Hawking temperature from tunnelling formalism

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Abstract

It has recently been suggested that the attempt to understand Hawking radiation as tunnelling across black hole horizons produces a Hawking temperature double the standard value. It is explained here how one can obtain the standard value in the same tunnelling approach.

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A classical black hole has a horizon beyond which nothing can leak out. But there is a relation between the area of the horizon and the mass (and other parameters like the charge) indicating a close similarity [1] with the thermodynamical laws, thus allowing the definition of an entropy and a temperature [2]. This analogy was surmised to be of quantum origin and made quantitative after the theoretical discovery of radiation from black holes [3]. For a Schwarzschild black hole, the radiation, which is thermal, has a temperature

\[ T_H = \frac{\hbar}{4\pi r_h} = \frac{\hbar}{8\pi M}. \]  

(1)

where \( r_h \) gives the location of the horizon in standard coordinates and \( M \) is the mass of the black hole. This was derived by considering quantum massless particles in a Schwarzschild background geometry. The derivation being quite complicated, attempts have been made to understand the process of radiation by other methods. In [4], a path integral study was made, and analytic continuation in complex time used to relate amplitudes for particle emission and absorption with the result that the ratio of emission and absorption probabilities for energy \( E \) is given by

\[ P_{\text{emission}} = \exp \left(-\frac{E}{T_H}\right) P_{\text{absorption}}. \]  

(2)

This “detailed balance” relation provides further evidence for the temperature \( T_H \). Furthermore, the propagator in the Schwarzschild background was shown [4] to have a periodicity in the imaginary part of time with period \( 4\pi r_h = 8\pi M \), again suggesting the same temperature. There is also an argument involving a conical singularity on passing to imaginary time, which can only be avoided if the standard Hawking temperature is chosen.

Later, other attempts were made to understand the emission of particles across the horizon as a quantum mechanical tunnelling process [5]. The approach of using (2) was followed in [6]. Different Hamilton–Jacobi treatments were used to reproduce the standard temperature \( T_H \) [7]. Recently, however, it has been pointed out [8] that this approach seems to produce a temperature that is double the standard value \( T_H \), which corresponds to a halving of the period in imaginary time. This is reminiscent of [9], where it was pointed out that the Hawking temperature could be doubled with a different interpretation of the gravitational field in quantum theory. However, such an interpretation is not used in [8]. So it becomes necessary to try to resolve the contradiction between this and the earlier analyses.

A massless particle in the Schwarzschild background is described by the Klein–Gordon equation

\[ \hbar^2 \left(-g\right)^{-1/2} \partial_\mu \left(g^{\mu\nu} \left(-g\right)^{1/2} \partial_\nu \phi\right) = 0. \]  

(3)

One expands

\[ \phi = \exp \left(\frac{i}{\hbar} S + \cdots\right) \]  

(4)

to obtain to leading order in \( \hbar \) the equation

\[ g^{\mu\nu} \partial_\mu S \partial_\nu S = 0. \]  

(5)
If we use separation of variables to write, provisionally,
\[ S = Et + S_0(r), \]
the equation for \( S_0 \) becomes
\[ -\frac{E^2}{1 - \frac{r}{r_0}} + \left(1 - \frac{r_h}{r}\right)S_0'(r)^2 = 0 \]
in the Schwarzschild metric. The formal solution of this equation is
\[ S_0(r) = \pm E \int_1^r \frac{dr}{1 - \frac{r}{r_0}}. \]
The sign ambiguity comes from the square root and corresponds to the fact that there can be incoming/outgoing solutions. There is, furthermore, a singularity at the horizon \( r = r_h \), which has to be handled if one tries to find a solution across it.

One way to skirt the pole is to change \( r - r_h \) to \( r - r_h - i\epsilon \). This yields
\[ S_0(r) = \pm E \left[r + r_h \cdot i\pi + r_h \int_1^r \frac{dr}{1 - \frac{r}{r_0}}\right]. \]
where \( P() \) denotes the principal value. For the outgoing solution,
\[ S_{\text{out}} = Et - E \left[r + r_h \cdot i\pi + r_h \int_1^r \frac{dr}{1 - \frac{r}{r_0}}\right], \]
the imaginary part yields a decay factor \( \exp(-\pi r_h E/\hbar) \) in the amplitude and hence a factor \( \exp(-2\pi r_h E/\hbar) \) in the probability. This has been interpreted to signal a temperature \[ C = -i\pi r_h E + (\text{Re} \, C), \]
\[ \text{Im} \, S_{\text{out}} = -2\pi r_h E \]
directly, yielding the expected decay factor \( \exp(-4\pi r_h E/\hbar) \) in the probability. There is no lack of consistency [8,10] between the Schwarzschild and the Painlevé formulations, and it is reassuring to note that \( \text{Im} \, S_{\text{out}} - \text{Im} \, S_{\text{in}} = -2\pi r_h E \) in both the Schwarzschild and the Painlevé cases, irrespective of the value of the complex constant \( C \).

Here we have restricted ourselves to the simplest black hole horizon. It is easy to check that these ideas work also in the case of, say, the de Sitter horizon and the Rindler horizon (see the second paper in [7]). In short, there is no problem with the standard value of the Hawking temperature. Hawking radiation at the standard temperature can be understood through tunnelling, contrary to the view of [8]. The crucial step is to note that the classical absorption probability is unity.
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References