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A natural bijection between permutations and a family of descending plane partitions

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ABSTRACT

We construct a direct natural bijection between descending plane partitions without any special part and permutations. The directness is in the sense that the bijection avoids any reference to nonintersecting lattice paths. The advantage of the bijection is that it provides an interpretation for the seemingly long list of conditions needed to define descending plane partitions. Unfortunately, the bijection does not relate the number of parts of the descending plane partition with the number of inversions of the permutation as one might have expected from the conjecture of Mills, Robbins and Rumsey, although there is a simple expression for the number of inversions of a permutation in terms of the corresponding descending plane partition.

1. Introduction

Descending plane partitions were introduced by George Andrews in order to prove the weak Macdonald conjecture [1] and are counted by the ASM numbers. When they were initially introduced by Andrews, the general sense was that these objects were extremely artificial and designed to specifically solve the conjecture.[1]

Descending plane partitions were later found to have remarkable connections to alternating sign matrices by Mills, Robbins and Rumsey in their proof of the Macdonald conjecture [6] which they refined further in many ways in a series of conjectures in [7]. Subsequently, they were also related to other structures in combinatorics. Many kinds of plane partitions are in natural bijection with classes of nonintersecting lattice paths [2] and Lalonde [5] has shown, in particular, that the antiautomorphism τ of descending plane partitions defined in [7] has a natural interpretation as

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1 See for example, Doron Zeilberger's paean to Dave Robbins [8].

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Gessel–Viennot path duality. Krattenthaler [4] has proved a bijection between descending plane partitions and rhombus tilings of a hexagon from which an equilateral triangle has been removed from the center.

We will prove that the number of descending plane partitions with no special part is the same as the number of permutations by constructing an explicit and very natural bijection between the two objects. Unfortunately this bijection does not naturally relate permutations with p inversions to descending plane partitions with p total parts. We hope that a generalization of these ideas will lead to a bijection between descending plane partitions and alternating sign matrices.

The outline of the rest of the article is as follows. We begin with the notations and relevant known results in Section 2. We will need a result about descending plane partitions with one row, which we describe in Section 3 and proceed to the proof of the bijection in Section 4. We shall give details of the other (known) bijection through lattice paths and some other remarks in Section 5.

2. Definitions

We begin with a series of definitions and known results about the objects considered here. This section is present mostly to set the notation and experts should feel free to skip it.

Definition 1. A *descending plane partition* (DPP) is an array $a = (a_{ij})$ of positive integers defined for $j \geq i \geq 1$ that is written in the form

$$\begin{array}{ccccccc} a_{11} & a_{12} & \cdots & \cdots & \cdots & \cdots & a_{1,\mu_1} \\ & a_{22} & \cdots & \cdots & \cdots & a_{2,\mu_2} & \\ & & \cdots & \cdots & \cdots & & \\ & & a_{rr} & \cdots & a_{r,\mu_r} & & \end{array} \tag{2.1}$$

where,

(1) $\mu_1 \geq \cdots \geq \mu_r$,
(2) $a_{i,j} \geq a_{i,j+1}$ and $a_{i,j} > a_{i+1,j}$ whenever both sides are defined,
(3) $a_{i,i} > \mu_i - i + 1$ for $i \leq i \leq r$,
(4) $a_{i,i} \leq \mu_{i-1} - i + 2$ for $1 < i \leq r$.

The second condition in the above definition means that terms are weakly decreasing along rows and strictly decreasing along columns. The third condition simply means that the diagonal entry is strictly greater than the number of entries in its row, and the fourth condition, that it is at most the number of entries in the row above it. Note that the last two conditions ensure that the diagonal entries are always greater than one.

Definition 2. A descending plane partition of *order n* is a descending plane partition all of whose entries are less than or equal to n.

Theorem 1 (*Andrews, 1979, [1]*). *The number of descending plane partitions of order n, $D(n)$ is given by*

$$D(n) = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}. \tag{2.2}$$

We now go on to discuss refined enumeration of DPPs.

Definition 3. An entry $a_{i,j}$ of the descending plane partition a is called a *special part* if $a_{ij} \leq j - i$.

This implies that diagonal elements can never be special parts. We have now all definitions needed for DPPs. We go on to define ASMs and their refinements. Another important statistic for us will be the $r(a)$, the number of rows of the DPP a.

Definition 4. A permutation π of the letters $\{1, \dots, n\}$ has an *ascent* at position k with $1 \le k < n$, if $\pi_k < \pi_{k+1}$.

The number of permutations on n letters with k ascents is the Eulerian number $E(n, k)$ [3], which satisfies the recurrence

$$E(n, k) = (k + 1)E(n - 1, k) + (n - k)E(n - 1, k - 1),$$

for $n \ge 0$, $0 \le k \le n$ with the initial condition $E(0, k) = \delta_{k,0}$.

Definition 5. The *non-inversion number* $I(\pi)$ of a permutation π on n letters is the number of pairs of elements i, j such that $i < j$ and $\pi_i < \pi_j$.

The non-inversion number is the number of elementary transpositions to convert a given permutation π to the totally descending permutation $n(n - 1) \dots 21$, as opposed to $12 \dots n$, hence the name. There is an obvious involution on permutations which turns inversion numbers into non-inversion numbers.

Theorem 2. *There is a natural one-to-one correspondence between descending plane partitions of order n with k rows and no special part, and permutations of size n with k ascents.*

Remark 1. In Theorem 2, k varies from 0 to $n - 1$. The empty DPP, $a = \phi$ counts as a permutation with zero rows, and vacuously, with no special part. There is also exactly one permutation with zero ascents, namely $\pi = n(n - 1) \cdots 21$.

This immediately leads to the refined count of DPPs.

Corollary 3. *The number of descending plane partitions of order n with k rows and no special parts is given by the Eulerian number $E(n, k)$.*

3. Descending plane partitions with one row

Before we can prove the main theorem, however, we need a simpler result. We fix notation for future use. We denote a DPP by $a = (a_{i,j})$ and the ith row of the DPP by $\alpha^{(i)}$. We will also use a different notation for permutations suited for the interest. We will denote a permutation with k ascents by $\beta^{(1)} \cdots \beta^{(k+1)}$, where each $\beta^{(i)}$ is decreasing. When $k = 1$, we will denote the permutation as $\beta\gamma$ to avoid unnecessary clutter of indices. We will also use a_i and b_i to denote pure numbers.

Lemma 4. *There is a natural one-to-one correspondence between descending plane partitions of order n with one row $a = (a_1, \dots, a_m)$ and permutations of size n with a single ascent $\beta\gamma$.*

Proof. We first associate a permutation with a single ascent to a DPP with a single row. From the basic definitions of the DPP, we know that

$$n \ge a_1 \ge a_2 \ge \cdots \ge a_m.$$

Since the DPP has no special parts, we know that $a_k \ge k$ for $1 \le k \le m$. But we also know that $a_1 > m$ from the third condition in the definition, which is stronger than the previous condition for $k = 1$.

From the DPP, we construct

$$\gamma = (a_1, a_2 - 1, \dots, a_m - (m - 1)).$$

From the weak decreasing condition above, we clearly see that

$$n \ge \gamma_1 > \cdots > \gamma_m.$$

From the no special part condition, $\gamma_k \ge 1$ for all k. Therefore, γ is a strictly decreasing sequence of elements belonging to $[n]$. We then define $\beta = [n] \setminus \gamma$ also sorted in decreasing order. From this, we obtain the required permutation by writing it as $\beta\gamma$. Notice that the single ascent occurs at the junction of β and γ because γ contains m elements, at least one of which is greater than m, forcing at least one element not in γ less than m. Finally, since the maximum value of a_1 is n, the third condition in Definition 1 forces $m < n$ and β is therefore necessarily nonempty.

The inverse procedure is quite clear. A permutation with a single ascent can be clearly uniquely decomposed into two nonempty descending lists β and γ such that the first element of γ is greater than the last element of β. We obtain the required DPP $a = (\gamma_1, \gamma_2 + 1, \dots, \gamma_m + (m-1))$. This satisfies the weak decrease condition since the elements of γ are strictly decreasing. Since $\gamma_k \geq 1$, we clearly have $a_k \geq k$. Lastly, notice that γ_1 has to be strictly greater than m, because if not, then we are forced to have $\gamma_2 = m-1, \dots, \gamma_m = 1$, but this would mean that the last element of β is $m+1$ violating the condition of a single ascent. The list a thus yields a DPP with one row and no special part. □

We can use this to calculate the non-inversion number for such permutations.

Corollary 5. *If a permutation π is in bijection with a descending plane partition $a = (a_1, \dots, a_m)$ with one row and no special part, then*

$$I(\pi) = \sum_{i=1}^{m} a_i - m^2. \tag{3.1}$$

Proof. We use the same notation as the proof of Lemma 4. $I(\pi)$ is simply the total number of elementary transpositions taken by the elements in γ to return to their original position in the completely descending permutation. We start from the rightmost entry in γ. Clearly γ_m will take $\gamma_m - 1$ steps, γ_{m-1} will take $\gamma_{m-1} - 2$ steps and so on. Thus

$$\begin{aligned} I(\pi) &= (a_m - (m-1) - 1) + (a_{m-1} - (m-2) - 2) + \dots + (a_1 - m) \\ &= \sum_{i=1}^{m} (a_i - m), \end{aligned} \tag{3.2}$$

which gives the desired result. □

Notice that γ and therefore $I(\pi)$ is determined independently of the order of the DPP. For example, suppose the DPP is $a = (6, 4, 3)$. Then $\gamma = (6, 3, 1)$, and we obtain the permutation 7542631 if $n = 7$ and 987542631 if $n = 9$. In both cases, the non-inversion number for the permutation is four, whereas a has three total parts. We will need some properties of the bijection in Lemma 4 for proving Theorem 2.

Lemma 6. *Using the same notation as Lemma 4 and assuming a has length m, the following hold:*

(1) $\beta_{n-m} = 1$ *occurs if and only if* $a_m > m$. *Assuming* $1 < p < n$,

$$\beta_{n-m} = p \Leftrightarrow \forall i > m - p + 1, \qquad a_i = m \quad \textit{and} \quad a_{m-p+1} > m.$$

(2) $\beta_1 = n$ *occurs if and only if* $a_1 < n$. *Assuming* $0 < p < m$,

$$\beta_1 = n - p \Leftrightarrow \forall i \leq p, \qquad a_i = n \quad \textit{and} \quad a_{p+1} < n.$$

Lastly, $\beta_1 = n - m$ *if and only if* $a_1 = \dots = a_m = n$.

Proof. (1) $\beta_{n-m} = p$ iff the letters $1, \dots, p-1$ belong to γ, and since γ is arranged in descending order,

$$a_m - (m-1) = 1, \dots, a_{m-(p-2)} - (m - (p-1)) = p - 1,$$

and furthermore $a_{m-(p-1)} - (m-p) > p$, which is precisely the condition stated, when $p > 1$. Notice that p cannot take the value n because that would violate the single ascent condition. In case $p = 1$, we can either have $a_m - (m-1) > 1$ or $m = 1$. The latter works because, if $m = 1$, $a_1 > 1$ in order for the permutation to have a single ascent.

(2) $\beta_1 = n-p$ iff the letters $n-p+1, \dots, n$ belong to γ and since γ is arranged in descending order,

$$a_1 = n, a_2 - 1 = n-1, \dots, a_p - (p-1) = n - (p-1),$$

and the reason $n-p$ does not belong to β is that either $m = p$ or $m > p$ and $a_{p+1} - p < n - p$. This is again exactly the condition stated, when $p > 0$. If $p = 0$, n does not belong to γ and thus $a_1 < n$.

4. The main result

We will construct the bijection inductively on the number of rows in the DPP. Before that, we make some remarks on the properties of DPPs, which follow from Definition 1 and will need a lemma which will be the workhorse of the proof.

Remark 2. (1) Any row of a DPP is, by itself, also a valid DPP. Moreover, a row which is part of a DPP with no special part is also a DPP with no special part. The latter follows from the shifted position of successive rows.

(2) Removing the last row from a DPP yields another valid DPP. Obviously, if the original DPP had no special part, neither will the new one.

Lemma 7. *Given a set S of positive integers of cardinality n, there exists a natural bijection between the DPPs, a, with one row and no special part whose length m satisfies $m < n$ and $a_1 \le n$, and sequences of all the elements of S with one ascent.*

Proof. We define a map ϕ from the set S to $[n]$ which takes the smallest element to 1, the next smallest to 2 and so on until it takes the largest element to n. Clearly, ϕ is invertible. Using Lemma 4 therefore, we obtain a bijection between DPPs of one row and order n and no special part, and the sequence of elements of S with a single ascent. Since the DPP has order n, we have $a_1 \le n$ and therefore, the length of a is strictly less than n. $\square$

For example, suppose $S = \{11, 10, 6, 3, 2\}$ and $a = (4, 3, 3)$. The bijection from Lemma 4 yields the permutation on $n = 5$ letters, 53421, which using the map ϕ gives the sequence 11, 6, 10, 3, 2.

Before we go on to the proof, we take an example to illustrate the idea. Consider the DPP with no special part

$$\begin{array}{ccccc} 7 & 7 & 6 & 5 & 5 \\ & 4 & 4 & 4 & \\ & & 3 & 2 & \end{array} \tag{4.1}$$

of order $n = 9$, say. Then we start with the permutation 987654321. We will now alter it by considering the DPP rowwise. In each row, we mark two vertical lines to separate β and γ using the notation in Lemma 4. The rightmost is γ and the one in the middle is β. The leftmost part is completely untouched.

$$\begin{array}{rcl} 77655 & \to & 98|53|76421 \\ 444 & \to & 9853|71|642 \\ 32 & \to & 985371|4|62 \end{array} \tag{4.2}$$

and we end up with the permutation 985371462, which has exactly three ascents. In lines two and three, we have used Lemma 7 for the rightmost part in the previous line.

Proof of Theorem 2. We will use induction on k, the number of rows of the DPP. The case $k = 1$ of the induction is precisely Lemma 4. We now assume we have constructed, in a one-to-one way, a permutation with $k-1$ ascents from a DPP a with $k-1$ rows,

$$\alpha^{(1)}, \dots, \alpha^{(k-1)}.$$

Write the permutation with $k-1$ ascents as

$$\beta^{(1)} \cdots \beta^{(k)},$$

where each $\beta^{(j)}$ is descending and write the kth row of the DPP as $\alpha^{(k)}$. Assume that the $k-1$th row of the DPP has length m_{k-1}. That is, the terms are from $a_{k-1,k-1}$ to $a_{k-1,k+m_{k-1}-2}$. Similarly, $\alpha^{(k)}$ has length m_k, $m_k \le m_{k-1}-1$ from Definition 1 comprising of terms $a_{k,k}$ to $a_{k,k+m_k-1}$.

The idea is to perform the operation on $\beta^{(k)}$ and create another ascent within it of length m_k from the right, while preserving the ascent from $\beta_{(k-1)}$, which we describe now. Let S be the set of numbers in $\beta^{(k)}$, which has cardinality m_{k-1}. $\alpha^{(k)}$ is a DPP with one row, no special part, of length less than m_{k-1} and whose first element $a_{k,k}$ satisfies $a_{k,k} \le m_{k-1}$. Therefore we are in a position to use Lemma 7 and obtain a sequence of the elements of S with a single ascent, which we call $\gamma^{(k)}$ and $\gamma^{(k+1)}$. The length of $\gamma^{(k+1)}$ is clearly m_k. We claim that this procedure is invertible and by repeated application yields the desired permutation with k ascents. What follows is a check of these claims.

It remains to show that $\gamma_1^{(k)}$ is larger than the last entry in $\beta^{(k-1)}$. Suppose this last entry is $p \in [n-1]$. If $p = 1$, we are done. If not, let $p \in \{2, \ldots, n-1\}$. Since the rule for creating an ascent is the same as that of creating the first one, we can use properties of the bijection for a single row. We will need them for the row $\alpha^{(k-1)}$. From Lemma 6(1), this implies that

$$a_{k-1,k+m_{k-1}-2} = \cdots = a_{k-1,k+m_{k-1}-p} = m_{k-1}$$

and $a_{k-1,m_{k-1}-p+1} \ge m_{k-1}+1$, and moreover that the last $p-1$ letters of $\beta^{(k)}$ are $p-1, \ldots, 1$. Thus

$$a_{k-1,k-1} \ge \cdots \ge a_{k-1,k+m_{k-1}-p-1} \ge m_{k-1}+1.$$

Notice that the first $m_{k-1}-(p-1)$ letters of $\beta^{(k)}$ are greater than p. For it to happen that $\gamma_1^{(k)} < p$, $\gamma_1^{(k)}$ must be one of the last $p-1$ letters of $\beta^{(k)}$. This implies that the action of $\alpha^{(k)}$ forces all the letters larger than $\gamma_1^{(k)}$ into $\gamma^{(k+1)}$, which can only happen if $a_{k,k} = \cdots = a_{k,k+m_{k-1}-p} = m_{k-1}$. But this would imply $a_{k,k+m_{k-1}-p} = a_{k-1,k+m_{k-1}-p}$, which violates condition (2) in Definition 1. Therefore the first entry of $\gamma^{(k)}$ is greater than the last entry of $\beta^{(k-1)}$. We have thus shown that each DPP with no special entries and with k rows gives rise to a permutation with k ascents.

For the reverse process, one has to read the permutation with k ascents from the right by looking at the part immediately after the $k-1$th ascent. Using Lemma 7, one obtains the kth row of the DPP with no special parts. One then is left with a permutation with $k-1$ ascents and one goes on recursively.

Everything except the columnwise descent is clearly ensured by this procedure. Essentially this occurs because of the condition that creation of a new ascent should not kill off an earlier ascent. We now describe the columnwise descent in some detail. We use the usual notation for the permutation with k ascents, we denote the lengths of $\beta^{(k-1)}$, $\beta^{(k)}$ and $\beta^{(k+1)}$ being $m_{k-2}-m_{k-1}$, $m_{k-1}-m_k$ and m_k respectively so that the last three rows for the DPP, denoted $\alpha^{(k-2)}$, $\alpha^{(k-1)}$ and $\alpha^{(k)}$ have lengths m_{k-2}, m_{k-1} and m_k in accord with the convention used before.

We will now analyze the structure of $\alpha^{(k)}$ and $\alpha^{(k-1)}$. In particular, we will denote the maps used in Lemma 7 as ϕ and ϕ' respectively.

We then use the modified form of Lemma 6(1) to note that $\beta^{(k-1)}_{m_{k-2}-m_{k-1}} = p$ implies

$$a_{k-1,k+m_{k-1}-2} = \cdots = a_{k-1,k+m_{k-1}-\phi'(p)} = m_{k-1},$$

and $a_{k-1,k+m_{k-1}-\phi'(p)-1} \ge m_{k-1}+1$. This is clear because the only change in using Lemma 6 directly is that relative positions are now specified using the map ϕ'. Similarly, $\beta_1^{(k)} = r$ implies using the modified form of Lemma 6(2), this time with map ϕ,

$$a_{k,k} = \cdots = a_{k,k+m_{k-1}-\phi(r)-1} = m_{k-1},$$

and either $m_{k-1}-m_k = r$ or $a_{k,k+m_{k-1}-\phi(r)} < m_{k-1}-1$. The ascent of the permutation implies $r > p$. This in turn implies $\phi(r) \ge \phi'(p)$ because it is possible that there are no elements between r and p. A violation of the descent condition of the DPP would entail the overlapping of the parts of the $k-1$th and kth rows of a which equal m_{k-1}. This means $k+m_{k-1}-\phi'(p) \le k+m_{k-1}-\phi(r)-1$ which implies that $\phi(r) \le \phi'(p)-1$. But this is a contradiction. Therefore a permutation with k ascents gives rise to a DPP with k rows.

We can also extend the result of Corollary 5 to calculate the non-inversion number for a permutation with k ascents.

Corollary 8. *If a permutation π has k ascents, then the non-inversion number is given by the corresponding descending plane partition a with k rows of sizes $m_1, \ldots, m_k$ as*

$$I(\pi) = \sum_{i=1}^{k} \sum_{j=i}^{m_i+i-1} a_{i,j} - \sum_{i=1}^{k} m_i^2. \tag{4.3}$$

Proof. Since the ith row of a has length m_i, the entries are written as $a_{i,i}, \ldots, a_{i,m_i+i-1}$.

Each successive row of the DPP is going to create more non-inversions because one shifts successively larger numbers to the right. Moreover, the action of each row is the same independent of the previous rows. Therefore, one obtains the same answer for each row as in Corollary 5. Thus, the required answer is the sum for all rows. □

5. Remarks

We should also mention that the existence of such a bijection is part of folklore and perhaps known to many experts, although this does not seem to have been noted explicitly anywhere. We conjecture that combining the bijection proposed by Gessel and Viennot [2] between permutations and nonintersecting lattice paths with Lalonde's bijection [5] between these paths and descending plane partitions, one can obtain an equivalent description of the bijection proved in this article.

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