Finite Dimensional Invariant Subspaces for a Semigroup of Linear Operators*

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1. INTRODUCTION

Let $E$ be a separated locally convex space and $X$ be a subset of $E$ containing an $n$-dimensional subspace. In [3], Ky Fan proved the following invariance property $P(n)$ for $n$-dimensional subspaces contained in $X$. If $\mathcal{S} = \{T_s : s \in S\}$ is a representation of a left amenable semigroup $S$ as continuous linear transformations from $E$ into $E$ such that $T_s(L)$ is an $n$-dimensional subspace contained in $X$ whenever $L$ is an $n$-dimensional subspace contained in $X$, and there exists a closed $\mathcal{S}$-invariant subspace $H$ in $E$ of codimension $n$ with the property that $x + H \cap X$ is compact and convex for each $x \in E$, then there exists an $n$-dimensional subspace $L_0$ contained in $X$ such that $T_s(L_0) = L_0$ for all $s \in S$ (see also Iversen [8, Theorem 2]).

In this paper, we prove, among other things, that if $S$ has property $P(1)$, then $S$ is left amenable. In particular, $S$ has property $P(n)$ for each positive integer $n$ (Theorem 1(b)). We generalize Ky Fan's theorem in [3] to the class of left amenable semi-topological semigroups (Theorem 1(a)). We also give analogue characterizations for the class of extremely left amenable semi-topological semigroups (Theorem 4) and for semi-topological semigroups for which the space of (weakly) almost periodic functions has a left invariant mean (Theorems 2 and 3).

2. LEFT AMENABLE SEMI-TOPOLOGICAL SEMIGROUPS

A semi-topological semigroup $S$ is a non-empty set with a Hausdorff topology and an associative binary operation such that for each $a \in S$, the mappings $s \rightarrow s \cdot a$ and $s \rightarrow a \cdot s$ from $S$ into $S$ are continuous. Let $C(S)$

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denote the Banach algebra of all bounded continuous real-valued functions on $S$ with the supremum norm. If $f \in C(S)$, $a \in S$, define $l_a(f)(s) = f(a \cdot s)$, $s \in S$. Let $LUC(S)$ denote all $f \in C(S)$ such that the mapping $s \mapsto l_a(f)$ from $S$ into $C(S)$ is continuous when $C(S)$ has the norm topology. Then $LUC(S)$ is a translation invariant closed subalgebra of $C(S)$ containing constants. When $S$ is a topological group, $LUC(S)$ is the space of bounded right uniformly continuous functions on $S$ as defined in [7, 19.23]. $S$ is called left amenable if $LUC(S)$ has a left invariant mean $m$, i.e., $m \in LUC(S)^*$ such that $\|m\| = m(1) = 1$ and $m(l_a(f)) = m(f)$ for each $f \in LUC(S)$, $a \in S$. The class of left amenable semi-topological semigroups includes all commutative semigroups, solvable groups and compact groups. The free group on two generators is not left amenable. For details, see Day [2] and Mitchell [12].

Let $E$ be a separated locally convex space. Let $\mathcal{X} = \{T_s; s \in S\}$ be a representation of $S$ as linear mappings from $E$ into $E$. We assume (throughout this paper) that the map $\psi: S \times E \to E$, $(s, x) \mapsto T_s(x)$, $s \in S$ and $x \in E$ is separately continuous, i.e., $\psi$ is continuous in each of the two variables when the other is kept fixed. Then $\mathcal{X}$ is jointly continuous on a subset $K \subseteq E$ if the map $\psi$ is continuous on $S \times K$ when $S \times K$ has the product topology; $\mathcal{X}$ is quasi-equicontinuous on $K$ if the closure of $\mathcal{X}$ in the product space $E^K$ consists of continuous mappings from $K$ to $E$; and $\mathcal{X}$ is equicontinuous on $K$ if for each $y \in K$, $U$ a neighbourhood of 0, there exist a neighbourhood $V$ of 0 such that whenever $x \in K$, $x - y \in V$, then $T_s(x) - T_s(y) \in U$ for all $s \in S$.

Given a subset $X \subseteq E$, $n = 1, 2, 3, \ldots$, $\mathcal{L}_n(X)$ will denote all $n$-dimensional subspaces of $E$ contained in $X$. $\mathcal{L}_n(X)$ is $\mathcal{X}$-invariant if $T_s(L) \subseteq \mathcal{L}_n(X)$ for each $L \in \mathcal{L}_n(X)$ and each $s \in S$.

**Theorem 1.** Let $S$ be a semi-topological semigroup.

(a) If $S$ is left amenable, then $S$ satisfies $P(n)$ for each positive integer $n$:

$P(n)$: Let $E$ be a separated locally convex space and $\mathcal{X} = \{T_s; s \in S\}$ be a representation of $S$ as linear operators from $E$ into $E$ jointly continuous on compact convex subsets of $E$. Let $X$ be a subset of $E$ such that there exists a closed $\mathcal{X}$-invariant subspace $H$ of $E$ with codimension $n$ and $x + H \cap X$ is compact convex for each $x \in E$. If $\mathcal{L}_n(X)$ is non-empty and $\mathcal{X}$-invariant, then there exists $L_0 \in \mathcal{L}_n(X)$ such that $T_s(L_0) = L_0$ for each $s \in S$.

(b) If $S$ satisfies $P(1)$, then $S$ is left amenable.

**Proof.** (a) Let $F$ denote the quotient space $E/H$ with quotient map $q: E \to F$. Following an idea of Fan [3], let $\mathcal{K}$ denote the collection of all linear maps $T$ from $F$ into $E$ such that $T(y) \in q^{-1}(y) \cap X$ for each $y \in F$. If $T \in \mathcal{K}$, then $T(F)$ is an $n$-dimensional subspace contained in $X$. Conversely each $n$-dimensional subspace of $E$ contained in $X$ is the range of a unique $T$.
In particular, $\mathcal{H}$ is non-empty. Also $T_s \circ T \in \mathcal{H}$ for each $s \in S$, $T \in \mathcal{H}$ by invariance of $H$ and $L_n(X)$.

Since $q^{-1}(y) \cap X$ is a compact convex set for each $y \in F$, an application of the Tychonoff theorem shows that $\mathcal{H}$, with the topology $\tau$ of pointwise convergence, is compact and convex. Also, since $\mathcal{S}$ is jointly continuous on compact convex subsets of $E$, the action of $S$ on $(\mathcal{H}, \tau)$ defined by $(s, T) \mapsto T_s \circ T$, $s \in S$, $T \in \mathcal{H}$, is jointly continuous. Hence by Mitchell's generalisation of Day's fixed point theorem [12, Theorem 21], there exists a $T_0 \in \mathcal{H}$ such that $T_s \circ T_0 = T_0$ for each $s \in S$. Let $L_0$ be the range of $T_0$. Then $T_s(L_0) = L_0$ for each $s \in S$.

(b) Let $E = LUC(S)^*$ with the weak*-topology and $\mathcal{S} = \{l_s^*; s \in S\}$. If $\Phi$ is a weak* compact subset of $LUC(S)^*$, then $\Phi$ is bounded. Let $K > 0$ such that $\|\phi\| \leq K$ for each $\phi \in \Phi$. Let $\{\phi_\alpha\}$ be a net of $\Phi$ converging to some $\phi$ in the weak*-topology and $\{s_\beta\}$ be a net in $S$ converging to some $s \in S$, then for each $f \in LUC(S)$,

$$|l_{s_\beta}^*\phi_\alpha(f) - l_s^*\phi(f)| \leq |l_{s_\beta}^*\phi_\alpha(f) - l_s^*\phi_\alpha(f)| + |l_s^*\phi_\alpha(f) - l_s^*\phi(f)|$$

$$\leq K \|l_{s_\beta}f - l_s f\| + |\phi_\alpha(l_s f) - \phi(l_s f)|$$

which converges to zero. Hence $\mathcal{S}$ is jointly continuous on weak*-compact convex subsets of $LUC(S)^*$. Let $X$ denote the union of one-dimensional subspaces of $LUC(S)^*$ spanned by the set of means $m$, i.e., $m \in LUC(S)^*$ and $\|m\| = m(1) = 1$. Let $H = \{\phi \in LUC(S)^*; \phi(1) = 0\}$. Then for each $\phi_\alpha \in LUC(S)^*$, $\phi_\alpha + H \cap X = \{\phi_\alpha(1)m; m \text{ is a mean on } LUC(S)^*\}$, which is weak*-compact and convex. If $L$ is a one-dimensional subspace of $X$, then $L = \{\lambda m; \lambda \in \mathbb{R}\}$ for some mean $m \in LUC(S)^*$. Hence $l_s^*(L) = \{\lambda l_s^*m; \lambda \in \mathbb{R}\}$ is a one-dimensional subspace contained in $X$ since $l_s^*m$ is also a mean for each $s \in S$. By $P(1)$, there exists $L_0 = \{\lambda m_0; \lambda \in \mathbb{R}\}$, $m_0$ a mean, such that $l_s^*(L_0) = L_0$ for each $s \in S$. If $s \in S$, then $l_s^*m_0 = \lambda m_0$, $\lambda \in \mathbb{R}$. But $(l_s^*m_0)(1) = 1 = \lambda$. So $l_s^*m_0 = m_0$, i.e., $m_0$ is a left invariant mean on $LUC(S)^*$.

**Corollary 1.** Let $S$ be a semi-topological semigroup with property $P(1)$. Then $S$ has property $P(n)$ for all positive integer $n$.

**Open Problem.** Does $P(n)$ imply $P(n + 1)$ for any positive integer $n$?

### 3. Almost Periodic Functions

A function $f \in C(S)$ is almost periodic (resp. weakly almost periodic) if $\{l_a f; a \in S\}$ is relatively compact in the norm topology (resp. weak
topology) of $C(S)$. Let $AP(S)$ (resp. $WAP(S)$) denote the space of almost periodic (resp. weakly almost periodic) functions on $S$.

**Theorem 2.** Let $S$ be a semi-topological semigroup.

(a) If $AP(S)$ has a left invariant mean, then $S$ satisfies $P_A(n)$ for each positive integer $n$:

$P_A(n)$: Let $E$ be a separated locally convex space and $\mathcal{S} = \{T_s; s \in S\}$ be a representation of $S$ as linear operators from $E$ into $E$ equicontinuous on compact convex subsets of $E$. Let $X$ be a subset of $E$ such that there exists a closed $\mathcal{S}$-invariant subspace $H$ of $E$ with codimension $n$ and $x + H \cap X$ is compact convex for each $x \in E$. If $\mathcal{L}_n(X)$ is non-empty and $\mathcal{S}$-invariant, then there exists $L_0 \in \mathcal{L}_n(X)$ such that $T_s(L_0) = L_0$ for each $s \in S$.

(b) If $S$ has property $P_A(1)$, then $AP(S)$ has a left invariant mean.

**Proof:** (a) Let $\mathcal{H}$ be as in the proof of Theorem 1(a). It is easy to see that $\mathcal{H}$ is separately continuous on $(K, \tau)$. By [10, Theorem 3.2], it suffices to show that $\mathcal{H}$ is equicontinuous on $K$. Let $T_0 \in \mathcal{H}$ be fixed. For each neighbourhood $\nu$ of the origin in $E$ and each finite subset $a$ of $E$, let $V_{\nu, a} = \{T \in \mathcal{H}(F, E), T(y) \in \nu, \forall y \in a\}$. (Here $\mathcal{H}(F, E)$ denotes all linear maps from $F$ to $E$.) Then sets of the form $V_{\nu, a}$ form a neighbourhood base of 0 in $\mathcal{H}(F, E)$ in the topology of pointwise convergence. For each $y \in \sigma$, $T(y) \in q^{-1}(y) \cap X$ for each $T \in \mathcal{H}$ and $\mathcal{S}$ is equicontinuous on $q^{-1}(y) \cap X$. Hence there exist $\nu_y$, a neighbourhood of 0 in $E$ such that $T_s(T(y) - T_0(y)) \in \eta$ for all $s \in S$ when $T(y) - T_0(y) \in \nu_y$. Let $\omega$ be the intersection of all $\nu_y$, $y \in \sigma$. Then $T_s \circ T - T_s \circ T_0 \in V_{\nu, \sigma}$ whenever $T - T_0 \in V_{\omega, \sigma}, T \in \mathcal{H}$.

(b) Let $E = AP(S)^*$ with the weak* topology and $\mathcal{S} = \{l_s^*; s \in S\}$. For each $f \in AP(S)$, let $p_f$ be a seminorm on $AP(S)^*$ defined by $p_f(\phi) = \sup(\|\phi(l_s f)\|, \|\phi(f)\|, a \in S)$ for each $\phi \in AP(S)^*$, and let $Q = \{p_f; f \in AP(S)\}$. Then clearly $\mathcal{S}$ is a equicontinuous on $AP(S)^*$ when $AP(S)^*$ has the locally convex topology $\tau$ determined by $Q$. Let $\Phi$ be a weak* compact subset of $AP(S)^*$. Then on $\Phi$ the weak* topology agrees with the topology of uniform convergence on totally bounded subsets of $AP(S)$. Hence $\tau$ and the weak* topology agree on $\Phi$. Consequently $\mathcal{S}$ is equicontinuous on $(\Phi, \text{weak*})$. An argument similar to that for Theorem 1(b) shows that $AP(S)$ has a left invariant mean.

**Corollary 2.** If $S$ is a semi-topological semigroup with finite intersection for closed right ideals, then $S$ has property $P_A(n)$ for each positive integer $n$. 
Proof. This follows from Theorem 2(a) and [6, Corollary 1].

Using the second theorem in [11, p. 123] and an argument similar to that for Theorem 2 shows:

**THEOREM 3.** Let $S$ be a semi-topological semigroup.

(a) If $WAP(S)$ has a left invariant mean, then $S$ satisfies $P_w(n)$ for each positive integer $n$:

$P_w(n)$: Let $E$ be a separated locally convex space and $\mathcal{S} = \{T_s; s \in S\}$ be a representation of $S$ as linear operators from $E$ into $E$ quasi-equicontinuous on compact convex subsets of $E$. Let $X$ be a subset of $E$ such that there exists a closed $\mathcal{S}$-invariant subspace $H$ of $E$ with codimension $n$ and $x + H \cap X$ is compact and convex for each $x \in E$. If $\mathcal{S}_n(X)$ is non-empty and $\mathcal{S}$-invariant, then there exists $L_0 \in \mathcal{S}_n(X)$ such that $T_s(L_0) = L_0$ for each $s \in S$.

(b) If $S$ has property $P_w(1)$, then $WAP(S)$ has a left invariant mean.

Since $WAP(S)$ has a left invariant mean when $S$ is a group [1, p. 151], we have

**COROLLARY 3.** If $S$ is a group, then $S$ has property $P_w(n)$ for each positive integer $n$.

Remark. Since equicontinuity of $\mathcal{S}$ on subsets of $E$ imply joint continuity and quasi-equicontinuity, $P(n)$ or $P_w(n)$ implies $P_A(n)$ for each positive integer $n$. If $S$ is the free group on two generators, then $S$ has property $P_w(n)$ (and hence $P_A(n)$) but not $P(n)$. However, we do not know of an example of a topological semigroup with property $P_A(n)$ but not $P_w(n)$.

4. **EXTREMELY LEFT AMENABLE SEMI-TOPOLOGICAL SEMIGROUPS**

A semi-topological semigroup $S$ is extremely left amenable if $LUC(S)$ has a left invariant mean $m$ which is multiplicative, i.e., $m(fg) = m(f)m(g)$ for all $f, g \in LUC(S)$. When $S$ is discrete, this is equivalent to for any two elements in $S$ has a common right zero (see Granirer [4]). If $S$ is a subsemigroup of a locally compact and $S$ is extremely left amenable, then $S$ has only one element [5, Theorem 3]. For examples of extremely left amenable semi-topological semigroups $S$ which are not left amenable as discrete semigroups see [9, p. 72]. The following is an analogue of Theorem 1 for the class of extremely left amenable semigroups.

**THEOREM 4.** Let $A$ be a semi-topological semigroup.
(a) If $S$ is extremely left amenable, then $S$ satisfies $Q(n)$ for each positive integer $n$:

$Q(n)$: Let $E$ be a separated locally convex space and $\mathcal{S} = \{T_s; s \in S\}$ be a representation of $S$ as linear operators from $E$ into $E$ jointly continuous on compact subsets of $E$. Let $X$ be a subset of $E$ such that there exists a closed $\mathcal{S}$-invariant subspace $H$ of $E$ with codimension $n$ and $x + H \cap X$ is compact for each $x \in E$. If $\mathcal{L}_n(X)$ is non-empty and $\mathcal{S}$-invariant, then there exists $L_0 \in \mathcal{L}_n(X)$ such that $T_s(L_0) = L_0$ for each $s \in S$.

(b) If $S$ satisfies $Q(1)$, then $S$ is extremely left amenable.

Proof: (a) With the same notation as the proof of Theorem 1, the set $(\mathcal{S}, \tau)$ is a compact and the action of $S$ on $\mathcal{S}$ is jointly continuous. Now apply Mitchell's fixed point theorem [12, Theorem 1].

The proof of (b) is also similar to that of Theorem 1(b). We omit the details.

References

10. A. T. Lau, Invariant means on almost periodic functions and fixed point properties, Rocky Mountain J. Math. 3 (1973), 69-76.