Non-submetrizable spaces of countable extent with $G_\delta$-diagonal

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Abstract

We construct two non-submetrizable spaces of countable extent that have a $G_\delta$-diagonal. Both spaces are locally-compact, locally-countable, separable, Tychonoff. One space is hereditary real-compact. The other is pseudocompact and consistent.

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1. Introduction

In [3], Chaber proved that a countably compact space with a $G_\delta$-diagonal is metrizable. It is known that countable compactness cannot be replaced by pseudocompactness as witnessed by the classical Mrowka space [11]. The Mrowka space is a pseudocompact locally compact space with a $G_\delta$-diagonal that fails to be submetrizable. However, while having all these strong compactness-type properties, it has a huge discrete which make the space far from being countably compact. Therefore, it is natural to ask if there are non-submetrizable spaces of countable extent that have a $G_\delta$-diagonal and possess strong compactness-type properties. In this paper we construct the following examples.

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Example I (Sections 2–3). A hereditary realcompact locally-compact locally-countable separable Tychonoff space with countable extent and a $G_\delta$-diagonal that fails to be submetrizable.

Example II (Section 4). A consistent example of a pseudocompact non-compact locally-compact locally-countable separable Tychonoff space that has countable extent and a $G_\delta$-diagonal.

Our examples suggest the following questions.

Question 1.1. Is there a ZFC example of a pseudocompact non-compact space with countable extent that has a $G_\delta$-diagonal?

Question 1.2. Let $X$ be a countably paracompact space with countable extent and a $G_\delta$-diagonal. Is then $X$ submetrizable? What if $X$ is first-countable (or locally compact)?

Recall that a space $X$ has countable extent if every closed discrete subset of $X$ is countable. Note that countable extent and non-metrizability imply that our spaces are not Moore spaces. The first example of a non-Moore locally-compact non-submetrizable space with a $G_\delta$-diagonal was constructed by Burke in [4].

Another possible direction where one might expect to have a theorem is relaxing Lindelöfness. Recall that a Lindelöf space with a $G_\delta$-diagonal is submetrizable. Recall that a Lindelöf space with a $G_\delta$-diagonal is submetrizable.

Question 1.3 (A.V. Arhangel’ skii). Let $X$ be an $\omega_1$-Lindelöf space with a $G_\delta$-diagonal. Is $X$ submetrizable? What if $X$ is linearly Lindelöf?

Recall that a space $X$ is called $\omega_1$-Lindelöf if every $\omega_1$-sized open cover of $X$ contains a countable subcover. That is, $\omega_1$-Lindelöfness is simply $[\omega_1, \omega_1]$-compactness according to Alexandroff–Urysohn terminology. And $X$ is called linearly Lindelöf if every open cover of $X$ that forms a chain contains a countable subcover. Notice that under CH the answer to Question 1.3 is “Yes”. This follows from the Ginsburg–Woods theorem [8] that the cardinality of a space with a $G_\delta$-diagonal and countable extent is at most $2^{\omega_1}$ (and apply the fact that an $\omega_1$-sized $\omega_1$-Lindelöf space is Lindelöf). Another Lindelöf-type property is discrete Lindelöfness. A space $X$ is called discretely Lindelöf if the closure of every discrete subspace of $X$ is Lindelöf. In [1], the authors proved that every discretely Lindelöf space of tightness $\omega_n$ is Lindelöf. In general, it is not known whether the class of discretely Lindelöf spaces differs from the class of Lindelöf spaces.

Question 1.4. Let $X$ be a discretely Lindelöf space with a $G_\delta$-diagonal. Is $X$ submetrizable?

It might be interesting to strengthen countable extent to $\omega_1$-Lindelöfness in the condition of Question 1.1 (or 1.2). In general, a pseudocompact $\omega_1$-Lindelöf, even linearly Lindelöf (see Mischenko’s example in [10]), space need not be countably compact. However, a pseudocompact space with a $G_\delta$-diagonal is first-countable and the author does not know an answer to the following question.
Question 1.5. Is there a first-countable pseudocompact (and locally-compact) $\omega_1$-Lindelöf space which is not countably compact?

It is easy to see that a pseudocompact space with a $G_\delta$-diagonal is Čech-complete. This suggests the following question.

Question 1.6. Let $X$ be an $\omega_1$-Lindelöf Čech-complete space with a $G_\delta$-diagonal. Is $X$ submetrizable? Is $X$ metrizable? What if $X$ is linearly Lindelöf?

We would like to make a couple of observations that speak in favour of “Yes” to this question. First, the square of an $\omega_1$-Lindelöf (or linearly Lindelöf) Čech complete space is $\omega_1$-Lindelöf (linearly Lindelöf) (see Karpov [9]). And if $X$ has a $G_\delta$-diagonal, then being an $F_\sigma$-subset, the co-diagonal part is $\omega_1$-Lindelöf (linearly Lindelöf). Second observation is that a locally compact $\omega_1$-Lindelöf space with a $G_\delta$-diagonal is metrizable. Indeed, a locally-compact space with a $G_\delta$-diagonal is locally metrizable. And as shown in [2], a locally metrizable $\omega_1$-Lindelöf space is metrizable. Of course, a counterexample will be also a very welcome addition to a small collection of nice $\omega_1$-Lindelöf (linearly Lindelöf) spaces.

The author recently showed that if $X^2$ has countable extent and $X$ has a zeroset diagonal, then $X$ is submetrizable. In particular, it follows that a Čech-complete $\omega_1$-Lindelöf space with a zeroset diagonal is submetrizable. However, in Questions 1.2–1.4, the author does not know answers even if we replace “$G_\delta$-diagonal” by “regular $G_\delta$-diagonal” or “zeroset diagonal”.

In notation and terminology we will follow [7]. All spaces above and below are Tychonoff. A space $X$ is said to have a $G_\delta$-diagonal if its diagonal $\Delta_X$ is a $G_\delta$-set in $X \times X$. This is known to be equivalent to the existence of a sequence $\{\mathcal{V}_n\}_n$ of open covers of $X$ such that $\bigcap_n \text{St}(x, \mathcal{V}_n) = \{x\}$ for every $x \in X$.

The set of countable sequences on a set $A$ is denoted by $[A]^\omega$. If $s$ is a sequence on a set $A$, then by $\text{ran}(s)$ we denote the range of $s$, that is, the set $\{s(n) : n \in \omega\}$.

Necessary definitions for each example will be given in Sections 2 and 4, respectively. In both examples, to achieve countable extent we use a technique discovered independently by van Douwen and Wicke [6,5] and Pytkeev [13] and based on famous Ostaszewski’s constructions (see, for example, [12]).

2. The key lemma for Example I

The ground sets for our example will be two copies of the Cantor Tree $T$. To construct our space, we first perform a construction of van Douwen–Wicke and Pytkeev (with some changes that serve our purpose) to achieve countable extent (namely, we use a construction of $\Sigma$ in [5]). And then we identify uncountable discretes of two Trees along a carefully chosen subset to ensure that the resulting quotient space is not submetrizable but keeps $G_\delta$-diagonal and all other promised properties.

Here is a brief description of the Cantor Tree (borrowed with some changes from [14]).
Cantor Tree. Let $C$ denote the Cantor set in the unit interval $[0, 1]$; the midpoints of the components of $[0, 1] \setminus C$ are $1/2, 1/6, 5/6, 1/18, 5/18$, etc. Let $D$ be the set of points in the form $(1/2, -1), (1/6, -1/2), (5/6, -1/2), (1/18, -1/4), (5/18, -1/4)$, etc. The set $D$ is naturally viewed as a discrete binary tree branching upward. Then the space $T$ is defined as $D \cup C$. A base neighborhood of a point $c \in C$ is $I^* \cup \{c\}$, where $I^*$ is a path in the tree $D$ whose upper limit is $c$.

The natural projection of $T$ to $R^2$ is obtained by identifying points of $C$ with the corresponding points in $[0, 1] \times \{0\}$. Whenever we refer to the distance $\rho$ between points in $T$ we mean that the distance is calculated in this projection with respect to the Euclidean topology of $R^2$. If the closure of a set $Y \subset T$ is taken in the Euclidean topology we write $\text{Cl}_{R^2}(Y)$.

Let $T'$ be a primed copy of the Cantor Tree $T$. If $y \in Y \subset T$, by $y'$ and $Y'$ we denote the corresponding twin copies in $T'$.

If $A \subset C$, by $TAT'$ we denote the quotient space defined by the partition on $T \oplus T'$ whose only nontrivial elements are $\{x, x'\}$, where $x \in A$. Note that no matter what $A \subset C$ is, $TAT'$ is always regular.

Lemma 2.1. There exists an $A \subset C$ with the following properties:

1. $TAT'$ is not submetrizable;
2. $A$ is a Bernstein set in $C$ with the Euclidean topology.

Proof. Let $F$ be the set of all continuous maps from $T \oplus T'$ to $R^\omega$. Since $T$ is separable, $|F| = 2^\omega$. Let $B$ be the set of all uncountable compact sets in $C$ with the Euclidean topology. Enumerate $B = \{B_\alpha: \alpha < 2^\omega\}$ and $F = \{f_\alpha: \alpha < 2^\omega\}$. Inductively, we will define $a_\alpha$ and $c_\alpha$ (when possible) and $A$ will be the set of all $a_\alpha$’s.

Definition of $a_\alpha$: Let $A_\alpha$ be the set of all $a \in C \setminus \{[a_\beta: \beta < \alpha] \cup [c_\beta: \beta < \alpha]\}$ such that $f_\alpha(a) \neq f_\alpha(a')$. If $A_\alpha \neq \emptyset$, let $\alpha^1$ be the smallest ordinal such that $B_{\alpha^1} \in B$ meets $A_\alpha$ and does not meet $[a_\beta: \beta < \alpha]$. Let us prove the existence of $\alpha^1$. Partition $C$ with the Euclidean topology into $2^\omega$ uncountable compacta. Such a partition can be obtained by taking homeomorphic images in $C$ of vertical threads of $C \times C$. Since $\alpha < 2^\omega$, at least one element of this partition does not meet $[a_\beta: \beta < \alpha]$. Fix this element. Take any $a \in A_\alpha$ and adjoin this $a$ to the fixed element of the partition. Clearly, we found an element of $B$ that meets $A_\alpha$ and does not meet $[a_\beta: \beta < \alpha]$.

Pick any $a_\alpha \in B_{\alpha^1} \cap A_\alpha$.

Definition of $c_\alpha$: Let $C_\alpha$ be the set of all $c \in C \setminus \{[a_\beta: \beta \leq \alpha] \cup [c_\beta: \beta < \alpha]\}$ such that $f_\alpha(c) = f_\alpha(c')$. If $C_\alpha \neq \emptyset$, let $\alpha^2$ be the smallest ordinal such that $B_{\alpha^2} \in B$ meets $C_\alpha$ and does not meet $[c_\beta: \beta < \alpha]$. Pick any $c_\alpha \in B_{\alpha^2} \cap C_\alpha$.

At each step $\alpha$, either $a_\alpha$, or $c_\alpha$, or both exist because either equality or inequality is satisfied for $2^\omega$ pairs, while at most $2\alpha < 2^\omega$ points are picked before step $\alpha$. Let $A = \{\text{all picked } a_\alpha\}$. Note that $A \cap \{\text{all picked } c_\alpha\} = \emptyset$. Let us show that $TAT'$ is not
submetrizable. Assume the contrary. Then there exists a continuous injection \( f \) of \( TAT' \) to \( R^\omega \). There exists \( f_\alpha \in F \) such that \( f_\alpha = f \circ p \), where \( p \) is the quotient map that defines \( TAT' \). If \( a_\alpha \) is defined then \( f_\alpha(a_\alpha) \neq f_\alpha(a'_\alpha) \), and therefore, \( f \) assigns the element \( \{a_\alpha, a'_\alpha\} \) to two different elements, a contradiction with \( f \) being a mapping. If \( \epsilon_\alpha \) is defined, then \( f_\alpha(\epsilon_\alpha) = f_\alpha(\epsilon'_\alpha) \), and therefore, \( f \) maps two different elements to one element, a contradiction with \( f \) being one-to-one.

Now let us show that \( A \) is a Bernstein set in \( C \) with the Euclidean topology. Assume that for \( \beta < \alpha \), \( B_\beta \in B \) meets both \( A \) and \( C \setminus A \). Let \( F_1 \subset F \) be the set of all one-to-one maps. Clearly \( |F_1| = 2^\omega \). By our assumption, there exists \( f_\beta \in F \) such that \( \{a_\beta : \beta < \gamma\} \) meets all \( B_\beta \) for \( \beta < \alpha \). Since \( |F_1| = 2^\omega \), we may assume that \( f_\gamma \in F_1 \). Since \( f_\gamma \) is one-to-one, \( A_\gamma \cap B \neq \emptyset \) for every \( B \in B \). Therefore, either \( \{a_\beta : \beta < \gamma\} \) meets \( B_\alpha \) or \( \alpha = \gamma + 1 \), and therefore, \( a_\gamma \in B_\alpha \cap A_\gamma \).

To show that \( C \setminus A \) meets \( B_\alpha \), define \( F_2 \) as the set of all \( f \in F \) such that \( f(c) = f(c') \) for all \( c \in C \) (simply those mappings that naturally identify \( C \) with \( C' \) and then map the results in \( R^\omega \)). The set \( F_2 \) has cardinality \( 2^\omega \). Further argument is the same as with the set \( A \). \( \square \)

Note that the argument of Lemma 2.1 can be applied to any separable space with \( 2^\omega \)-sized closed discrete. It is clear that \( TAT' \) is a Moore realcompact space. It is not submetrizable as it is so constructed and it certainly has a \( G_\delta \)-diagonal as all Moore spaces do. To show that \( TAT' \) is a Moore space, for each \( x \in C \) fix \( \Gamma_n(x) \) a path in \( D \) with upper limit \( x \) and with the Euclidean diameter less than \( 1/n \). Define the family \( \mathcal{V}_n \) as follows: \( V \in \mathcal{V}_n \) iff it is one of the following types:

1. \( \Gamma_n(x) \cup \{x\} \), where \( x \in C \setminus A \);
2. \( \Gamma'_n(x') \cup \{x'\} \), where \( x \in C \setminus A \);
3. \( \Gamma_n(x) \cup \Gamma'_n(x') \cup \{x, x'\} \), where \( x \in A \);
4. \( \{x\} \), where \( x \in D \cup D' \).

Obviously the sequence \( \{\text{St}(z, \mathcal{V}_n)\}_n \) is a base at \( z \in TAT' \).

3. Example I

Let \( A \) be as in Lemma 2.1. Let \( \{A_n\}_n, \{A^*_n\}_n \) be partitionings of \( A \) and \( C \setminus A \) into Bernstein sets with respect to the Euclidean topology (exist since \( A \) is a Bernstein set). In the following lemma, convergence and closures are taken in the Euclidean topology.

**Lemma 3.1.** For each \( n \) there exist families \( S_n = \{s_\alpha \in [A_n]^\omega : \alpha < 2^\omega \} \) and \( S^*_n = \{s^*_\alpha \in [A^*_n]^\omega : \alpha < 2^\omega \} \) of sequences with the following properties:

1. \( s_\alpha \to x \in A_{n+1} \) and \( s^*_\alpha \to x^* \in A^*_{n+1} \);
2. If \( \alpha \neq \beta \), \( \lim s_\alpha \neq \lim s_\beta \) and \( \lim s^*_\alpha \neq \lim s^*_\beta \);
3. For any countable \( B \subset A_n \) (\( B \subset A^*_n \)) such that \( |\text{Cl}_{R^2}(B)| = 2^\omega \) there exists \( s \in S_n \) (\( s^* \in S^*_n \)) such that \( \text{ran}(s) \subset B \) (\( \text{ran}(s^*) \subset B \)).
**Proof.** Let $\mathcal{B}$ and $\mathcal{B}^*$ be the families of all countable subsets of $A_n$ and $A_n^*$, respectively, with uncountable closures. Enumerate $\mathcal{B} = \{B_\alpha: \alpha < 2^\omega\}$ and $\mathcal{B}^* = \{B_\alpha^*: \alpha < 2^\omega\}$. We will define $s_\alpha$ and $s_\alpha^*$ inductively.

**Definition of $s_\alpha$, $s_\alpha^*$:** Pick any $x \in \overline{B}_\alpha \cap A_{n+1} (x^* \in \overline{B}^*_\alpha \cap A_{n+1}^*)$ such that $s_\beta \not\rightarrow x (s_\beta^* \not\rightarrow x^*)$ for all $\beta < \alpha$. This can be done because both $A_{n+1}$ and $A_{n+1}^*$ are Bernstein sets. Choose $s_\alpha \in \overline{B}_\alpha^\omega$ ($s_\alpha^* \in \overline{B}^*_\alpha^\omega$) such that $s_\alpha \rightarrow x (s_\alpha^* \rightarrow x^*)$. □

Let $T_r, T_e$ be the Tree and Euclidean topologies on $T$. Using van Douwen’s $\Sigma$-construction, we will define a new topology $T$ on $T$ with the following properties.

P1. $T_e \subset T \subset T_r$;

P2. $(T, T)$ is regular, locally compact, locally countable, separable;

P3. $(T, T)$ has countable extent;

P4. For any $x \in T, n > 0$ there exists $x \in V(x) \in T$ such that the diameter of $V(x)$ in $R^2$ is less than $1/n$;

P5. $A$ and $C \setminus A$ are closed in $(T, T)$. In fact, P4 is a direct consequence of P1 and P2 but we will keep it for further reference.

**Definition of $T$.** Let $X_0 = D$ and $X_n = T \setminus \{x \in A_m \cup A_n^*: m > n\}$. Let $T_0$ be the Euclidean topology on $D$ (simply discrete). Suppose $T_m$ is defined on $X_m$. $T_k \subset T_m$ for all $k < m < n$, and $(X_m, T_m)$ is locally compact. Suppose that $T_m$ is finer than the Euclidean topology and coarser than the Tree topology on $X_m$.

**Definition of $T_n$:** Take any $x \in X_n$. If $x \in X_{n-1}$, then base neighborhoods at $x$ are those from $T_{n-1}$. Otherwise $x \in A_n \cup A_n^*$. If no $s \in S_{n-1} \cup S_{n-1}^*$ converges to $x$ in the Euclidean topology, then base neighborhoods at $x$ are in form $\{x\} \cup \Gamma(x)$, where $\Gamma(x)$ is a path in $D$ with upper limit $x$. Otherwise, there exists $s \in S_{n-1} \cup S_{n-1}^*$ that converges to $x$ in the Euclidean topology. For each $s(i)$ fix a compact base neighborhood $V(s(i)) \in T_{n-1}$ with the Euclidean diameter less than $\rho(x, s(i))$. Such neighborhoods exist since $(X_{n-1}, T_{n-1})$ is locally compact and $T_{n-1}$ is finer than the Euclidean topology on $X_{n-1}$. For each $k > 0$, fix $\Gamma_k(x) \subset D$ a path to $x$ with the Euclidean diameter less than $1/k$. Define local base at $x$ as follows:

$$B_x = \{x\} \cup \Gamma_k(x) \cup \left[ \bigcup_{i > k} V(s(i)) \right] : k \in \omega \}.$$

Let $T$ be the topology on $T$ with a base $\bigcup_n T_n$.

For P1, notice that a base neighborhood of each $x \in C$ contains $\Gamma(x) \cup \{x\}$. Properties P2–P4 are all proved in [5] and, in fact, are clear from construction. Nevertheless, let us repeat van Douwen’s argument for P3. Take any uncountable subset $M \subset T$. Then there exists $n$ such that $M \cap A_n$ (or $M \cap A_n^*$) is uncountable. There exists a countable $K \subset M \cap A_n$ such that $\text{Cl}_{R^2}(K)$ is uncountable. Then, by our construction, there exists
\( s \in S_h \) such that \( \text{ran}(s) \subset K \) and \( s \to x \in A_{n+1} \) in the Euclidean topology. By the definition of a local base at \( x, \) \( x \) is a limit point for \( s, \) and therefore for \( M \) in \( \mathcal{T}. \)

Let us prove P5. If \( x \in A \) then either its base neighborhood is \( \{x\} \cup T(x), \) which does not meet \( C \setminus A, \) or \( \{x\} \cup T(x) \cup \bigcup_{i>k} V(s(i)), \) where \( \text{ran}(s) \subset A, \) which does not meet \( C \setminus A \) either (simple induction on \( n \)). And since \( D \) is open in \((T, T),\) \( C \setminus A \) is closed in \((T, T).\) Similarly, \( A \) is closed in \((T, T).\)

We are finally ready to construct our space. Let \( \mathcal{T}A\mathcal{T}' \) be the quotient topology defined by the partition on \((T, T) \oplus (T', T')\) whose only nontrivial elements are \( \{x, x'\} \), where \( x \in A. \) Since \( A \) and \( C \setminus A \) are closed in \((T, T),\) the quotient map is perfect and therefore \((\mathcal{T}A\mathcal{T}', \mathcal{T}A\mathcal{T}'))\) is locally compact and regular. It is also locally countable and has countable extent. In addition, our space is hereditary realcompact since it admits a two-to-one continuous map to \( R^2. \) The space \((\mathcal{T}A\mathcal{T}', \mathcal{T}A\mathcal{T}'))\) is not submetrizable because \( \mathcal{T}A\mathcal{T}' \) is not by Lemma 2.1 and the tree topology on \( T \) is finer than \( T. \) We only need to show that \((\mathcal{T}A\mathcal{T}', \mathcal{T}A\mathcal{T}'))\) has a \( G_\delta \)-diagonal.

**Lemma 3.2.** \((\mathcal{T}A\mathcal{T}', \mathcal{T}A\mathcal{T}'))\) has a \( G_\delta \)-diagonal.

**Proof.** Let \( p \) be the quotient map that defines \((\mathcal{T}A\mathcal{T}', \mathcal{T}A\mathcal{T}')).\) For each \( x \in C \setminus A \) fix \( V_n(x) \in \mathcal{T} \) such that \( V_n(x) \) has Euclidean diameter less than \( 1/n \) and \( V_n(x) \cap A = \emptyset \) (recall that \( A \) is closed in \((T, T)). \) Since \( V_n(x) \cap A = \emptyset, \) \( V_n(x) \in \mathcal{T}A\mathcal{T}'. \) Let \( V_n'(x') \) be the corresponding twin neighborhood for \( x'. \)

For every \( x \in A \) fix its neighborhood \( V_n(x) \in \mathcal{T} \) with the Euclidean diameter less than \( 1/n \) such that \( V_n(x) \cap (C \setminus A) = \emptyset. \) Define the family \( \mathcal{V}_n \) as the union of the following collections:

1. \( \{V_n(x): x \in C \setminus A\}; \)
2. \( \{V_n'(x'): x \in C \setminus A\}; \)
3. \( \{p(V_n(x)) \cup p(V_n'(x')): x \in A\}; \)
4. \( \{\{x\}: x \in D \cup D'\}. \)

Clearly, \( \mathcal{V}_n \) is an open cover of \((\mathcal{T}A\mathcal{T}', \mathcal{T}A\mathcal{T}')). \) It suffices to show now that \( \bigcap_n \text{St}(x, \mathcal{V}_n) = \{x\} \) for every \( x \in \mathcal{T}A\mathcal{T}'. \) Since the diameters approach 0 it is enough to show that for any \( x \in C \setminus A \) there exists \( n \) such that no element of \( \mathcal{V}_n \) contains both \( x \) and \( x' \) at the same time. But it is so for \( n = 1! \)

**Remark.** Note that the constructed space is neither normal nor countably paracompact. This follows from the fact that \( A \) and \( C \setminus A \) are closed in \( \mathcal{T} \) while dense in \( \mathcal{T} \) with the Euclidean topology. But \( A \) to be closed to make the quotient space regular and closedness of \( C \setminus A \) guarantees that the quotient space keeps \( G_\delta \)-diagonal. One possible way to achieve normality and countable paracompactness is to identify \( T \) and \( T' \) in a random manner along all of \( C \) and on the identified Cantor sets perform the van Douwen’s \( A \)-construction. However, there appear to be difficulties with controlling convergence from the side of \( T' \) and the author does not know whether they can be overcome.
4. (CH) Example II

To construct our example we combine two known techniques. The space we construct resembles Mrowka’s space except that the huge discrete will be turned into a closed subspace with countable extent using the technique of van Douwen–Wicke and Pytkeev. As in Mrowka’s space, we achieve pseudocompactness using a mad family concept. A maximal almost disjoint (abbreviated as mad) family \( M \) on a countable set \( A \) is a maximal family of infinite subsets of \( A \) with the property that \( |M \cap N| < \omega \) for any \( M, N \in M \).

In this section after Lemma 4.1 we assume CH. By \( R \) and \( Q \) we denote the real line and the rational numbers.

**Lemma 4.1.** There exists a family \( S \) of convergent sequences in \( R \setminus Q \) and an enumeration \( R \setminus Q = \{x_\alpha : \alpha < 2^\omega\} \) with the following properties:

1. \( \lim s_1 \neq \lim s_2 \) if \( s_1 \neq s_2 \in S \);
2. For any countable \( A \subset R \setminus Q \) with uncountable closure there exists \( s \in S \) such that \( \text{ran}(s) \subset A \);
3. If \( s \in S \) and \( x_\alpha = \lim s \) then \( \text{ran}(s) \subset \{x_\beta : \beta < \alpha\} \);
4. For uncountably many \( \alpha \), no \( s \in S \) converges to \( x_\alpha \).

**Proof.** Let \( A = \{A_\alpha : \alpha < \omega_1\} \) be an infinite mad family on \( Q \). Let \( N = \{N_\alpha : \alpha < \omega_1\} \) be the family of all infinite subsets of elements of \( \mathcal{M} \). That is, \( N \in \mathcal{N} \) iff \( N \) is an infinite subset of some \( M \in \mathcal{M} \). Let us view each \( N_\alpha \) as a sequence and by \( N_\alpha^k \) we denote \( \{N_\alpha(i) : i > k\} \).

The underlying set for our space is \( R \). We will define a new topology \( T \) on \( R \) with the following properties:

P1. \( (R, T) \) is regular, locally compact, locally countable, separable;
P2. \( (R, T) \) has countable extent;
P3. \( (R, T) \) is pseudocompact and not compact;
P4. \( (R, T) \) has a \( G_\delta \)-diagonal.

All points of \( Q \) are declared isolated. Local bases \( B_{x_\alpha} \) at \( x_\alpha \) are defined inductively. Let \( X_\alpha = Q \cup \{x_\beta : \beta \leq \alpha\} \) and let \( T_\alpha \) be the topology on \( X_\alpha \) generated by local bases at \( x_\beta \) and singletons from \( Q \).

Assume that local bases \( B_{x_\beta} \) are defined for all \( \beta < \alpha \) and the following hold.
A1. \((X_\beta, T_\beta)\) is regular, locally compact, locally countable, not compact;
A2. If \(\gamma < \beta\), \(T_\gamma \subset T_\beta\) and \(T_\beta|_{X_\gamma} = T_\gamma\);
A3. All elements of \(B_{x_\beta}\) are compact in \((X_\beta, T_\beta)\);
A4. \(Q\) is dense in \((X_\beta, T_\beta)\);
A5. For each \(n > 0\) there exists \(B \in B_{x_\beta}\) such that the Euclidean diameter of \(B \cap (R \setminus Q)\) is less than \(1/n\).

The following remark can be found in [5,6,12], or [13]. However, we will prove it here for completeness.

**Remark.** If A1–A3 hold then the space \(X' = \bigcup_{\beta < \alpha} X_\beta\) with the union topology \(T'\) is locally-compact, locally-countable, not compact, regular. Indeed, for Hausdorffness take any \(x, y \in X'\). There exists \(\beta < \alpha\) such that \(x, y \in X_\beta\). Since \(T_\beta \subset T'\), and \(T_\beta\) is Hausdorff, \(x, y\) are Hausdorff separated. Local countability follows from A2. For local compactness, take any \(x_\gamma \in X'\) and \(B \in B_{x_\gamma}\). By A1, \(B\) is compact and open in \(X_\gamma\) and, by A2, \(B\) is compact and open in \((X', T') = (X_\beta, T_\beta)\) and, hence, is not compact. Otherwise, there exists a strictly increasing sequence \(\{\alpha_n\}_n\) converging to \(\alpha\). By A2, \(\{x_{\alpha_n}\}_n\) is a closed and discrete subspace of \((X', T')\).

For \(\beta < \alpha\), let \(N_\beta = \{N \in N' : N \text{ meets infinitely many elements of } B_{x_\beta}\}\).

**Definition of local base \(B_{x_\alpha}\) at \(x_\alpha\).** By Remark, \(\bigcup_{\beta < \alpha} X_\beta\) with the union topology is locally-compact and not compact. Since this union is countable, it is not pseudocompact. Since \(Q\) is dense in it, there exists an infinite subset of \(Q\) that is closed in \(\bigcup_{\beta < \alpha} X_\beta\). Hence, \(N' \setminus \bigcup_{\beta < \alpha} N_\beta\) is not empty. Take the first \(N \in N' \setminus \bigcup_{\beta < \alpha} N_\beta\).

**Case 1.** For no \(s \in S\), \(s \to x_\alpha\). Let \(N'\) be an infinite and co-infinite subset of \(N\). Define a local base at \(x_\alpha\) as follows:

\[
B_{x_\alpha} = \{\{x_\alpha\} \cup N^k : k \in \omega\}.
\]

**Case 2.** For some \(s \in S\), \(s \to x_\alpha\). Since \(\text{ran}(s) \subset \{x_\beta : \beta < \alpha\}\), local bases at \(s(i)\) are already defined. Re-enumerate \(\{x_\beta : \beta < \alpha\} \setminus \text{ran}(s)\) as \(\{x_{\alpha_n} : n \in \omega\}\). For each \(x_{\alpha_n}\) fix \(B(x_{\alpha_n}) \in B_{x_{\alpha_n}}\) such that \(B(x_{\alpha_n}) \cap (R \setminus Q)\) does not meet \(s\). Such \(B(x_{\alpha_n})'s\) exist because \(\lim s \neq x_{\alpha_n}\) and by assumption the diameter of \(B(x_{\alpha_n}) \cap (R \setminus Q)\) can be made small. For each \(n\) fix \(B(s(n)) \in B_{s(n)}\) with the following properties:

1. \(B(s(n))\) does not meet \(B(x_{\alpha_n})\) for \(i \leq n\);
2. \(B(s(n))\) does not meet \(B(s(i))\) for \(i < n\);
3. The Euclidean diameter of \(B(s(n)) \cap (R \setminus Q)\) is less than \(\rho(s(n), x_\alpha)\);
4. \(B(s(n))\) does not meet \(N\).

The existence of such \(B(s(n))'s\) with (1)–(3) follows from the inductive assumptions. Property (4) can be achieved because the definition of \(N_\beta'\)'s implies that
\[ N \text{ meets only finitely many base elements of } s(n). \] Define a local base at \( x_\alpha \) as follows:
\[
B_{x_\alpha} = \left\{ \{x_\alpha\} \cup \left( \bigcup_{n > k} B(s(n)) \right) : k \in \omega \right\}.
\]
In both cases, put \( N_\alpha = \{ N \in \mathcal{N} : N \text{ meets infinitely many elements of } B_{x_\alpha} \} \). Let us verify the induction requirements. The space \((X_\alpha, T_\alpha)\) is Hausdorff due to (1), (2). Each \( B \in B_{x_\alpha} \) is compact in \((X_\alpha, T_\alpha)\) as a convergent sequence of compacta with its limit. Clearly, \((X_\alpha, T_\alpha)\) is locally-countable. Finally it is not compact due to the choice of \( N' \) in case 1 and property (4) in case 2. Requirement A4 is clear and A5 follows from (3) and convergence of \( s \) to \( x_\alpha \).

Let \( T \) be the topology on \( R \) generated by just defined local bases. By Remark, \((R, T)\) is locally compact, locally countable, not compact, and regular. By definition of local bases, \( Q \) is dense in \((R, T)\). The space has countable extent due to property (2) in Lemma 4.1 and the definition of local bases in case 2. In the following two lemmas we will prove that \((R, T)\) has the rest of the desired properties.

**Lemma 4.2.** \((R, T)\) is pseudocompact.

**Proof.** First notice that \( Q \) is dense in \((R, T)\). It suffices to show that any infinite subset \( A \subset Q \) has a limit point in \((R, T)\). Since \( \mathcal{M} \) is mad, \( A' = A \cap \mathcal{M} \) is infinite for some \( M \in \mathcal{M} \). Then \( A' = N_\alpha \in \mathcal{N}' \). Due to condition (4) of Lemma 4.1, at uncountably many steps, inductive definition runs through case 1. Therefore, either at some step \( \gamma \), \( N_\alpha \in \mathcal{N}'_\gamma \), and therefore \( x_\gamma \) is a limit point for \( N_\alpha \). Or there exists \( x_\beta \) such that \( N_\alpha \) is the first in \( \mathcal{N} \setminus \bigcup_{\gamma < \beta} \mathcal{N}'_{\gamma} \) and case 1 takes place. And then \( x_\beta \) is a limit point for \( N_\alpha \). \( \square \)

**Lemma 4.3.** \((R, T)\) has a \( G_\delta \)-diagonal.

**Proof.** Let \( Q = \{ q_n : n \in \omega \} \). For each \( x \in R \setminus Q \) and \( n \in \omega \) fix \( B_n(x) \in B_x \) such that the Euclidean diameter of \( B_n(x) \cap (R \setminus Q) \) is less than \( 1/n \) and \( q_i \notin B_n(x) \) for all \( i \leq n \). Define an open cover \( \mathcal{V}_n \) of \((R, T)\) as follows:
\[
\mathcal{V}_n = \left\{ B_n(x) : x \in R \setminus Q \right\} \cup \left\{ \{q_i\} : i \in \omega \right\}.
\]
Since the diameters go to 0, \( \bigcap_n \text{St}(x, \mathcal{V}_n) = \{x\} \) for every \( x \in R \). \( \square \)

**References**

