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Minimum-time multidrop broadcast

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Abstract

The multidrop communication model assumes that a message originated by a sender is sent along a path in a network and is communicated to each site along that path. In the presence of several concurrent senders, we require that the transmission paths be vertex-disjoint. The time analysis of such communication includes both start-up time and drop-off time terms. We determine the minimum time required to broadcast a message under this communication model in several classes of graphs.

1. Introduction

One of the basic information dissemination tasks in a communication network is that of broadcasting: one site originates a message to be sent to all other sites in the network. Depending on the technology of the network, different models of information dissemination are used. The *telephone* model assumes that only two sites connected by a direct link can communicate at any time, precluding any other communication involving the two sites (although other sites may communicate in the same manner concurrently). The *line* communication model assumes a similar exclusive pair-wise communication where communicating sites can be connected by a succession of adjacent links, and no link is used in two simultaneous transmissions. In this paper, we consider a multidrop version of this line communication model, whereby all sites along the path followed by a call become informed of the message.

We model the topological aspects of a communication network by a simple, undirected graph, $G=(V,E)$, in which vertices V correspond to network sites and edges E correspond to communication links between them. We consider a synchronous *multidrop line* communication model. Communication takes place in rounds, where a round corresponds to a collection of vertex-disjoint paths in the graph. In this model,

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a site initiates a call that communicates a message to all sites along the path followed by the call. Several such calls can occur in the same round if they do not interfere by using the same vertex or edge.

The time required by a multidrop call from one vertex to x other vertices is $s + dx$, where s is the transmission start-up time and d is the message drop-off rate. We simplify this expression to the form $1 + cx$, where c represents the relative message drop-off rate, i.e., d/s. The time to complete a round is the maximum time of all calls in that round. The time of a multidrop broadcast scheme is the sum of durations of all rounds in the scheme. Thus, it is equal to $r + ct$, where r is the number of line-call rounds (called the *start-up term*) and t is the sum of the maximum number of drop-offs in each round (ct is called the *drop-off term*).

We denote by $b_v(G)$ the minimum time to complete a multidrop broadcast from a vertex v in a connected graph G. A graph G is *connected* if there exists a path between every pair of vertices in G, a necessary condition for broadcast to be feasible in G. The question we address in this paper is: for a given connected graph G including a vertex v, what is the value of $b_v(G)$ under the timing model introduced above?

Feldmann et al. [1] and Hromkovič et al. [3] initiated the formal study of multidrop broadcasting. They adopt a simpler timing model whereby each call requires one unit of time, regardless of the number of vertices along the path of a call. This measure simply counts the number of rounds to complete a multidrop broadcast. With this measure of time, the existence of a Hamiltonian path from a given source assures a broadcast requiring one time unit.

Our model of the time required to complete a call better approximates properties of existing wormhole routing protocols in more general networks. On the MasPar multiprocessor using the *xnetc* communication primitive, the relative drop-off rate c is quite small, approximately .005 for 8-bit data transfers (cf. [2]). In general, the relative drop-off rate associated with wormhole routing can be expected to vary with implementation and technology [4].

For different values of the relative drop-off rate c (as determined by network technology), schemes with differing numbers of rounds may yield the overall minimum time for the same network topology. Thus, we explore optimum r-round broadcast (minimum time over all r-round broadcasts) for several values of r.

The goal of this paper is to extend research on the multidrop model by using our more detailed timing model and to explore schemes leading to minimum-time multidrop broadcast in several classes of graphs. Specifically, we will study multidrop broadcasting in trees, cycles and grids.

2. Preliminary results

In this section we establish some preliminary, general results that will be useful in establishing lower bounds on the minimum multidrop broadcast time.

Lemma 1. *At any point during a multidrop broadcast in a graph G, the set of informed vertices induces a connected subgraph of G.*

Proof. Any call informing a new vertex also informs every vertex on the path connecting that vertex with the set of previously informed vertices. □

The following lemma has a straightforward inductive proof and provides a lower bound on the minimum drop-off term for k-round broadcast schemes.

Lemma 2. *Consider a k-round multidrop broadcast scheme S in a graph G. The maximum number of vertices informed by S is $\prod_{1\leqslant i\leqslant k}(1+x_i)$, where x_i is the maximum number of vertices informed by a call in round i.*

Corollary. *The minimum drop-off term in any k-round broadcast scheme in a graph G with n vertices is $ck(\sqrt[k]{n}-1)$.*

A limiting factor on multidrop broadcast time is the eccentricity of the originating vertex in the given graph. The *eccentricity* e of a vertex v in a graph G is the maximum (shortest path) distance from v to any other vertex of G.

Lemma 3. *In any graph G, the drop-off term for any multidrop broadcast scheme from a vertex v with eccentricity e is at least ce.*

Proof. The existence of a scheme with drop-off term less than ce would imply the existence of a path of length less than e from v to any vertex, contradicting the definition of eccentricity. □

When there is more than one vertex at eccentricity distance from the originator, the drop-off term increases.

Lemma 4. *In any graph G, the minimum drop-off term for a multidrop broadcast scheme from a vertex v with eccentricity e, for which there are two vertices at distance e from v, is $c(e+1)$.*

Proof. Suppose that u and w are two vertices at distance e from v and that there is a multidrop broadcast scheme with drop-off term ce. Consider intermediate nodes used to inform each of u and w. Let v_k be the last common intermediate node. Thus, the intermediate nodes for u are $v_1,\ldots,v_k,u_1,\ldots,u_l$ and the intermediate nodes for w are $v_1,\ldots,v_k,w_1,\ldots,w_s$, where w_1 is different from u_1. Calling u requires $k+l+1$ rounds, while calling w requires $k+s+1$ rounds. The total length of the path from v to u via $v_1,\ldots,v_k,u_1,\ldots,u_l$ is at least e and from v to w via $v_1,\ldots,v_k,w_1,\ldots,w_s$ is also at least e. Without loss of generality, suppose that v_k calls u_1 before calling w_1. Suppose that the drop-off term in the first k rounds is x, the drop-off term in the call from v_k to u_1 is $y>0$ and the drop-off term in the $s+1$ rounds used to make calls from v_k to

w_1, then from w_1 to w_2, and finally from w_s to w is z. However, $x+z \geqslant ce$, hence, the total drop-off term is at least $ce+y>ce$. This contradicts our assumption. □

Throughout the rest of the paper, we will denote by e the eccentricity of the broadcast originator in a given graph. If we can find a multidrop broadcast in s rounds having a drop-off term of ce, then we need not consider broadcasts with greater number of rounds in our search for an optimal scheme.

Lemma 5. *In any graph G, if there exists a multidrop broadcast from a vertex v of eccentricity e in time $s+ce$, then no minimum-time multidrop broadcast from v can involve more than s rounds.*

Proof. It follows from Lemma 3 that any broadcast from v with $s'>s$ rounds would require at least $s'+ce>s+ce$ time. □

3. Multidrop broadcast in trees

We begin our study of multidrop broadcasting by considering several restricted classes of graphs. We first consider trees, which have the fewest edges over all connected graphs. Each vertex of a tree is either a leaf (i.e., a degree 1 vertex) or separates the tree into unconnected subtrees. This property leads to the following general lemmas regarding multidrop broadcasting in trees.

Lemma 6. *The minimum number of rounds required to complete a multidrop broadcast in a tree is not less than the greater of* (i) *the degree of the originating vertex, or* (ii) *one less than the maximum vertex degree in the tree.*

Proof. (i) The vertex originating the broadcast constitutes a separator (or is a leaf) of the tree; thus, its neighbors must become informed in different rounds as it can be involved in at most one call at a time. (ii) Let v be a vertex of maximum degree d that is not the originator. In the first round of a broadcast, at most two neighbors of v can be informed; informing the remaining neighbors requires at least $d-2$ additional rounds, as argued above. □

We observe two simple facts about broadcasting in small trees. Obviously, broadcasting in the trivial tree with one vertex takes no time. Broadcasting in a tree of two vertices from either vertex takes time $1+c$.

Lemma 7. *Given a tree T with vertex v. The minimum time $b_v(T)$ of a multidrop broadcast from v is equal to the minimum of $(1+cp)+\max_u(b_u(T_P^u))$ over all paths P in T starting in v, where p is the length of P and T_P^u is the subtree of T consisting of vertex $u \in P$ and the vertices separated from the path P by u.*

Proof. Any broadcast from v must have as its first round a multidrop call from v along some path P in T starting with v. Each now informed vertex u of the path P must broadcast the message as originator in its corresponding subtree T_P^u. Each such partial broadcast can be considered independently, as each vertex u is a separator and can be involved in at most one call at a time. Any assistance by a call made from outside the subtree must pass through vertex u and thus could less expensively be made by u. $\square$

Lemma 8. *Given a tree T with more than two vertices and a leaf vertex v adjacent to vertex r. Let T' denote the subtree of T obtained by removing leaf v. Then, $b_v(T)$ is equal to $b_r(T') + c$.*

Proof. The first call of any broadcast from v must inform vertex r. Let that call pass through r and the broadcast scheme continue as if it were initiated by r. This gives us the value indicated. Clearly, we can do no better without contradicting the optimality of $b_r(T')$. $\square$

In the following subsections, we will consider multidrop broadcast in paths, binomial trees, and complete binary trees. We design minimum-time multidrop broadcast schemes for these classes of graphs, exploiting the above lemmas.

3.1. Path P_n

Consider multidrop broadcasting in paths of n vertices, P_n, being the unique trees with just 2 leaves for all $n \geqslant 2$. There are two cases to consider, one when the originator is an end-vertex (Leaf) and the other when the originator is an interior vertex of the path.

Theorem 1. *Given path P_n, $n \geqslant 2$. When the originator v is an end-vertex, $b_v(P_n)$ equals $1 + c(n-1)$. When the originator v is an interior (degree 2) vertex, $b_v(P_n)$ equals $2 + ce$, except in the case where n is odd and v is the middle vertex of the path, when $b_v(P_n)$ equals $2 + c(e+1) = 2 + c\lceil n/2 \rceil$.*

Proof. If the originator is an end-vertex of the path, then one round informing all other nodes takes time $1 + c(n-1) = 1 + ce$. This is optimal in view of Lemmas 3 and 5.

When the originator is an interior vertex of the path, any broadcast requires at least two rounds, by Lemma 6. The obvious broadcast involving two calls from the originator, one in each direction, requires $2 + c(n-1)$ time units but is not optimal. A better scheme increases the degree of parallellism among calls. The following scheme proves to be optimal. The originator calls its neighbor in the direction of a furthest vertex. In the second round, the broadcast is completed with the originator and its informed neighbor calling their respective subpaths. With the exception of the originator being the middle vertex of P_{2k+1}, our scheme requires time $(1 + c) + (1 + c(e - 1)) = 2 + ce$. Since broadcast from an interior vertex requires at least two rounds, this is a minimum-time broadcast, by Lemma 3. For the exceptional originator, a minimum-time broadcast takes at least $2 + c(e + 1)$ time, by Lemma 4, and is realized by the given scheme. □

3.2. Binomial trees

Next, we consider multidrop broadcasting in a binomial tree. Binomial trees are of interest in the context of communications in multiprocessor systems because they represent, on the one hand, the structure of minimum time broadcast trees (in the classical model of broadcasting, see, e.g., [5]), and on the other hand, the structure of spawning parallel processes.

The *binomial tree* of rank 0 is the single vertex graph. The binomial tree of rank $i > 0$ is the rooted tree obtained by addition, to the root vertex of the binomial tree of rank $i - 1$, of a principal subtree that is a copy of the binomial tree of rank $i - 1$. As such, the rank i binomial tree, T_i, has depth i, has 2^i vertices, and has i principal subtrees $S^{i-1}, \ldots, S^0$ (their roots are children of the root vertex), which are the binomial trees of ranks $i-1, \ldots, 0$, respectively. We also define a particular, connected subgraph of the binomial tree of rank $i > 1$, the *branch-deficient* binomial tree T_i^{-j}, as being T_i without its principal subtree S^j (isomorphic with T_j). It follows from the definition that $T_i^{-(i-1)}$ is isomorphic with T_{i-1}. We will use this terminology throughout our discussions to follow in this section.

The binomial tree of rank $i \geqslant 2$, T_i, has two vertices of degree i and two vertices of degree $i - 1$. These vertices induce a path P_4 in T_i. These vertices, called the *broadcast core*, can be viewed as roots of four disjoint subtrees, each isomorphic with the binomial tree of rank $i - 2$. This view will assist us in determining the number of rounds necessary to complete a minimum-time multidrop broadcast in T_i. We will refer to the root of a binomial tree and the root of its largest subtree (i.e., the two highest degree vertices) as *broadcast centers* of the tree.

Theorem 2. *Given the binomial tree T_i with root r. Then $b_r(T_i)$ equals $i + ce$.*

Proof. In the first round, the root calls the other broadcast center. During each successive round, an informed vertex calls the root of its largest uninformed subtree. By this scheme, the time required by the broadcast is $i + ci = i + ce$, since the eccentricity of root r is i. By Lemma 6, the number of rounds cannot be decreased. By Lemma 3, the drop-off term cannot be decreased. Hence, the scheme is optimal. □

Theorem 3. *Given the binomial tree T_i, $i \geqslant 2$, with originator v in subtree S^{i-2}. Then $b_v(T_i)$ equals $(i - 1) + ce$.*

Proof. In the first round, the originator calls the degree $i - 1$ vertex in S^{i-1}. This call informs all four vertices of the broadcast core, while requiring time $1 + c(e - i + 2)$. In the subsequent $(i - 2)$ rounds, these four informed vertices broadcast the message as roots of binomial trees of order $i-2$. By Theorem 2, this requires time $(i-2)+c(i-2)$. Hence, the total time is $(i-1)+ce$. This is optimal by Lemmas 3 and 6. □

Optimal broadcast time from other originators will be established in terms of the time to broadcast from the root of a branch-deficient binomial tree.

Theorem 4. *Given the branch-deficient binomial tree T_i^{-j} with root r. Then, $b_r(T_i^{-j})$ equals $(i-1)+c(2e-j-2)$, for $c \leqslant 1/(e-j-2)$, and $i+ce$, otherwise.*

Proof. By Lemma 6, the minimum number of rounds to complete a multidrop broadcast is $i-1$. First, suppose that $j=i-1$. Recall that $T_i^{-(i-1)}$ is isomorphic with T_{i-1}. We see that the hypothesized time in this case equals $(i-1)+c(2i-(i-1)-2)=(i-1)+ce$, which is indeed $b_r(T_i^{-(i-1)})$ by Theorem 2. Note that in this case $e=i-1$. This constitutes the base case, where $i=j+1$, for an inductive proof of the theorem. Let us now consider the case where $i>j+1$ and assume inductively that the theorem holds for all smaller values of k, $i>k \geqslant j+1$.

If the first call does not inform the degree $(i-1)$ child of the root of the largest principal subtree S_{i-1} in T_i^{-j} then, by Lemma 7, the broadcast cannot be completed in $i-1$ rounds. Thus, the first round must inform at least two vertices, for a minimum cost of $1+2c$. By Lemma 7, this initial call of length 2 results in three subtrees that must complete broadcast from their roots independently, in parallel: two copies of T_{i-2} and a single T_{i-1}^{-j}. By Theorem 2, the broadcast can be completed in $(i-2)$ rounds in these remaining three subtrees. By the inductive assumption, the time required for this completion is $(i-2)+c(\max\{i-2, 2i-j-4\})=(i-2)+c(2i-j-4)$. Together with the time of the first call, this achieves the hypothesized time bound, where $e=i$.

If i rounds are used in the overall broadcast, then a scheme that follows that for the binomial tree T_i, with the obvious modification ignoring the missing subtree, completes broadcast in the minimum time of $i+ci$, which is $i+ce$. □

The $(i-1)$ round scheme implied by the proof above has the root of T_i^{-j} making calls of length 2 into successive subtrees until reaching the missing S_j, after which all calls are of length 1. This result can be extended to vertices in other subtrees of T_i; the eccentricity e is now equal to i plus the depth of the originating vertex in T_i.

Theorem 5. *Given the binomial tree T_i with vertex v in S^j, for $0 \leqslant j < i-2$. Then, $b_v(T_i)$ equals $(i-1)+c(2e-j-2)$, if $c \leqslant 1/(e-j-2)$, and $i+ce$, otherwise.*

Proof. The optimal schemes are based upon those presented in Theorem 4 for branch-deficient trees. As noted, to complete broadcast in the desired number of rounds, the first call must inform either the degree $(i-1)$ child of the root of subtree S_{i-1} for any $(i-1)$ round scheme or the root of subtree S_{i-1} for any i round scheme. Following this call, the time to complete broadcast in all subtrees separated by the path of the first call, as per Lemma 7, is dominated by broadcast time in T_{i-1}^{-j}. Thus, we can view the overall process as a modification of broadcast from the root of the branch-deficient tree T_i^{-j}. In these modified schemes, the first call from v is now of length either $e-i+2$ or $e-i+1$. This gives us the postulated results. □

3.3. Complete binary tree B_h

Now, we consider multidrop broadcasting in a complete binary tree B_h with h levels (i.e., $n=2^h-1$ nodes). The following lemma was proved in [1].

Lemma 9. *The minimum number of rounds necessary to complete a multidrop broadcast from the root of B_h is h, for $h>1$.* □

With our more detailed timing model, we find the following results.

Theorem 6. *Given the complete binary tree B_h with root r. Then, $b_r(B_h)$ equals $h+c(2h-2)$, for $h>1$.*

Proof. We prove the theorem by induction on h. As a base case, consider B_2, which is isomorphic with the path P_3. The originator r is the interior vertex of an odd length path equidistant from the ends. Thus, by Theorem 1, we have $b_r(B_2)$ equal to $2+2c$. Now, assuming that the statement holds for all k, $2\leqslant k<i$, consider the case of B_i. A first call of length 1 is optimal, because time to complete the broadcast will be determined, as per Lemma 7, by the broadcast time in the larger subtree, i.e., the one not called. The root r is now a leaf of that subtree. By Lemma 8, the overall time is $(1+c)+(b_{r'}(B_{h-1})+c)$, where r' is the root of an uncalled subtree isomorphic with B_{h-1}. By our inductive assumption, this equals $(1+2c)+((h-1)+c(2(h-1)-2))=h+c(2h-2)$, as postulated. □

We now analyze the case of a general originator v at level $i\geqslant 2$. If $h=2$, the minimum time is clearly $1+2c$, since this is equivalent to a broadcast in P_3 from an end-vertex.

Theorem 7. *Given the complete binary tree B_h, $h>2$, and a vertex v. Then $b_v(B_h)$ equals $h+c(2h-2)$, for v at level 2, and $(h-1)+c(2h+i-4)$, for v at level $i\geqslant 3$.*

Proof. When v is at level 2, the first call can be either of length 1 or of length 2, to or through the root of B_h. A longer call would not be useful, as the remaining broadcast would be dominated in time by the uninformed subtree of which v is the root, by Lemma 7. In either case, we get the time indicated, the same as from the root.

For a vertex v at level $i\geqslant 3$, consider a vertex w at level 3 in the other principal subtree of B_h. If the first call of length $(i+1)$ is placed to w, then the time to complete broadcast is dominated by two isomorphic trees, each being an informed leaf appended to the root of an uninformed B_{h-2}. By Lemmas 7 and 8, this requires a total time of $(1 + c(i+1)) + (b_r(B_{h-2}) + c)$, where r is the root of the uninformed subtree. By Theorem 6, this equals $(1+c(i+2))+((h-2)+c(2(h-2)-2)) = (h-1)+c(2h+i-4)$.

By inspection, any shorter first round call leads to more rounds of calls and not lesser drop-off term. Any longer first round call would leave a remaining broadcast time dominated by time in the B_{h-2} subtree discussed above. □

We can generalize the above results for the root of a complete k-ary tree, as follows.

Theorem 8. *Given the complete k-ary tree of height h, L_h^k, with root r. Then, $b_r(L_h^k)$ equals $(k + (k-1)(h-2)) + c(kh-k)$, for $h>1$.*

Proof. As a base case, we consider $h=2$. We have $b_r(L_2^k)$ equal to $k + kc$ by our general lemmas for trees; the root must make each call of length 1. This case satisfies the general statement.

We assume the statement holds for all $i, h>i>1$, and consider L_h^k. The root must make the first $k-1$ calls of length 1, followed by an optimal call into the kth subtree. By Lemma 8 and our inductive assumption, this requires a time of $(k-1) + c(k-1) + b_{r'}(L_{h-1}^k) + c$, where r' is root of the last, uninformed subtree. By our inductive assumption, this time equals $((k-1) + c(k-1)) + ((k + (k-1)(h-3)) + (c + c(k(h-1)-k)))$, which gives the postulated value. □

Our study of multidrop broadcasting in trees shows that, even in highly regular tree structures, minimum multidrop broadcast time depends on the position of the originator. For arbitrary trees, a precise formula linking time to the eccentricity of an originator and to degrees of vertices seems difficult to establish. While our general lemmas provide bounds, it remains an open problem to determine an optimal, multidrop broadcasting algorithm for arbitrary trees, such as the one given in [6] for the classical model of broadcasting.

4. Cycle C_n

We will now consider broadcasting in the cycle of n vertices, C_n. In such a cycle, the eccentricity of any vertex is $e = \lfloor n/2 \rfloor$.

Since C_n contains a Hamiltonian path starting at any vertex, there exists a one-round broadcast that requires $1 + c(n-1)$ time. However, we can increase the degree of parallelism with a two-round broadcast. A call in the first round to i vertices, $0<i<n$, leaves the necessity of a call to $\lceil (n-i-1)/2 \rceil$ vertices, for the total time of at least $2 + c(i + \lceil (n-i-1)/2 \rceil)$. This is minimized by $i=1$ and achieved by the following scheme. The originator calls a neighbor in the first round, and the two informed vertices complete the broadcast by calling approximately the same number of vertices in the second round. This broadcast takes $2 + c\lceil n/2 \rceil$ time units. In a cycle with even number n of vertices, $n/2 = e$. For a cycle with odd number n of vertices we have $e+1 = \lceil n/2 \rceil$.

As there are two vertices at distance e from any vertex in an odd-cycle, $c(e+1)$ is the minimum drop-off term, by Lemma 4. By Lemma 5, we need not consider broadcast schemes with more rounds. Hence, the optimal scheme is the better one of the two above.

Theorem 9. *The minimum time for multidrop broadcast from any vertex v in cycle C_n is*

$$\begin{array}{ll} 1+c(n-1) & \text{for } c<\dfrac{1}{\lfloor n/2-1\rfloor}, \\ 2+c\lceil n/2\rceil & \text{otherwise.} \end{array}$$

5. Grid $M_{p\times q}$

Next, we consider the grid $M_{p\times q}$ with p rows and q columns $p,q>1$. Since there is a Hamiltonian path starting at any vertex of the grid (except for the 3×3 case), the corresponding broadcast in the grid requires $1+c(pq-1)$ time. In most cases, $pq-1$ is much greater than the eccentricity of any originating vertex, which is of the order of $p+q$.

We must consider three general locations of the originating vertex v: at a corner, on a side, and at an interior point of the grid. Such vertices differ in degree and possible eccentricities. We will consider two general classes of multidrop broadcast schemes: unidimensional and general. For the first class, optimal schemes can be established; for the general case, we can only approximate optimal time with regularly structured schemes.

5.1. Unidimensional schemes

We first consider multidrop broadcasting in grids where calls must be unidimensional, as per the *netc* call in a *MasPar* multiprocessor. A multidrop call in a grid graph is *unidimensional* if only proceeds in one direction. The grid $M_{p\times q}$ can be viewed as the product of two paths, P_p and P_q. This perspective gives us optimal broadcasting schemes in terms of optimal path schemes when calls must be unidimensional.

When the originator v is a corner vertex, eccentricity is $e=p+q-2$. A two-round broadcast scheme involving successive, single-round, multidrop broadcasts from the ends of paths P_p and P_q (in either order) proves to be optimal, realizing a drop-off term equal to vertex eccentricity. One-round schemes are obviously excluded.

Theorem 10. *The minimum time to complete unidimensional, multidrop broadcast in $M_{p\times q}$ from a corner vertex is $2+ce$.*

Next, we consider the originator at a non-corner vertex on the side of the grid, say in the top row. In this case, a unidimensional scheme involves two rounds to complete a broadcast in this row and one round to inform the columns. This requires three rounds and realizes the minimum drop-off term equal to ce or (in the middle vertex case) to $c(e+1)$ (cf. Lemmas 3 and 4).

Theorem 11. *The minimum time to complete unidimensional, multidrop broadcast in $M_{p\times q}$ from a non-corner side vertex v is $3+ce$, or $3+c(e+1)$ when v is in the middle of an odd-length side.*

We have a similar result for interior vertices. In this case a minimum of 4 rounds must be used. The optimal unidimensional scheme, realizing minimum possible drop-off term, uses two rounds to inform the row of the originator and then two rounds to inform all columns.

Theorem 12. *The minimum time to complete unidimensional, multidrop broadcast in $M_{p\times q}$ from an interior vertex v is $4+ce$, $4+c(e+1)$, or $4+c(e+2)$, depending on whether the originator is: the middle vertex of neither its row nor its column, of one of them, or of both of them, respectively.*

Proof. In the first two cases the schemes are optimal by Lemmas 3 and 4. In the last case we must verify that $c(e+2)$ is the minimum drop-off term from the middle of an odd-by-odd grid. Thus, the proof is completed by the following lemma.

Lemma 10. *The minimum drop-off term in an optimal multidrop broadcast from the middle vertex in $M_{p\times q}$, when p and q odd, is $c(e+2)=c(1+(p+q)/2)$.*

Proof. Let $k=(q-1)/2$ and $l=(p-1)/2$. Thus, the eccentricity of the middle vertex is $e=k+l$; there are four vertices at that distance. Consider the middle vertex of the grid to be the point $(0,0)$ and assign coordinates to other vertices in the natural way. Consider the first round. Let A be the largest coordinate of a column, $-C$ the smallest coordinate of a column, B the largest coordinate of a row and $-D$ the smallest coordinate of a row, reached in this round. Without loss of generality, assume that $A\leqslant C$ and $B\leqslant D$. The drop-off term in this round is at least $c(2A+C+2B+D)$. The remaining part of the broadcast must have the drop-off term at least $c(k-A+l-B)$, for a total of $c(k+l+A+B+C+D)=c(e+A+B+C+D)$. If $A+B+C+D\geqslant 2$, we are done. Otherwise, only one vertex becomes informed in the first round with drop-off term c. Hence, after this round there are two vertices at distance e from the closest informed vertex. By Lemma 4, these vertices cannot become informed in less than $c(e+1)$ additional drop-off term, for the total drop-off term of at least $c(e+2)$, as postulated. □

5.2. General schemes

In the general case we lift the restriction that calls must be placed along paths in one direction. We consider the single-round, Hamiltonian path scheme as one option, as well as schemes with fewer rounds than the unidimensional schemes considered above. The optimal scheme will then depend on the value of the relative drop-off factor c.

When the originator v is a corner vertex, its eccentricity is $e = p + q - 2$. As noted, a two-round, unidimensional broadcast scheme gives an optimal drop-off term. Hence, either the single round scheme or the two-round scheme is optimal, depending on c.

Theorem 13. *The minimum time for multidrop broadcast in $M_{p\times q}$ from a corner vertex v is*

$$\begin{array}{ll} 1 + c(pq - 1) & \text{for } c < ((p-1)(q-1))^{-1}, \\ 2 + c(p + q - 2) & \text{otherwise.} \end{array}$$

Next, we consider the originator v at a non-corner vertex on the side of the grid. We have seen that a three-round, unidimensional scheme provides the optimal drop-off term. Thus an optimal scheme has 1, 2 or 3 rounds. It remains to consider two-round schemes. Without loss of generality, we may assume that the originator is situated on the side of length q. Let a denote the distance of the originator from the farthest end of its row and let $k = \lfloor q/2 \rfloor$. Thus, the eccentricity of the originator is $e = a + p - 1$ and $a \geqslant k$.

First suppose that $a > k$, i.e., the originator is not in the center of its row or in one of the two middle vertices in case of even q. We construct two types of broadcasting schemes.

Scheme I. In the first round the originator informs all vertices between itself and the closest end w of its row, as well as all vertices of the column of w. In the second round the originator informs the remainder of its row and all informed vertices in other rows inform their entire row. The first round takes time $1 + c(q - 1 - a + p - 1)$ and the second round takes time $1 + c(q - 1)$, for a total time of $u_1 = 2 + c(2q + p - a - 3)$.

Scheme II. In the first round the originator informs all vertices between itself and the center v of its row as (or the closest middle vertex v) as well as the entire column of v (see the dashed path in Fig. 1). In the second round, each vertex in this column calls the appropriate parts of their row and an adjacent row. The details of the call can be described by defining cycles of length $2q$ formed by adjacent pairs of rows, starting with the top most row. Vertices call their cycle segments in the clockwise direction (see the solid path in Fig. 1).

The first round takes time $1 + c(a - k + p - 1)$ and the second round takes time $1 + c(2k)$ for a total time of $u_2 = 2 + c(a + p + k - 1)$.

If p is odd, the first row will not be covered by any cycle. In this case the originator and the vertex v call each its uninformed part of the first row (see Fig. 2).

Scheme III. In the first round the originator informs all vertices in its column and in the neighboring column on the side of the farthest vertex. In the second round all informed vertices call in parallel appropriate parts of their rows (see Fig. 3). The total time is $u_3 = 2 + c(2p + a - 2)$.

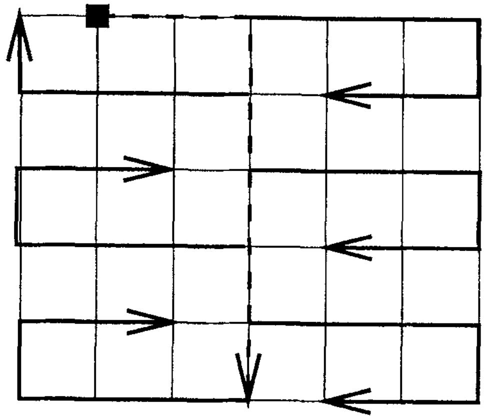

Fig. 1.

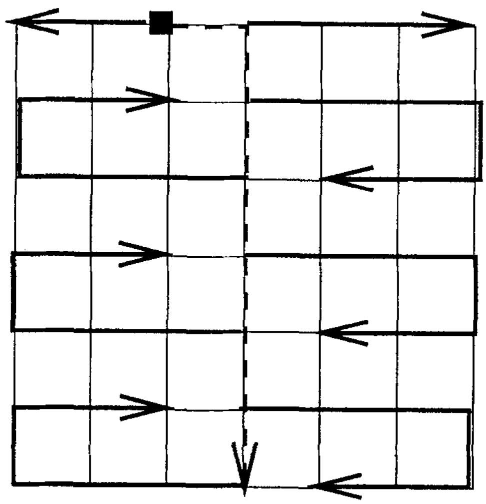

Fig. 2.

In the special case when $a=k$, Scheme II can be slightly modified, depending on the parity of p and q.

(1) If p is even, the scheme is as before. In this case the first round takes time $1+c(p-1)$ and the second round takes time $c(2k)$ for a total time of $2+c(2k+p-1)$.

(2) If p is odd and q is odd, in the first round the originator informs its column and the neighboring vertex of the last row. In the second round informed vertices in all rows except the last inform their portions of cycles, similarly as above, and the two informed vertices of the last row inform their parts of this row. In this case the first round takes time $1+cp$ and the second round takes time $c(2k)$ for a total time of $2+c(2k+p)$.

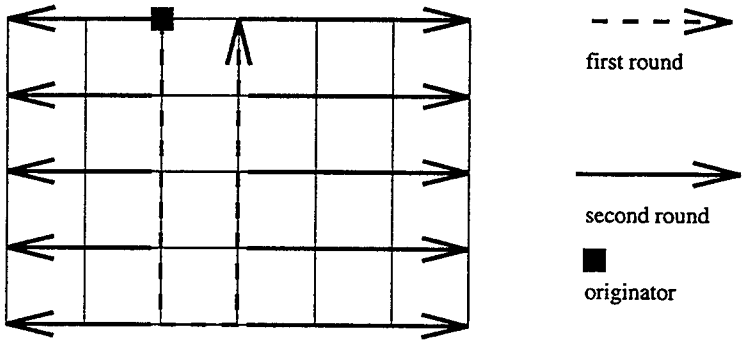

Fig. 3.

(3) If p is odd and q is even, in the first round the originator informs its column and the neighboring middle vertex of the last row. The second round proceeds as in case 2 and time in each round is as before.

Hence, the total time of Scheme II for $a=k$ is always at most u_2+c.

Scheme III in case $a=k$ remains as before and its total time is $2+c(2p+k-1)$ if q is odd and $2+c(2p+k-2)$ if q is even. Hence, it is always at most u_3+c.

For each position of the originator the most efficient of Schemes I–III should be taken.

Theorem 14. *Let t_i, for $i=1,2,3$, be the minimum time for i-round multidrop broadcast in $M_{p\times q}$ from a non-corner vertex v on the side of length q. Let a be the distance from v to the further corner of its side and let u_1,u_2,u_3 be as above. Then*

$t_1 = 1+c(pq-1)$;

if $a>\lfloor q/2\rfloor$ *then* $t_2 \leqslant \min(u_1,u_2,u_3)$

if $a=\lfloor q/2\rfloor$ *then* $t_2 \leqslant \min(u_1,u_2+c,u_3+c)$;

$$t_3=\begin{cases} 3+c(a+p) & \text{if the originator is in the middle of its side} \\ 3+c(a+p-1) & \text{otherwise.} \end{cases} \qquad \square$$

As for lower bounds in the case of two-round schemes, the one yielded by Lemma 2 is $2+c(2\sqrt{pq}-2)$. This lower bound is usually not tight. However, when the originator is in the center of the first row of a $2k\times(2k+1)$ grid, then it matches the performance of our scheme, giving exact minimum broadcast time $2+c(4k-1)$ for two-round schemes. On the other hand, for any side vertex of the square grid, the ratio of drop-off terms of our scheme and of this lower bound is at most $\frac{9}{8}$, hence our scheme is at most 12.5% slower than optimal. The problem of determining a tighter lower time bound on optimal two-round broadcast schemes in arbitrary grids remains open.

Finally, we consider multidrop broadcast from an interior vertex of the grid $M_{p\times q}$. We have seen that the four-round, unidimensional schemes provide optimal drop-off terms. It remains to consider two- and three-round schemes. Without loss of generality,

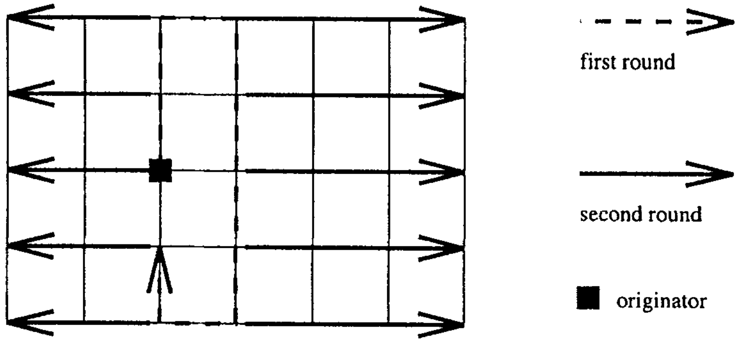

Fig. 4.

assume that $p \leqslant q$, i.e., columns are not longer than rows. Let $k = \lfloor q/2 \rfloor$ and $l = \lfloor p/2 \rfloor$. Let a denote, the distance of the originator from the farther end of its row, a' the distance from the closer end, b the distance of the originator from the farther end of its column and b' the distance from the closer end; hence, the eccentricity is $e = a + b$.

An efficient three-round broadcast is realized by an easy modification of the unidimensional scheme. The originator calls all three vertices of a unit square in the first round, placing a call requiring time $(1 + 3c)$. If there is exactly one vertex at distance e from the originator, the two remaining, unidimensional rounds take time $2 + c(e - 2)$ for a total of $3 + c(e + 1)$. If there are two such vertices, the remaining rounds take time $2 + c(e - 1)$ for a total of $3 + c(e + 2)$. Finally, if the originator is a center of an odd-by-odd grid, the remaining rounds take time $2 + ce$ for a total of $3 + c(e + 3)$. Thus, while saving one round of calls, this scheme adds only c to the drop-off term of the broadcast time.

We now turn our attention to two-round schemes. We present four type of schemes that essentially emulate corresponding schemes for a side originator.

Scheme A. In the first round, the originator calls all vertices between itself and the closest end w of its column, as well as all vertices that w would call in the first round of Scheme I. The second round is an in Scheme I. The first round takes time $1 + c(b' + a' + p - 1)$ and the second round takes time $1 + c(q - 1)$, for a total time $z_1 = 2 + c(2p + 2q - a - b - 4)$.

Scheme B. In the first round, the originator calls all vertices between itself and the closest end w of its column, as well as all vertices that w would call in the first round of Scheme II. The second round is as in Scheme II. The first round takes time $1 + c(b' + a - k + p - 1)$ and the second round takes time $1 + c(2k)$, for a total time $z_2 = 2 + c(2p + 2 + a - b + k)$.

Scheme C. This is Scheme B with the role of rows and columns exchanged. The total time is $z_3 = 2 + c(2q - 2 - a + b + l)$.

Scheme D. The scheme is shown in Fig. 4. It takes time at most $z_4 = 2 + c(2p + a - 1)$.

As before, for each position of the originator the most efficient of Schemes A–D should be taken. It can be verified that, in case of square grids, the ratio of drop-off terms of our scheme and of the lower bound yielded by Lemma 2 is at most $\frac{5}{4}$, hence, for any location of the originator in a square grid, our scheme is at most 25% slower than the least lower bound.

Theorem 15. *Let t_i, for $i = 1,2,3,4$, be the minimum time for an i-round multidrop broadcast from an interior vertex v of $M_{p\times q}$ with $p \leqslant q$. Then*

$$t_1 = 1 + c(pq-1)$$
$$t_2 \leqslant \min(z_1,z_2,z_3,z_4)$$

$$\left.\begin{array}{l} 3+ce \leqslant t_3 \leqslant 3+c(e+1) \\ t_4 = 4+ce \end{array}\right\} \text{ if the originator is neither in the middle of a row nor of a column,}$$

$$\left.\begin{array}{l} 3+c(e+1) \leqslant t_3 \leqslant 3+c(e+2) \\ t_4 = 4+c(e+1) \end{array}\right\} \text{ if the originator is in the middle of a row (column) but not both,}$$

$$\left.\begin{array}{l} 3+c(e+1) \leqslant t_3 \leqslant 3+c(e+3) \\ t_4 = 4+c(e+2) \end{array}\right\} \text{ if the originator is in the center of the grid.}$$

where z_1, z_2, z_3, z_4 are as above.

For some interior vertex positions of the originator, the eccentricity lower bound on the drop-off term actually can be achieved in three-round broadcasts. This happens if a two-edge call in the first round result in three informed vertices that can complete two-round broadcasts in their respective subgrids in the remaining $2 + c(e-2)$ time. This can be achieved when the originator vertex is not more than $\frac{1}{3}$ in from either side of the grid. (Note that in the limiting case of a side vertex, the three-round broadcast has been shown to meet this lower bound in Theorem 11.)

6. Conclusions

We have presented minimum-time, multidrop broadcast algorithms for paths, binomial and binary trees, cycles and from certain locations in grids. We have estimated this time for interior vertices of grids. In each case, we propose a scheme realizing optimal time or corresponding to an upper bound on the optimal time.

In our model, we require simultaneous calls to be vertex-disjoint. This avoids the issue of what message is dropped-off when more than one call passes through a vertex. In broadcasting, since all messages are the same, this issue is moot as long as multiple reception does not cause an error. Adopting a model only requiring edge-disjoint calls can have a significant impact on the performance of a network. This is especially true in trees, where the originator no longer would be a communication separator. Consider, for instance, the star with $n+1$ vertices, i.e., $K_{1,n}$. In our original model, any broadcast from the center would require n rounds of length one calls. However, allowing calls "crossing" at the center (i.e., not vertex-disjoint) the broadcast can be completed in $\log n$ rounds with calls of length at most two.

A natural line of future research is to investigate optimal multidrop broadcast schemes and their running time for other important architectures, such as hypercubes, cube-connected cycles or tori. For example, in the d-dimensional hypercube H_d, the standard broadcasting scheme involving d rounds of length one calls in all dimensions takes time $d + ce$, where $e = d$ is the eccentricity of any vertex, and thus schemes of more than d rounds need not be considered. A simple and efficient i-round multidrop broadcast $(1 \leqslant i \leqslant d)$ can be obtained as follows: Let $d_1, d_2, \ldots, d_i$ be integers differing by at most 1, whose sum is d. H_d can be represented as the product $H_{d_1} \times \cdots \times H_{d_i}$. In the jth round, for $j \leqslant i$, every informed vertex calls all vertices in its copy of H_{d_j} via a hamiltonian path. This scheme takes time $i + c(2^{d_1} + \cdots + 2^{d_i} - i)$. It is easy to show that this time is always at most twice the optimal, and that it is optimal if i divides d.

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