \( l(n) \) to denote the length of the algebra \( M_n(F) \) (see [8] for the definition). Pappacena proved that \( l(n) < n^2/(n - 1) + 1/4 + n/2 - 2 \) (see [8, Corollary 3.2]). In the next two results (Corollary 2.4 and 2.5), we give slight generalizations of triangularizability results due to Radjavi [12] and Guralnick [4] respectively.

**Corollary 2.4** (Radjavi’s Trace Theorem). (i) Let \( n > 1 \), \( F \) a field with \( \text{ch}(F) = 0 \) or \( > n/2 \), \( m \in \mathbb{N} \), and \( \mathcal{F} \) a family of triangularizable matrices in \( M_n(F) \). Then \( \mathcal{F} \) is triangularizable if and only if \( \text{tr}((AB - BA)S) = 0 \) for all \( A, B \in \mathcal{F} \) and all words \( S \) in \( \mathcal{F} \) of length at least \( m \).

(ii) Let \( n > 1 \), \( F \) a field with \( \text{ch}(F) = 0 \) or \( > n/2 \), and \( \mathcal{F} \) a family of triangularizable matrices in \( M_n(F) \). Then \( \mathcal{F} \) is triangularizable if and only if \( \text{tr}((AB - BA)S) = 0 \) for all \( A, B \in \mathcal{F} \) and all words \( S \) in \( \mathcal{F} \) of length at most \( l(n) \).

**Proof.** (i) Necessity is obvious. To prove sufficiency, in view of Corollary 1.3, we may, without loss of generality, assume that the ground field \( F \) is algebraically closed. Now the proof is easily settled by The Block Triangularization Theorem [13, Theorem 1.5.1] and Theorem 2.2.

(ii) It suffices to prove sufficiency. To this end, it follows from the definition that \( \text{tr}((AB - BA)S) = 0 \) for all \( A, B \in \mathcal{F} \) and all words \( S \) in \( \mathcal{F} \) of any length. Thus (i) applies, finishing the proof. □

**Corollary 2.5.** (i) Let \( F \) be a field, \( n, m \in \mathbb{N} \), and \( \mathcal{F} \) a family of triangularizable matrices in \( M_n(F) \). Then \( \mathcal{F} \) is triangularizable if and only if \( (AB - BA)S \) is nilpotent for all \( A, B \in \mathcal{F} \) and all words \( S \) in \( \mathcal{F} \) of length at least \( m \).

(ii) Let \( F \) be a field, \( n \in \mathbb{N} \), and \( \mathcal{F} \) a family of triangularizable matrices in \( M_n(F) \). Then \( \mathcal{F} \) is triangularizable if and only if \( (AB - BA)S \) is nilpotent for all \( A, B \in \mathcal{F} \) and all words \( S \) in \( \mathcal{F} \) of length at most \( l(n) \).

**Proof.** (i) Note that, in view of Corollary 1.3, we may assume that \( F \) is algebraically closed. Now, the proof is easily established by Theorem 2.2 in view of The Triangularization Lemma. We omit the proof for the sake of brevity.

(ii) From the definition of \( l(n) \) and Theorem 1 of [14] (or see [16, Theorem 2.2.13]), we see that \( (AB - BA)S \) is nilpotent for all \( A, B \in \mathcal{F} \) and all words \( S \) in \( \mathcal{F} \). Thus (i) applies, completing the proof. □

For a subset \( C \subseteq K \), the symbol \( \langle C \rangle_F \) is used to denote the linear manifold spanned by \( C \) over \( F \).
Lemma 2.6. Let \( n > 1 \), \( K \) a field, \( F \) a subfield of \( K \), \( \mathcal{S} \) an irreducible semigroup in \( M_n(K) \) such that \( [0] \neq \text{tr}(\mathcal{S}) \leq F \). Suppose that \( \{S_1, \ldots, S_m\} \) is a subset of \( \mathcal{S} \) that is linearly independent over \( F \). If \( A \in \text{Alg}_F(\mathcal{S} \cup \{I\}) \) and \( A = c_1S_1 + \cdots + c_mS_m \) where \( c_i \in K \) for all \( 1 \leq i \leq m \), then \( c_i \in F \) for all \( 1 \leq i \leq m \). Therefore, a subset \( \{S_1, \ldots, S_m\} \) of \( \mathcal{S} \) is linearly independent over \( F \) if and only if it is linearly independent over \( K \).

Proof. By the hypothesis we have
\[
A = c_mS_m + \cdots + c_1S_1,
\]
where \( c_i \in K \) for all \( 1 \leq i \leq m \). Set \( c_{m+1} := 1 \in F \). By proving that \( c_j \in \langle c_{j+1}, \ldots, c_m, 1 \rangle_F \) for each \( j = 1, \ldots, m \), we show that \( c_j \in F \) for all \( 1 \leq j \leq m \). First note that \( c_1 \in \langle c_2, \ldots, c_m, 1 \rangle_F \). To see this, since \( 0 \neq S_1 \subseteq \mathcal{S} \subseteq \text{Alg}_F(\mathcal{S} \cup \{I\}) \), it follows from Theorem 2.2 that there exists \( S \in \mathcal{S} \) such that \( \text{tr}(S) \neq 0 \). This together with (\#) easily implies that \( c_1 \in \langle c_2, \ldots, c_m, 1 \rangle_F \). Let \( j_0 \) be the largest \( j \) for which \( c_i \in \langle c_{i+1}, \ldots, c_m, 1 \rangle_F \) for \( i = 1, \ldots, j \). If \( j_0 = m \), we are done. Suppose, otherwise, that \( j_0 < m \), we show that \( c_{j_0+1} \in \langle c_{j_0+2}, \ldots, c_m, 1 \rangle_F \) contradicting the fact that \( j_0 \) is the largest index having the aforementioned property. It is plain that
\[
c_i \in \langle c_{j_0+1}, \ldots, c_m, 1 \rangle_F
\]
for all \( 1 \leq i \leq j_0 \). So for each \( i = 1, \ldots, j_0 \) we can write
\[
c_i = r_i c_{j_0+1} + n_i,
\]
where \( r_i \in F, n_i \in \langle c_{j_0+2}, \ldots, c_m, 1 \rangle_F \subseteq K \). Thus we can write
\[
A = c_mS_m + c_{m-1}S_{m-1} + \cdots + c_{j_0+1}(S_{j_0+1} + r_{j_0}S_{j_0} + \cdots + r_1S_1) + n_{j_0}S_{j_0} + \cdots + n_1S_1.
\]
We have \( 0 \neq S_{j_0+1} \subseteq \mathcal{S} \) such that \( \text{tr}(B) \neq 0 \). This along with the above equality easily yields \( c_{j_0+1} \in \langle c_{j_0+2}, \ldots, c_m, 1 \rangle_F \), a contradiction. Therefore, \( j_0 = m \) and so \( c_i \in F \) for all \( 1 \leq i \leq m \), finishing the proof. \( \square \)

Corollary 2.7. Let \( n > 1 \), \( K \) a field, \( F \) a subfield of \( K \), \( \mathcal{S} \) an irreducible semigroup in \( M_n(K) \) such that \( [0] \neq \text{tr}(\mathcal{S}) \leq F \). Then \( \text{Alg}_F(\mathcal{S} \cup \{I\}) = \text{Alg}_F(\mathcal{S}) \), the minimal polynomial of every element of \( \text{Alg}_F(\mathcal{S}) \) is in \( F[X] \), and \( \text{Alg}_F(\mathcal{S}) \) is semisimple and simple both as a ring and an \( F \)-algebra.

Proof. To prove \( \text{Alg}_F(\mathcal{S} \cup \{I\}) = \text{Alg}_F(\mathcal{S}) \), it suffices to show that \( I \in \text{Alg}_F(\mathcal{S}) \). To this end, first note that \( \text{Alg}(\mathcal{S}) = \langle \mathcal{S} \rangle \) is an irreducible algebra in \( M_n(K) \). By Lemma 2.1, \( I \in \langle \mathcal{S} \rangle \). Now this together with the fact that \( I \in \text{Alg}_F(\mathcal{S} \cup \{I\}) \) yields \( I \in \langle \mathcal{S} \rangle_F = \text{Alg}_F(\mathcal{S}) \) in view of the preceding lemma. To prove the rest of the
assertion, set $\mathcal{A}_F := \text{Alg}_F(\mathcal{S})$ and let $A \in \mathcal{A}_F$ be given. From the hypothesis we easily see that $\mathcal{A}_F$ is an irreducible $F$-algebra and $\text{tr}(\mathcal{A}_F) \subseteq F$. Suppose that
\[ m = x^k - m_{k-1}x^{k-1} - \cdots - m_1x - m_0 \]
with $k \leq n$ is the minimal polynomial of the given $A$. We need to show that $m_i \in F$ for each $i = 0, \ldots, k - 1$. We have
\[ A^k = m_{k-1}A^{k-1} + \cdots + m_1A + m_0I. \] (*)&

By minimality of $m$, the set $\{A^{k-1}, \ldots, A, I\} \subset \mathcal{A}_F$ is independent over $K$, hence over the subfield $F$. On the other hand, $A^k \in \mathcal{A}_F$. This together with (*) shows that $m_i \in F$ for each $i = 0, \ldots, k - 1$ in light of the preceding lemma. Finally, since the minimal polynomial of every element of the $F$-algebra $\text{Alg}_F(\mathcal{S})$ is in $F[X]$, it follows that $\text{Alg}_F(\mathcal{S})$ is an algebraic $F$-algebra. From the proof of Lemma 2.1, we see that $\text{Alg}_F(\mathcal{S})$ is semisimple and simple as an $F$-algebra, hence as a ring, for $\text{Alg}_F(\mathcal{S})$ contains the identity matrix, completing the proof. ∎

Remark. Let $n > 1$, $K$ a field that is $k$-closed for each $k$ dividing $n$ with $k > 1$, and $F$ a subfield of $K$. Then every irreducible $F$-algebra $\mathcal{A}$ in $M_n(K)$ with traces in $F$ is central, i.e., the center of $\mathcal{A}$ consists of $cI_n$’s where $c \in F$. To see this, by Theorem 2.2.21 of [16], $\mathcal{A}' = \{cI_n : c \in K\}$ and that trace is not identically zero on the $F$-algebra $\mathcal{A}$. So, due to the fact that the $F$-algebra $\mathcal{A}$ is unital by the preceding corollary, it suffices to show that if $cI_n \in \mathcal{A}$ for some nonzero $c \in K$, then $c \in F$. To prove this, from Theorem 2.2 we see that there exists $A_0 \in \mathcal{A}$ such that $\text{tr}(cI_nA_0) = c\text{tr}(A_0) \neq 0$. Therefore, $c \in F$, as desired.

Corollary 2.8. Let $n > 1$, $K$ a field, $F$ a subfield of $K$, $\mathcal{S}$ an irreducible semigroup in $M_n(K)$ such that $[0] \neq \text{tr}(\mathcal{S}) \subseteq F$. Then, $\text{Alg}_F(\mathcal{S})$ is a finite-dimensional $F$-algebra and
\[ \dim_F \text{Alg}_F(\mathcal{S}) = \dim_K \text{Alg}_K(\mathcal{S}). \]

Proof. It suffices to prove the equality. We note that $\text{Alg}_K(\mathcal{S}) = \langle \mathcal{S} \rangle_K$ and $\text{Alg}_F(\mathcal{S}) = \langle \mathcal{S} \rangle_F$ for $\mathcal{S}$ is a semigroup. Let $\{S_1, \ldots, S_m\} \subset \mathcal{S}$ be a basis for $\langle \mathcal{S} \rangle_K$. It suffices to show that $\{S_1, \ldots, S_m\}$ is a basis for $\langle \mathcal{S} \rangle_F$ as well. The subset $\{S_1, \ldots, S_m\}$ is linearly independent over the subfield $F$, for it is independent over $K$. To show that $\{S_1, \ldots, S_m\}$ spans $\langle \mathcal{S} \rangle_F$, suppose that $A \in \langle \mathcal{S} \rangle_F$ is given. Since $\langle \mathcal{S} \rangle_F \subset \langle \mathcal{S} \rangle_K$, we can write
\[ A = c_1S_1 + \cdots + c_mS_m. \]
for some $c_i \in K$ ($1 \leq i \leq m$). By Corollary 2.6 we obtain $c_i \in F$ for $1 \leq i \leq m$, completing the proof. ∎

Motivated by Theorem 4 of [7] and Theorem 3.4 of [10] we were able to prove the following theorem which is the main result of this note. Theorem 2.9 below, and
its consequences as explained in the remarks following the theorem, can be regarded as Wedderburn–Artin type theorems as follows: (a) for irreducible \( F \)-algebras of matrices in \( M_n(K) \) with traces in the subfield \( F \) but not identically zero, (b) for irreducible algebras of matrices in \( M_n(K) \) with zero trace, and (c) for irreducible algebras of matrices in \( M_n(K) \). Recall that by the Wedderburn–Artin Theorem every simple algebra \( \mathcal{A} \) of matrices is isomorphic to \( M_m(D) \) where \( m \) is a unique integer and \( D \) is a division algebra that is unique up to isomorphism. However, the theorem does not say how \( m \) and \( D \) are related to the simple algebra \( \mathcal{A} \). In comparison to the Wedderburn–Artin Theorem, Theorem 2.9 below and its consequences give a more precise description of irreducible \( (F-) \) algebras of types (a)–(c). To be more precise, by Theorem 2.9 below and its consequences, every irreducible \( (F-) \) algebra \( \mathcal{A} \) in \( M_n(K) \) of types (a)–(c) is, up to a similarity, equal to \( M_{m/r}(\mathcal{D}_r) \). Here \( r \) is the smallest nonzero rank in \( \mathcal{A} \) and divides \( n \), \( \mathcal{D}_r \oplus I_{n-r} \subset \mathcal{A} \), and \( \mathcal{D}_r \) is an irreducible division \( (F-) \) algebra in \( M_r(K) \) of types (a)–(c) respectively.

**Theorem 2.9.** Let \( n \in \mathbb{N} \), \( K \) a field, \( F \) a subfield of \( K \), and \( \mathcal{A} \) an irreducible semigroup in \( M_n(K) \) such that \( [0] \neq \text{tr}(\mathcal{A}) \subset F \). Let \( \mathcal{A} = \text{Alg}_F(\mathcal{A}) \) and \( r \in \mathbb{N} \) be the smallest nonzero rank present in \( \mathcal{A} \). Then:

(i) After a similarity, \( \mathcal{A} \) contains an idempotent \( E = I_r \oplus 0_{n-r} \) where \( I_r \) is the identity matrix of size \( r \) and \( 0_{n-r} \) is the zero matrix of size \( n-r \).

(ii) The integer \( r \) divides \( n \) and after a similarity \( E \mathcal{A} E = \mathcal{D}_r \oplus 0_{n-r} \) where \( \mathcal{D}_r \) is an irreducible division \( F \)-algebra in \( M_r(K) \) with \( [0] \neq \text{tr}(\mathcal{D}_r) \subset F \). Furthermore, the minimal polynomial of every \( D \in \mathcal{D}_r \), which is an element of \( F[X] \), is irreducible over \( F \).

(iii) After a similarity, \( \mathcal{A} = M_{n/r}(\mathcal{D}_r) \) where \( \mathcal{D}_r \) is the irreducible division \( F \)-algebra of (ii). Conversely, let \( K \) be an arbitrary field, and \( F \) a subfield of \( K \). If \( \mathcal{A} \subset M_n(K) \) is similar to \( M_{n/r}(\mathcal{D}_r) \) where \( \mathcal{D}_r \) is an irreducible division \( F \)-algebra in \( M_r(K) \) with \( \text{tr}(\mathcal{D}_r) \subset F \), then \( \mathcal{A} \) is an irreducible unital \( F \)-algebra in \( M_n(K) \) with \( \text{tr}(\mathcal{A}) \subset F \) and \( r \) is the smallest nonzero rank present in \( \mathcal{A} \).

(iv) After a similarity, \( \mathcal{A} = M_n(F) \) if and only if \( r = 1 \).

**Proof.** (i) Assume with no loss of generality that \( r < n \). Next note that by Corollary 2.7, the minimal polynomial of every \( T \in \mathcal{A} \), denoted by \( m_T \), is in \( F[X] \). In view of Lemma 1.1 and Levitzki’s Theorem, we can assume that there exists a nonnilpotent element \( T \) of \( \mathcal{A} \) with rank \( r \). Now it follows from the Primary Decomposition Theorem that there exist complementary \( T \)-invariant subspaces \( \mathcal{A} \) and \( \mathcal{N} \) such that \( K^n = \mathcal{A} \oplus \mathcal{N}, T = T_1 \oplus T_2 \) where \( T_1 = T|_{\mathcal{N}}, T_2 = T|_{\mathcal{A}}, m_T = x^{m_0} f, m_{T_1} = f, m_{T_2} = x^{m_0}, f \in F[X] \) and \( f(0) \neq 0 \) and such that \( \deg(f) \geq 1 \) (thus \( T_1 \) is invertible). Since \( T \) has minimal rank, it follows that \( T_2 = 0_{n-r} \), for otherwise

\[
\text{rank}(T^{m_0}) = \text{rank}(T_1^{m_0} \oplus T_2^{m_0}) = \text{rank}(T_1^{m_0} \oplus 0_{n-r}) = \text{rank}(T_1^{m_0})
\]

\[
= \text{rank}(T_1) < \text{rank}(T_1) + \text{rank}(T_2) = \text{rank}(T) = r,
\]

\[\text{rank}(T^{m_0}) = \text{rank}(T_1^{m_0} \oplus T_2^{m_0}) = \text{rank}(T_1^{m_0} \oplus 0_{n-r}) = \text{rank}(T_1^{m_0}) = \text{rank}(T_1) + \text{rank}(T_2) = \text{rank}(T) = r,
\]
contradicting the minimality of $r$. Thus $T_2 = 0_r$ and therefore $n_0 = 1$ for $x = m_{0_+} = t_{0_+} = x^n$ and so $r = \text{rank}(T_1) = \dim(\mathcal{A})$. Setting $p = -((f - f(0))/f(0)$, we have $p(T) = l_{\mathcal{A}} \oplus 0_r \in \mathcal{A}$. Therefore, after a similarity $\mathcal{A}$ contains the desired idempotent $E = I_r \oplus 0_{n-r}$.

(ii) To show that $r$ divides $n$, we use induction on $n$. If $n = 1$, we have nothing to prove. Suppose that the assertion holds for all irreducible $F$-algebras of matrices of size less than $n$ with traces in $F$ but not identically zero. For a given irreducible $F$-algebra $\mathcal{A}$ of matrices in $M_n(K)$ with $[0] \neq \text{tr}(\mathcal{A}) \subseteq F$, find $E = I_r \oplus 0_{n-r}$ as described in (i). Without loss of generality assume that $r \geq 2$ and $E \in \mathcal{A}$ (note that rank is invariant under similarity). Then $n - r \geq 2$ since $I = E \in \mathcal{A}$. We can write

$$\mathcal{A} : = (I - E) \mathcal{A} (I - E) = 0_r \oplus \mathcal{A}_r,$$

where $\mathcal{A}_r \subseteq M_{n-r}(K)$. Since $\mathcal{A}$ is an irreducible $F$-algebra, by Lemma 1.2, we see that $\mathcal{A}_r$ is an irreducible $F$-algebra in $M_{n-r}(K)$. Since trace is not identically zero on $\mathcal{A}$, $I - E \in \mathcal{A}$ and $I - E \neq 0$, in light of Theorem 2.2, we conclude that $[0] \neq \text{tr}(\mathcal{A}_r) \subseteq F$. Now let $r'$ be the smallest nonzero rank present in $\mathcal{A}_r \subseteq M_{n-r}(K)$.

It follows from the induction hypothesis that $r'$ divides $n - r$. So to prove that $r$ divides $n$, it suffices to show that $r' = r$. Since $0_r \oplus \mathcal{A}_r = (I - E) \mathcal{A} (I - E) \subseteq \mathcal{A}$, it suffices to show that $r' \leq r$. To this end, first we claim that $(I - E) \mathcal{A} E \neq 0$. Suppose $(I - E) \mathcal{A} E = 0$. Then, it is easily checked that $\mathcal{A} : = E K^n$ is a nontrivial subspace of $K^n$ which is invariant under $\mathcal{A}$. That is, $\mathcal{A}$ is reducible, a contradiction. So there exists $A \in \mathcal{A}$ such that $(I - E) AE \neq 0$. Now since $0 \neq (I - E)AE \in \mathcal{A}$, in view of Corollary 2.7, it follows from Exercise IX.2.5(i) of [5] that there exists $B \in \mathcal{A}$ such that $(I - E)AEB$ is not nilpotent. Therefore, $(I - E)AEB(I - E)$ is not nilpotent either. Hence $(I - E)AEB(I - E) \neq 0$, and we can write

$$0 < \text{rank}((I - E)AEB(I - E)) \leq \text{rank}(E) = r.$$

Since $0 \neq AEB \in \mathcal{A}$, we conclude that

$$r' \leq \text{rank}((I - E)AEB(I - E)) \leq r.$$

So $r' \leq r$, and hence $r = r'$. The rest easily follows from the hypotheses together with Lemma 1.2, Theorem 2.2, and Corollary 2.7.

(iii) We prove the assertion by induction on $n$. If $n = 1$, we have nothing to prove. Suppose that the assertion holds for all $F$-algebras of matrices of size less than $n$ with traces in $F$. We prove the assertion for all $F$-algebras of matrices of size $n$ with traces in $F$. Let an irreducible $F$-algebra $\mathcal{A}$ in $M_n(K)$ with $[0] \neq \text{tr}(\mathcal{A}) \subseteq F$ be given. Applying (i) and (ii) after a similarity we can write

$$E \mathcal{A} E = \mathcal{D}_r \oplus 0_{n-r}, \quad (I - E) \mathcal{A} (I - E) = 0_r \oplus \mathcal{A}_r,$$

where $E = I_r \oplus 0_{n-r} \in \mathcal{A}$ and $\mathcal{D}_r \subseteq M_r(K)$, $\mathcal{A}_r \subseteq M_{n-r}(K)$ are, respectively, an irreducible division $F$-algebra and an irreducible $F$-algebra with traces in $F$ but not identically zero. By the proof of (ii), the smallest nonzero rank present in $\mathcal{A}_r$ is $r$. Since $n - r < n$, it follows from the induction hypothesis that after a similarity of the form $T_r^{-1}(\cdot)T_r$ with $T_r \in M_{n-r}(K)$, we have $\mathcal{A}_r = M_{n-r}(\mathcal{D}_r)$ where
$\mathcal{D}' \subseteq M_r (K)$ is an irreducible division $F$-algebra with traces in $F$ but not identically zero. Applying the similarity $I_r \oplus T_r$ to $\mathcal{A}$ we may assume that

$E : \mathcal{A} = \mathcal{D}_r \oplus 0_{n-r} \subset \mathcal{A}$, \hspace{1cm} (I - E) \mathcal{A} (I - E) = 0_r \oplus \mathcal{A}_r \subset \mathcal{A},$ \hspace{1cm} (*)

where $E = I_r \oplus 0_{n-r} \in \mathcal{A}$, $\mathcal{A}_r = M_{n/r} (\mathcal{D}_r')$ and $\mathcal{D}_r$, $\mathcal{D}_r'$, $\mathcal{A}_r$ are as described above. Note that every element of $A \in \mathcal{A}$ can be represented in the block form, i.e., $A = (a_{ij})_{2 \times 2}$ where the blocks, i.e., $a_{ij}$'s, are matrices of size $r$ over $K$. For $A = (a_{ij}) \in \mathcal{A}$, $A_{ij} \in M_{n_r} (M_r (K))$ is used to denote the block matrix with $a_{ij} \in M_r (K)$ in the $ij$ place and $0_r \in M_r (K)$ elsewhere. Let $E_{ij} \in M_{n/r} (M_r (K))$ denote the block matrix with the identity $I_r \in M_r (K)$ in the $ij$ place and $0_r \in M_r (K)$ elsewhere. It follows from $(*)$ that $E_{ij} \in \mathcal{A}$ for $i = j = 1$ and for all $2 \leq i, j \leq n/r$. Thus if $A \in \mathcal{A}$, then $A_{1j} = E_{11} A E_{jj}, A_{1i} = E_{ii} A E_{11} \in \mathcal{A}$ for all $1 \leq i, j \leq n/r$.

As we saw in the proof of (ii), it follows from irreducibility of $\mathcal{A}$ that $(I_1 - E_{11}) \mathcal{A} E_{11} \neq 0$ (note that in fact $E_{11} = E = I_r \oplus 0_{n-r} \in \mathcal{A}$). Since $I_n - E_{11} = E_{22} + \cdots + E_{rr} \in \mathcal{A}$ it follows that there exists $2 \leq i_0 \leq n/r$ such that $E_{i_0 i_0} \mathcal{A} E_{11} \neq 0$. That is, there exists $A \in \mathcal{A}$ such that $0 \neq A_{i_0 1} = E_{i_0 i_0} A E_{11} \in \mathcal{A}$. This along with minimality of $r$ implies that $A_{i_0 1} \in M_r (K)$ is invertible (note that $A = (a_{ij}) \in \mathcal{A}$). Similarly, it follows from irreducibility of $\mathcal{A}$ that $E_{11} \mathcal{A} (I_n - E_{11}) \neq 0$. Therefore, there exist $2 \leq j_0 \leq n/r$ and $A' \in \mathcal{A}$ such that $0 \neq A_{1j_0}' = E_{11} A E_{j_0 j_0} \in \mathcal{A}$. This together with minimality of $r$ implies that $A_{1j_0}' \in M_r (K)$ is invertible (note that $A' = (a_{ij}') \in \mathcal{A}$). It is not difficult to see that we have $T^{-1} \mathcal{A} T = M_{n/r} (\mathcal{D}_r)$ where $T = \text{diag} (I_r, a_{i_0 1}, \ldots, a_{i_0 1}) \in M_r (K)$ and $\mathcal{D}_r$ is as in (ii) (see [16, Theorem 2.3.11] for a detailed proof).

We omit the proof of the converse since it is easy (see [16, p. 49] for a detailed proof).

(iv) The “only if” part trivially holds. To see the “if” part, it is plain that if $r = 1$, then $\mathcal{D}_1 = F$, and hence after a similarity $\mathcal{A} = M_n (F)$.

\textbf{Remarks}

1. In view of Proposition 1.5 of [11] and the proof of the preceding theorem, it is not difficult to see that under the hypothesis that the irreducible $F$-algebra $\text{Alg}_F (\mathcal{D})$ is algebraic over $F$ (this is certainly the case whenever $K$ is a finite extension of $F$), the assumption that trace is not identically zero on the semigroup is redundant and yet the theorem holds true except that we cannot be sure that the minimal polynomial of every member of $\text{Alg}_F (\mathcal{D})$ belongs to $F [X]$ unless $F = K$.

2. Let $n > 1$, $K$ a field, and $F$ a subfield of $K$. If the ground field $K$ is perfect; or $\text{ch} (K)$ is not a divisor of $n$; or $k$-closed for each $k$ dividing $n$ with $k > 1$, then in light of Theorems 2.2.19 and 2.3.3 of [16], we see that the trace functional cannot be identically zero on an irreducible semigroup in $M_n (K)$. Therefore, the conclusions of Lemma 2.6 and Corollary 2.11 below (together with those results that are stated as remarks) hold for every irreducible semigroup or $F$-algebra with traces in the subfield $F$ provided that the ground field $K$ is perfect; or $\text{ch} (K)$ is not a divisor of $n$; or the field $K$ is $k$-closed for each $k$ dividing $n$ with $k > 1$. 

}\
3. In light of Lemmas 1.1, 1.2, and Theorem 2.3.3 of [16], if the semigroup in Lemma 2.6 and Theorem 2.9 happens to be absolutely irreducible, then the same arguments show that, with no condition imposed on the field $K$, similar conclusions hold except that everywhere “irreducible” should be replaced by “absolutely irreducible”.

In case the subfield $F$ happens to be finite, we can give a more precise description of irreducible ($F$-)algebras of types (a), (b), and (c) as described on page 24 following Corollary 2.8. The following two results serve this purpose.

**Lemma 2.10.** Let $n > 1$, $K$ a field, $F$ a finite subfield of $K$, and $D$ an irreducible division $F$-algebra in $M_n(K)$ with $\{0\} \neq \text{tr}(D) \subseteq F$. Then $D$ is a field and there exists a $K$-irreducible matrix $A \in M_n(F)$ such that after a similarity $D = F[A]$. Therefore, $D$ is indeed a simple extension field of $F$.

**Proof.** Let $\{I_n, D_1, \ldots, D_p\}$ be a basis in $D$ for $\langle D \rangle_K$, the linear manifold spanned by $D$ over $K$. By Corollary 2.8 we have $D = \langle I_n, D_1, \ldots, D_p \rangle_F$. As $F$ is a finite field, we see that $D$ is a finite division ring. Now by Wedderburn’s Theorem (see [5, Corollary IX.6.9]), $D$ must be a field. That is, $D$ is a finite-dimensional extension field of the finite, hence perfect, field $F$. Thus, there exists a matrix $B \in M_n(K)$ such that after a similarity $D = F[B]$. The irreducibility of $D$ yields that of the matrix $B$ in $M_n(K)$. It is now plain that the characteristic polynomial of the matrix $B \in M_n(K)$ equals its minimal polynomial which is an element of $F[X]$ by Corollary 2.7. Therefore, there exists a $K$-irreducible $A \in M_n(F)$ such that after a similarity $D = F[A]$ (in fact $A$ can be taken as the companion matrix of $B$). □

**Remark.** In the preceding lemma if $K$ happens to be $k$-closed for each $k$ dividing $n$ with $k > 1$ (ch($K$) $\neq 0$), then it follows from the proof of the lemma that there is no irreducible division $F$-algebra $D$ in $M_n(K)$ with tr($D$) $\subseteq F$.

The following result is a quick consequence of Theorem 2.9 and Lemma 2.10.

**Corollary 2.11.** Let $n > 1$, $K$ a field, $F$ a finite subfield of $K$, and $S$ an irreducible semigroup in $M_n(K)$ with $\{0\} \neq \text{tr}(S) \subseteq F$. Let $r \in \mathbb{N}$ be the smallest nonzero rank present in $\text{Alg}_F(S)$. Then $r$ divides $n$ and

- (i) if $r = 1$, then after a similarity $\text{Alg}_F(S) = M_n(F)$;
- (ii) if $r > 1$, then there exists a $K$-irreducible matrix $A \in M_n(F)$ such that after a similarity $\text{Alg}_F(S) = M_{n/r}(F[A])$;

Therefore, in any case $\text{Alg}_F(S)$ is indeed a finite irreducible $F$-algebra in $M_n(K)$. Furthermore, after a similarity, the commutant of $\text{Alg}_F(S)$ in $M_n(K)$, which is the same as that of $S$ in $M_n(K)$, is equal to $B \oplus \cdots \oplus B$ where $B \in F[A]$ is arbitrary.
Proof. Theorem 2.9 together with Lemma 2.10 easily settles the proof. □

Remark. Theorem 2.9, its proof, and the preceding corollary are used in [17] to prove, among other things, Burnside type theorems for irreducible $F$-algebras of matrices with traces (resp. spectra) in the subfield $F$.

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