Linear maps preserving group majorization

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Abstract

A necessary and sufficient condition for a linear map to preserve group majorizations is given. The condition is applied to prove some preservation results. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

Given a closed subgroup $G$ of the orthogonal group $O(V)$ acting on a linear space $V$, group majorization induced by $G$ (in short, $G$-majorization) is the preordering $\leq_G$ on $V$ defined by

$$y \leq_G x \iff y \in \text{conv } Gx,$$

where conv $Gx$ denotes the convex hull of the orbit $Gx := \{gx : g \in G\}$ (cf. [6]).

Throughout the paper $V$ and $W$ are finite-dimensional real inner product spaces. In the sequel, we assume that $V$ and $W$ are provided with group majorization pre-orderings $\leq_G$ and $\leq_H$, respectively, where $G \subset O(V)$ and $H \subset O(W)$ are closed groups. We say that a map $T : V \to W$ is isotone with respect to $\leq_G$ and $\leq_H$ (in other words, $T$ preserves group majorizations $\leq_G$ and $\leq_H$) if

$$y \leq_G x \Rightarrow Ty \leq_H Tx$$
for all \(x, y \in V\). Many researchers are interested in such maps defined on various spaces of practical interest. Niezgoda and Otachel [11] have characterized differentiable isotone maps w.r.t. cone preorderings. The problem of the isotonicity w.r.t. so-called group induced cone orderings was discussed in [9]. Ando [1] has studied linear maps preserving classical majorization induced by permutation group acting on \(\mathbb{R}^n\). Dean and Verducci [4] have examined maps of this type in the context of applications in probability and statistics. Recently, Beasley and Lee [3] have given sufficient and necessary conditions for a linear map to preserve multivariate majorization on matrix space \(M_{m,n}(\mathbb{R})\).

In the present paper, we develop some ideas of the above-mentioned authors from the group of permutations to a closed group of orthogonal operators. In Section 2, we present an approach to the problem of characterization of isotone linear maps on general linear spaces equipped with group majorizations. Section 3 is devoted to a discussion of our results for the coordinate sign changes group. In Section 4, as an application, we provide unified proofs of some known preservation theorems from [1,3,4] related to subgroups of the permutation group.

2. Isotone linear maps

We begin our discussion with the following result which is based on an idea included in [1, Theorem 2.1].

**Theorem 2.1.** Let \(T\) be a linear map from \(V\) to \(W\). Then the following conditions are equivalent:

(i) \(T\) is isotone w.r.t. \(\preceq_G\) and \(\preceq_H\).

(ii) For any \(x \in V\) and \(g \in G\) there exists \(h \in H\) such that

\[
Tgx = hTx.
\]

**Proof.** (i) \(\Rightarrow\) (ii): If \(T\) is isotone, then for any \(x \in V\) and \(g \in G\) we have \(Tx \preceq_H Tgx\) and \(Tgx \preceq_H Tx\), since \(x \preceq_G gx\) and \(gx \preceq_G x\). Now by virtue of Steerneman [12, Proposition 2.1(iii)], there exists \(h \in H\) such that \(Tgx = hTx\).

(ii) \(\Rightarrow\) (i): Let \(y \preceq_G x\) for some \(x, y \in V\). Then \(y = \sum_{i=1}^{n} \lambda_i g_i x\) for some \(\lambda_i > 0\), \(\sum_{i=1}^{n} \lambda_i = 1\), and \(g_i \in G\). Hence, \(Ty = \sum_{i=1}^{n} \lambda_i Tg_i x = \sum_{i=1}^{n} \lambda_i h_i Tx\), where \(h_i \in H\) is such that \(Tg_i x = h_i Tx\) by (1). Accordingly, \(Ty\) belongs to the convex hull of the orbit \(HTx\), which means \(Ty \preceq_H Tx\). □

Condition (1) says that image of any \(G\)-orbit \(Gx\) under \(T\) is included in \(H\)-orbit of \(Tx\):

\[
TGx \subset HTx.
\]
In general, the map $h \in H$ in (1) depends on $g \in G$ and $x \in V$. However, $h$ does not depend on $x$ for finite or countable groups $H$. In this way we can obtain a simpler condition on the isotonicity of a linear map (cf. [1, Theorem 2.6]).

**Theorem 2.2.** Let $T : V \to W$ be a linear map. Suppose that $H$ is finite or countable. Then the following conditions are equivalent:

(i) $T$ is isotone w.r.t. $\preceq_G$ and $\preceq_H$.

(ii) For any $g \in G$ there exists $h \in H$ such that

$$Tg = hT. \tag{2}$$

**Proof.** Fix any $g \in G$. By Theorem 2.1, it is enough to prove that the following two statements are equivalent:

(a) For any $x \in V$ there exists $h \in H$ such that $Tgx = hTx$.

(b) There exists $h \in H$ such that $Tg = hT$.

The implication (b) $\Rightarrow$ (a) is trivial.

(a) $\Rightarrow$ (b): For $h \in H$, let $V_h$ denote the kernel of the linear map $Tg - hT$ from $V$ to $W$. By (a), we have $V = \bigcup_{h \in H} V_h$. We shall show that there exists $h \in H$ such that $\dim V_h = \dim V$. On the contrary, suppose that $\dim V_h < \dim V$ for all $h \in H$. Then each $V_h$ is a closed boundary set. So $V$ is a set of the first category, since $V = \bigcup_{h \in H} V_h$ and $H$ is finite or countable. This is impossible by Baire’s theorem.

In consequence, $\dim V_h = \dim V$ and $V_h = V$ for some $h \in H$. Hence, $(Tg - hT)x = 0$ for all $x \in V$, which gives $Tg = hT$, as required. \(\square\)

Any linear map $T : V \to W$ can be represented in the form

$$Tx = \langle x, a_1 \rangle e_1 + \cdots + \langle x, a_n \rangle e_n, \quad x \in V, \tag{3}$$

for some vectors $a_1, \ldots, a_n \in V$ depending on $T$, where $\langle \cdot, \cdot \rangle$ is inner product on $V$ and $e_1, \ldots, e_n$ is an orthonormal basis in $W$. For example, if $T : \mathbb{R}^m \to \mathbb{R}^n$, then $a_1^T, \ldots, a_n^T$ can be treated as rows of the matrix $M$ corresponding to $T$ w.r.t. the standard bases.

Thus, the problem of finding the isotone linear maps $T$ leads to finding suitable vectors $a_1, \ldots, a_n \in V$ satisfying (3). In Theorem 2.3, we rewrite condition (2) in terms of the vectors. As can be seen in this theorem, for each $g \in G$ the vector $a := [a_1^T, \ldots, a_n^T]^T \in V \times \cdots \times V = V^n$ should be a solution of the following linear equation: $(I_n \otimes g - h \otimes I_m)a = 0$ with some unknown $h \in H$, where $m = \dim V$ and $n = \dim W$, $\otimes$ is the Kronecker product, and $I_m$ and $I_n$ are the identity maps (matrices) on $V$ and $W$, respectively.

**Theorem 2.3.** Under the assumptions of Theorem 2.2, the following conditions are equivalent:

(i) $T$ is isotone w.r.t. $\preceq_G$ and $\preceq_H$.

(ii) For any $g \in G$ there exists $h \in H$ such that
where $h$ is represented by $n \times n$ matrix $[h_{ij}]$, that is,

$$he_i = h_{1i}a_1 + \cdots + h_{ni}a_n, \quad i = 1, \ldots, n. \quad (5)$$

**Proof.** Applying (3), we see that statement (ii) of Theorem 2.2 can be equivalently presented as:

(iii) For any $g \in G$ there exists $h \in H$ such that for each $x \in V$

$$\langle gx, a_1 \rangle e_1 + \cdots + \langle gx, a_n \rangle e_n = \langle x, a_1 \rangle he_1 + \cdots + \langle x, a_n \rangle he_n.$$

Because $G \subset O(V)$, we have $G^T = G^{-1} = G$, and therefore, by (5), (iii) can be rewritten in the form as follows.

(iv) For any $g \in G$ there exists $h \in H$ such that for each $x \in V$

$$\langle x, ga_1 \rangle e_1 + \cdots + \langle x, ga_n \rangle e_n = \langle x, h_{11}a_1 + \cdots + h_{1n}a_n \rangle e_1 + \cdots + \langle x, h_{n1}a_1 + \cdots + h_{nn}a_n \rangle e_n.$$

It is easily seen that (iv) is equivalent to the following condition.

(v) For any $g \in G$ there exists $h \in H$ such that

$$ga_1 = h_{11}a_1 + \cdots + h_{1n}a_n,$$

$$\vdots$$

$$ga_n = h_{n1}a_1 + \cdots + h_{nn}a_n. \quad (6)$$

Now observe that (6) and (4) are the same, which proves the theorem. \qed

3. Case of diagonal groups

In this section, we discuss the case when $W = \mathbb{R}^n$ and $H$ is a subgroup of the group $C_n$ of all coordinate sign changes. Recall that $h \in C_n$ iff $h$ is the diagonal matrix $\text{diag}(h_{11}, \ldots, h_{nn})$ for some $h_{ii} = \pm 1$, $i = 1, \ldots, n$, on the main diagonal. Then (4) takes the form

$$ga_1 = h_{11}a_1,$$

$$\vdots$$

$$ga_n = h_{nn}a_n. \quad (7)$$
Corollary 3.1. Under the assumptions of Theorem 2.3, let \( H_i := \{ \lambda \in \mathbb{R} : \lambda = h_{ii} \text{ for some } h \in H \}, i = 1, \ldots, n \). If \( H = H_1 \oplus \cdots \oplus H_n \), then the following conditions are equivalent:
(i) \( T \) is isotone w.r.t. \( \preceq_G \) and \( \preceq_H \).
(ii) Each vector \( a_i \) in (3), \( i = 1, \ldots, n \), is a joint eigenvector of all \( g \in G \) with an eigenvalue belonging to \( H_i \) (or \( a_i = 0 \)), i.e.,
\[
a_i \in V_i := \bigcap_{g \in G} \bigcup_{\lambda \in H_i} \{ v \in V : gv = \lambda v \}.
\]

Example 3.1. Let \( V = W = \mathbb{R}^n \) and \( G = H = \mathcal{C}_n \). We shall show that the only isotone (w.r.t. \( \preceq_{\mathcal{C}_n} \) and \( \preceq_{\mathcal{C}_n} \)) linear maps \( T : \mathbb{R}^n \to \mathbb{R}^n \) are represented by \( n \times n \) matrices having at least \( n - 1 \) zeros in each row:
\[
M = \begin{bmatrix}
0 & \cdots & \cdots & c_1 & \cdots & 0 \\
\vdots & & & \vdots & & \vdots \\
0 & \cdots & c_n & \cdots & 0
\end{bmatrix},
\]
where \( c_1, \ldots, c_n \in \mathbb{R} \).

Notice that \( H_i = \{1, -1\}, i = 1, \ldots, n \). On account of Corollary 3.1, it is enough to show that each \( V_i \) consists of all vectors \( \{0, \ldots, 0, c, 0, \ldots, 0\}^T \) containing at least \( n - 1 \) zeros, where \( c \in \mathbb{R} \). In fact, every such a vector is an eigenvector with eigenvalue \( \pm 1 \) for each \( g \in G = \mathcal{C}_n \). On the other hand, let \( v = [v_1, \ldots, v_n]^T \in V_i \). Assume \( v \neq 0 \). Then its \( j \)th entry \( v_j \neq 0 \) for some \( j \in \{1, \ldots, n\} \). Consider \( g := \text{diag}(1, \ldots, -1, \ldots, 1) \in G \) whose \( j \)th diagonal entry is \( -1 \) and remaining ones are \( 1 \). Since \( v_j \neq 0 \), we get \( gv \neq v \). Therefore, \( gv = -v \), and next \( v_1 = \cdots = v_{j-1} = v_{j+1} = \cdots = v_n = 0 \), which completes the proof.

Example 3.2. Put \( V = W = \mathbb{R}^n \) and \( H = \mathcal{C}_n \). Let \( G = \mathcal{P}_n \), the group of all \( n \times n \) permutation matrices, \( n > 2 \). We shall prove that a linear map \( T : \mathbb{R}^n \to \mathbb{R}^n \) is isotone w.r.t. \( \preceq_{\mathcal{P}_n} \) and \( \preceq_{\mathcal{C}_n} \) if and only if its \( n \times n \) matrix has the following form:
\[
M = \begin{bmatrix}
c_1 & \cdots & c_1 \\
\vdots & & \vdots \\
c_n & \cdots & c_n
\end{bmatrix},
\]
where \( c_1, \ldots, c_n \in \mathbb{R} \).

By Corollary 3.1, it is sufficient to show that each element of \( V_i, i = 1, \ldots, n \), has equal all entries, and vice versa. Let \( v = [v_1, \ldots, v_n]^T \in V_i \), where \( H_i = \{1, -1\} \). Suppose that \( v_j \neq v_k \) for some \( j \neq k \), where \( j, k \in \{1, \ldots, n\} \). Take \( g \in G \) to be the permutation matrix interchanging \( v_j \) and \( v_k \) and remaining \( v_l \) in its place for \( l \neq j, k \). Then \( gv \neq v \). Consequently, \( gv = -v \), since \( v \in V_i \). Hence, \( v_i = -v_i \) for \( l \neq j, k \), and \( v_j = -v_k \). Therefore, \( v = [0, \ldots, 0, c, 0, 0, \ldots, 0, -c, 0, 0, \ldots, 0]^T \) for some real number \( c \). If \( c \neq 0 \), then the same reasoning as previously for entries 0 and \( c \) instead of \( v_j \) and \( v_k \) permits us to deduce that \( 0 = -c \), a contradiction. In this way
$v = [0, \ldots, 0]^T$. This contradicts the assumption and proves that $v_1 = \cdots = v_n$ for $v = [v_1, \ldots, v_n]^T \in V_i$, as claimed. Conversely, it is easily seen that each vector having equal all entries must be a member of $V_i$.

**Example 3.3.** Again put $V = W = \mathbb{R}^n$ and $H = \mathcal{C}_n$. For even $n$, let $G$ be the group consisting of all $n \times n$ permutation matrices of type

\[
\begin{bmatrix}
P & 0 \\
0 & \tilde{P}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 \\
\tilde{Q} & 0
\end{bmatrix},
\]

where $P$, $\tilde{P}$, $Q$ and $\tilde{Q}$ are $\frac{n}{2} \times \frac{n}{2}$ permutation matrices. We shall prove that a linear map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is isotone w.r.t. $\preceq_G$ and $\preceq_{\mathcal{C}_n}$ if and only if it has an $n \times n$ matrix

\[
M = \begin{bmatrix}
c_1 & \cdots & c_1 & \pm c_1 & \cdots & \pm c_1 \\
& \ddots & \vdots & \ddots & \vdots & \vdots \\
c_n & \cdots & c_n & \pm c_n & \cdots & \pm c_n
\end{bmatrix},
\]

where $c_1, \ldots, c_n \in \mathbb{R}$, and each row has the same signs in its $\frac{n}{2} \times 1$ right side.

As previously, we are interested in sets $V_i$. It is not hard to check that vectors of the type $[c, \ldots, c, \pm c, \ldots, \pm c]^T$, $c \in \mathbb{R}$, belong to $V_i$. By Corollary 3.1, it remains to show that such vectors are the only elements of $V_i$. Let $[v^T, w^T]^T \in V_i$, where $v$ and $w$ are $\frac{n}{2} \times 1$ vectors. Since $H_i = \{1, -1\}$, we have

\[
\begin{bmatrix}
P & 0 \\
0 & \tilde{P}
\end{bmatrix} \begin{bmatrix}
v \\
w
\end{bmatrix} = \begin{bmatrix}
v \\
w
\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}
P & 0 \\
0 & \tilde{P}
\end{bmatrix} \begin{bmatrix}
v \\
w
\end{bmatrix} = -\begin{bmatrix}
v \\
w
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
0 & Q \\
\tilde{Q} & 0
\end{bmatrix} \begin{bmatrix}
v \\
w
\end{bmatrix} = \begin{bmatrix}
v \\
w
\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}
0 & Q \\
\tilde{Q} & 0
\end{bmatrix} \begin{bmatrix}
v \\
w
\end{bmatrix} = -\begin{bmatrix}
v \\
w
\end{bmatrix}
\]

for all $\frac{n}{2} \times \frac{n}{2}$ permutation matrices $P$, $\tilde{P}$, $Q$ and $\tilde{Q}$. Therefore, in particular,

\[
Qw = v, \quad \tilde{Q}v = w \quad \text{or} \quad Qw = -v, \quad \tilde{Q}v = -w. \tag{9}
\]

From this $Q\tilde{Q}v = v$ and $\tilde{Q}Qw = w$ for all $\frac{n}{2} \times \frac{n}{2}$ permutation matrices $Q$ and $\tilde{Q}$. Hence, $v = [c, \ldots, c]^T$ and $w = [d, \ldots, d]^T$ for some $c, d \in \mathbb{R}$. It follows from (9) that $d = c$ or $d = -c$. In consequence, $[v^T, w^T]^T = [c, \ldots, c, \pm c, \ldots, \pm c]^T$, where the signs on the right-hand side are all the same. This is exactly our assertion.

### 4. Case of permutation groups

We now study subgroups $H$ of the permutation matrices group $\mathcal{S}_n$ acting on $W = \mathbb{R}^n$. Each $h \in H$ is an $n \times n$ matrix $[h_{ij}]$ such that

\[
h_{ij} = \begin{cases}
1 & \text{for } j = \sigma(i), \\
0 & \text{for } j \neq \sigma(i)
\end{cases}
\]
for some $\sigma \in \Omega_n$, in symbol $h = R(\sigma)$. Here $\Omega_n$ stands for the group of all permutations of $\{1, \ldots, n\}$. Thus, there is a subgroup $\Omega_n(H)$ of $\Omega_n$ generated by $H$. By virtue of Theorem 2.3, we obtain:

**Corollary 4.1.** Under the assumptions of Theorem 2.3, let $H \subset P_n$. Then the following two conditions are equivalent:

(i) $T$ is isotone w.r.t. $\preceq_G$ and $\preceq_H$.

(ii) For any $g \in G$ there exists $\sigma \in \Omega_n(H)$ such that

$$ga_1 = a_{\sigma(1)},$$

$$\vdots$$

$$ga_n = a_{\sigma(n)}. \quad (10)$$

In particular, the $G$-orbit of each vector $a_k$ in (3), $k = 1, \ldots, n$, is included in the set $\{a_1, \ldots, a_n\}$, whenever $T$ is isotone w.r.t. $\preceq_G$ and $\preceq_H$.

A preliminary method to determine vectors $a_1, \ldots, a_n$ in $V$ generated by isotone linear maps is based on the last statement of the above result. Namely, if $a \in V$ is a member of the set $\{a_1, \ldots, a_n\}$, then the whole $G$-orbit of $a$ should be contained in this set. In addition, $\{a_1, \ldots, a_n\}$ should be a union of orbits of this type. More precisely, if each member of $H$ is of the type $h = I_r \otimes P$ for some $\mu \times \mu$ permutation matrix $P$ and some positive integers $r, \mu$ such that $r\mu = n$, then $\Omega_n(H) = \{\sigma = \tau \oplus \cdots \oplus \tau : \tau \in \Omega_\mu\}$, where the symbol $\tau \oplus \cdots \oplus \tau$ denotes the permutation in $\Omega_n$ satisfying $(\tau \oplus \cdots \oplus \tau)(j\mu + i) = j\mu + \tau(i)$ for $j = 0, \ldots, r - 1, i = 1, \ldots, \mu$.

Therefore, by (10), for each $g \in G$ there should exist $\tau \in \Omega_\mu$ such that

$$ga_{j\mu + i} = a_{j\mu + \tau(i)}, \quad j = 0, \ldots, r - 1, \ i = 1, \ldots, \mu. \quad (11)$$

Hence,

$$Ga_{j\mu + i} \subset \{a_{j\mu + 1}, \ldots, a_{j\mu + \mu}\}, \quad j = 0, \ldots, r - 1, \ i = 1, \ldots, \mu, \quad (12)$$

so the cardinality of the orbit $Ga_k$, $k = 1, \ldots, n$, should be not greater than $\mu$ (see Example 4.2). In particular, if $r = 1$, then $\mu = n = \dim W$ (see Example 4.1). So, in forthcoming examples, we shall be interested in vectors $a \in V$ satisfying the following necessary condition:

$$\text{card } G: \text{card } S(a) \leq \mu, \quad (13)$$

where $S(a) := \{g \in G : ga = a \text{ for all } g \in G\}$ is the stabilizer of $a$, because $\text{card } Ga = \text{card } (G/S(a)) = \text{card } G: \text{card } S(a)$ (for finite $G$), where $G/S(a) := \{gS(a) \in G : g \in G\}$ is the quotient set.

**Example 4.1.** In this example, our goal is to derive some results of [4] and [1] from Corollary 4.1. Let $V = \mathbb{R}^m$ and $W = \mathbb{R}^n$. Put $G = P_m$ and $H = P_n$. It is well known that the group majorization $\preceq_{P_m}$ is the classical majorization on $\mathbb{R}^m$ (see [8,
p. 23] and [5, pp. 3 and 4]). (For $\preceq_{\pi_n}$ analogously.) Then Theorem 2.2 reduces to Theorem 3 of [4] and, by the above discussion, Corollary 4.1 gives Theorem 1 of [4]. Namely, a linear map $T : \mathbb{R}^m \to \mathbb{R}^n$ is isotone iff its $n \times m$ matrix $M$ can be presented in the form

$$M = R(\sigma) \begin{bmatrix} M_1 \\ \vdots \\ M_p \end{bmatrix}$$

(14)

for some $\sigma \in \Omega_n$, where each matrix $M_i$ consists of rows being full permutation orbit of its first row.

For instance, if $m = n$, then rows of $M_i$, $i = 1, \ldots, p$, are

$$a^T = [c, \ldots, c] \text{ or } a^T = [c, \ldots, c, d, \ldots, c]$$

(15)

for some $c, d \in \mathbb{R}, \ c \neq d$. Indeed, in the first case $S(a) = G$ and card $(G/S(a)) = 1 \leq n = \mu$, and in the second card $(G/S(a)) = n!$: $(n - 1)! = n = \mu$. Other situations give card $(G/S(a)) > n$.

Thus, by (14), in the first case $p = n$ and $M_i = [c_i, \ldots, c_i]$ for some $c_i \in \mathbb{R}, \ i = 1, \ldots, n$. Then

$$M = \begin{bmatrix} c_1 & \cdots & c_1 \\ \vdots & \ddots & \vdots \\ c_n & \cdots & c_n \end{bmatrix}$$

In other words,

$$Tx = \left( \sum_{i=1}^{n} x_i \right) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \quad x \in \mathbb{R}^n.$$  
(16)

And the second case, by (14), leads to $p = 1$ and

$$M = R(\sigma)M_1 = R(\sigma) \begin{bmatrix} d & c & \cdots & c \\ c & d & \cdots & c \\ \vdots & \vdots & \ddots & \vdots \\ c & c & \cdots & d \end{bmatrix}$$

for some $\sigma \in \Omega_n$ and $c, d \in \mathbb{R}, \ c \neq d$. So

$$Tx = (d - c)R(\sigma)x + cJ_nx, \quad x \in \mathbb{R}^n,$$

(17)

where $J_n$ is $n \times n$ matrix of all ones.

Summarizing, each isotone linear map $T : \mathbb{R}^m \to \mathbb{R}^n$ has form (16) or (17). This is in accordance with Ando’s result [1, Corollary 2.7] (cf. also [3, Lemma 2.6] and [4, Corollary 1.2]).

Example 4.2. Beasley and Lee [3] characterized the isotonicity w.r.t. multivariate majorizations. In this example, we show how to obtain their main result from Corollary 4.1.
Let \( \mathbb{M}_{n,m}(\mathbb{R}) \) be the set of all \( n \times m \) real matrices, \( n > 2 \). For \( Z \in \mathbb{M}_{n,m}(\mathbb{R}) \) we say that \( Y \) is \((column)\) multivariate majorized by \( X \) if \( Y = DX \) for some \( n \times n \) doubly stochastic matrix \( D \) (cf. [3]). Since \( D \) is a convex combination of some \( n \times n \) permutation matrices (see [8, p. 19]), the relation of multivariate majorization can be treated as a group majorization ordering. Namely, identifying \( \mathbb{M}_{n,m}(\mathbb{R}) \) and \( \mathbb{R}^{nm} \) by \( Z = [Z_{\bullet 1}, \ldots, Z_{\bullet m}] \leftrightarrow \begin{bmatrix} Z_{\bullet 1} \\ \vdots \\ Z_{\bullet m} \end{bmatrix} \), we can consider the group \( G \) of all \( nm \times nm \) permutation matrices of the form \( I_m \otimes P \), where \( P \) is an \( n \times n \) permutation matrix. Then a simple calculation shows that the multivariate majorization on \( \mathbb{M}_{n,m}(\mathbb{R}) \) can be treated as the \( G \)-majorization on \( \mathbb{R}^{nm} \).

We shall investigate isotone linear maps \( T : V \to W \), where \( V = W = \mathbb{M}_{n,m}(\mathbb{R}) \) and \( G = H \). According to Corollary 4.1, at first we shall look for matrices \( A \in \mathbb{M}_{n,m}(\mathbb{R}) \) satisfying necessary condition of type (13). Here, \( G = n! \) and \( \mu = n \). So we want to have \( A \) such that card \( S(A) \geq \frac{n!}{\mu} = (n - 1)! \).

Consider an \( n \times m \) matrix

\[
A = \begin{bmatrix} c_1 & \cdots & c_m \\ \vdots & & \vdots \\ c_1 & \cdots & c_m \\ \vdots & & \vdots \\ c_1 & \cdots & c_m \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} c_1 & \cdots & c_m \\ \vdots & & \vdots \\ d_1 & \cdots & d_m \\ \vdots & & \vdots \\ c_1 & \cdots & c_m \end{bmatrix}
\]

(18)

In the former case the stabilizer \( S(A) \) of \( A \) is equal to \( G \) and card \( S(A) = n! \geq (n - 1)! \). And the latter gives card \( S(A) = (n - 1)! \), whenever \( d_i \neq c_i \) for some \( i = 1, \ldots, m \). This means that (13) is fulfilled for such type of matrices. Moreover, these are the unique members of \( \mathbb{M}_{n,m}(\mathbb{R}) \) with property (13). Indeed, if card \( S(A) \geq (n - 1)! \), then each column of \( A \) should be of the form \([c_i, \ldots, c_i, d_i, c_i, \ldots, c_i]^T \) (cf. Example 4.1, (15)). In addition, all elements \( d_i, i = 1, \ldots, m \), should be placed in the same row, because otherwise card \( S(A) < (n - 1)! \).

In the sequel, to simplify notation, we use the same symbol \( \langle \cdot, \cdot \rangle \) for the standard inner product on both \( \mathbb{M}_{n,m}(\mathbb{R}) \) and \( \mathbb{R}^n \). Let \( E_{ij} \) be the \( n \times m \) matrix whose \((i, j)\) entry is 1 and all others are 0, and let \( e \) be the \( n \times 1 \) vector of all ones. A linear map \( T : \mathbb{M}_{n,m}(\mathbb{R}) \to \mathbb{M}_{n,m}(\mathbb{R}) \) can be written as follows:

\[
TX = \langle X, A_{11} \rangle E_{11} + \cdots + \langle X, A_{1m} \rangle E_{1m} + \cdots + \langle X, A_{n1} \rangle E_{n1} + \cdots + \langle X, A_{nm} \rangle E_{nm}
\]

(19)
for some $A_{11}, \ldots, A_{nm} \in \mathbb{M}_{n,m}(\mathbb{R})$. By the above considerations, if $T$ is isotone, then $A_{kl}$ are matrices of form (18). Our purpose now is to assign to each $A_{kl}$ such a matrix so that Corollary 4.1 holds.

**Case (I)** Let $A_{11}, \ldots, A_{nm} \in \mathbb{M}_{n,m}(\mathbb{R})$ be defined by

$$A_{kl} := \begin{bmatrix} c_{1k}^{l} & \cdots & c_{mk}^{l} \\ \vdots & & \vdots \\ c_{1k}^{l} & \cdots & c_{mk}^{l} \end{bmatrix}, \quad k = 1, \ldots, n, \quad l = 1, \ldots, m,$$

(20)

for arbitrary real numbers $c_{ik}^{l}, i = 1, \ldots, m$. We shall derive a matrix form of isotone $T$ corresponding to (19) and (20).

There exist matrices $K_{l}^{i} = [K_{ki}^{l}] \in \mathbb{M}_{n,m}(\mathbb{R})$ and $L_{l} = [L_{il}] \in \mathbb{M}_{m,m}(0, 1)$ satisfying

$$A_{kl} = \begin{bmatrix} K_{k1}^{l}L_{1l} & \cdots & K_{km}^{l}L_{ml} \\ \vdots & & \vdots \\ K_{k1}^{l}L_{1l} & \cdots & K_{km}^{l}L_{ml} \end{bmatrix}.$$  

Then a bit of algebra gives

$$\langle X, A_{kl} \rangle = \left( \begin{bmatrix} X_{\bullet1} & \cdots & X_{\bullet m} \end{bmatrix}, \begin{bmatrix} K_{k1}^{l}L_{1l} & \cdots & K_{km}^{l}L_{ml} \\ \vdots & & \vdots \\ K_{k1}^{l}L_{1l} & \cdots & K_{km}^{l}L_{ml} \end{bmatrix} \right)$$

$$= \langle X_{\bullet1}, K_{k1}^{l}L_{1l}e \rangle + \cdots + \langle X_{\bullet m}, K_{km}^{l}L_{ml}e \rangle$$

$$= \langle X_{\bullet1}, e \rangle K_{k1}^{l}L_{1l} + \cdots + \langle X_{\bullet m}, e \rangle K_{km}^{l}L_{ml}$$

$$= K_{k\bullet}^{l}\left( \begin{bmatrix} \langle X_{\bullet1}, e \rangle \\ \vdots \\ \langle X_{\bullet m}, e \rangle \end{bmatrix} \circ L_{\bullet l} \right),$$

where $\circ$ is the Hadamard product on $\mathbb{R}^{m}$. Denoting

$$d(X) := \begin{bmatrix} \langle X_{\bullet1}, e \rangle \\ \vdots \\ \langle X_{\bullet m}, e \rangle \end{bmatrix} \quad \text{and} \quad D(X) := \begin{bmatrix} \langle X_{\bullet1}, e \rangle & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \langle X_{\bullet m}, e \rangle \end{bmatrix},$$

we obtain from (19)

$$TX = \begin{bmatrix} K_{1\bullet}^{1}(d(X) \circ L_{\bullet1}) & \cdots & K_{m\bullet}^{m}(d(X) \circ L_{\bullet m}) \\ \vdots & & \vdots \\ K_{n\bullet}^{1}(d(X) \circ L_{\bullet1}) & \cdots & K_{n\bullet}^{m}(d(X) \circ L_{\bullet m}) \end{bmatrix}.$$
\[
= \sum_{l=1}^{m} \begin{bmatrix}
  0 & \cdots & 0 & K_{1l}^{\bullet}(d(X) \circ L_{\bullet}) & 0 & \cdots & 0 \\
  \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & 0 & K_{nl}^{\bullet}(d(X) \circ L_{\bullet}) & 0 & \cdots & 0
\end{bmatrix}
\]

\[
= \sum_{l=1}^{m} \begin{bmatrix}
  K_{1l}^{\bullet} \\
  \vdots \\
  K_{nl}^{\bullet}
\end{bmatrix} \begin{bmatrix}
  0 & \cdots & 0 & d(X) \circ L_{\bullet} & 0 & \cdots & 0 \\
  \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & 0 & d(X) \circ L_{\bullet} & 0 & \cdots & 0
\end{bmatrix}
\]

\[
= \sum_{l=1}^{m} K_{l[D(X]} \begin{bmatrix}
  0 & \cdots & 0 & L_{\bullet} & 0 & \cdots & 0 \\
  \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & 0 & L_{\bullet} & 0 & \cdots & 0
\end{bmatrix}.
\]

In this way, we get

\[
TX = \sum_{l=1}^{m} K_{l[D(X)}LD_{l}, \quad X \in \mathbb{M}_{n,m}(\mathbb{R}),
\]

for some matrices \( K_{l} \in \mathbb{M}_{n,m}(\mathbb{R}) \) and \( L \in \mathbb{M}_{m,m}(0, 1) \), where \( D_{l} \) is the \( m \times m \) matrix whose \((l, l)\) entry is 1 and the remaining ones are 0.

**Case (II)** Similarly, as in Case (I), our interest is to find a matrix form of isotone \( T \) using (19) and the second type matrices in (18), that is, matrices

\[
A_{kl} := \begin{bmatrix}
  c_{1l} & \cdots & c_{ml} \\
  \vdots & \ddots & \vdots \\
  d_{1l} & \cdots & d_{ml} \\
  \vdots & \ddots & \vdots \\
  c_{1l} & \cdots & c_{ml}
\end{bmatrix} \quad \text{(} \leftarrow \text{ikt}\text{h row}),
\]

\[
k = 1, \ldots, n, \quad l = 1, \ldots, m,
\]

for arbitrary real numbers \( c_{il} \) and \( d_{il} \), \( i = 1, \ldots, m \), and for \( i_{kl} \in \{1, \ldots, n\} \). We suppose that at least one matrix \( A_{kl} \) has its row of \( d_{il} \)'s different from row of \( c_{il} \)'s, \( i = 1, \ldots, m \). (Otherwise Case (II) is a particular part of Case (I).) It follows from (10)–(13) (with \( n \) and \( r \) replaced by \( nm \) and \( m \), respectively) that the rows of \( d_{il} \)'s of \( A_{kl} \) for \( k = 1, \ldots, n \) and fixed \( l \) are the same but they are placed in different rows. So there exists permutation \( \pi_{l} \in \Omega_{n} \) such that \( i_{kl} = \pi_{l}(k) \).
We shall employ (10) to conclude that the rows of \( d_{il}' \)'s of \( A_{kl} \) for \( l = 1, \ldots, m \) and fixed \( k \) must be placed in rows with the same numeration. Namely, the action of \( g = I_n \otimes P \in G \) on \( A_{kl} \) is the following:

\[
gA_{kl} = A_{\zeta_l(k)l}, \tag{23}
\]

where \( P = R(\varrho) \in \mathcal{P}_n, \varrho \in \Omega_n \) and \( \zeta_l := \pi_l^{-1} \varrho^{-1} \pi_l \in \Omega_n \). In fact,

\[
gA_{kl} = [Pa_{1l}, \ldots, Pa_{ml}] = \begin{bmatrix} c_{1l} & \cdots & c_{ml} \\ \vdots & \ddots & \vdots \\ d_{1l} & \cdots & d_{ml} \\ \vdots & \ddots & \vdots \\ c_{1l} & \cdots & c_{ml} \end{bmatrix} \quad \text{(} \leftarrow \varrho^{-1}(ik) \text{th row}),
\]

where \( a_{il} \) is the \( i \)th column of \( A_{kl} \), \( i = 1, \ldots, m \). However, \( \varrho^{-1}(ik) = \varrho^{-1} \pi_l(k) = \pi_l \zeta_l(k) = i \zeta_l(k) \). So (23) holds by the above and (22).

We shall now prove that permutations \( \pi_l \in \Omega_n \) are the same for all \( l \) belonging to the set \( L := \{ l \in \{1, \ldots, m\} : d_{il} \neq c_{il} \text{ for some } i \in \{1, \ldots, n\} \} \). Since \( T \) is isotone, by virtue of Corollary 4.1, for each \( g = I_m \otimes P \), where \( P = R(\varrho) \) and \( \varrho \in \Omega_n \), there exists \( \sigma \in \Omega_{mn}(H) \) satisfying (10). In other words, there exists \( \tau \in \Omega_n \) such that \( \sigma = \tau \oplus \cdots \oplus \tau \) and \( gA_{il} = A_{\tau(1)l}, \ldots, gA_{nl} = A_{\tau(n)l} \) for all \( l = 1, \ldots, m \). (See the discussion after Corollary 4.1 with \( n \) and \( r \) replaced by \( nm \) and \( m \), respectively.) By (23), we get \( A_{\zeta_l(1)l} = A_{\tau(1)l}, \ldots, A_{\zeta_l(n)l} = A_{\tau(n)l} \). Since matrices \( A_{il}, \ldots, A_{nl} \) are mutually different for \( l \in \mathcal{L} \), we derive \( \zeta_l(1) = \tau(1), \ldots, \zeta_l(n) = \tau(n) \). Hence, \( \zeta_l = \tau \) for all \( l \in \mathcal{L} \). In consequence, \( \zeta_l = \zeta_p \) for all \( l, p \in \mathcal{L} \). Therefore, \( \pi_l^{-1} \varrho^{-1} \pi_l = \pi_p^{-1} \varrho^{-1} \pi_p \), and next \( (\pi_p \pi_l^{-1}) \varrho^{-1} = \varrho^{-1} (\pi_p \pi_l^{-1}) \) for each \( \varrho \in \Omega_n \). This means that \( \pi_p \pi_l^{-1} \) commutes with whole group \( \Omega_n \), and gives \( \pi_p \pi_l^{-1} = id \), because \( n > 2 \). Thus, \( \pi_l = \pi_p \) for all \( l, p \in \mathcal{L} \), which is our claim.

For this reason, (10) guarantees via (22) that

\[
A_{kl} = \begin{bmatrix} c_{1l} & \cdots & c_{ml} \\ \vdots & \ddots & \vdots \\ d_{1l} & \cdots & d_{ml} \\ \vdots & \ddots & \vdots \\ c_{1l} & \cdots & c_{ml} \end{bmatrix} \quad \text{(} \leftarrow \pi_l \text{th row}),
\]

\[
k = 1, \ldots, n, \quad l = 1, \ldots, m,
\]

where \( i_k = \pi(k) \) and a permutation \( \pi \in \Omega_n \) is independent of \( l \).

Conversely, the form of \( A_{kl} \) in (24) implies easily that (10) is met.

We now are going to present \( T \) in terms of \( A_{kl} \) from (24). For \( i, l = 1, \ldots, m \) and \( k = 1, \ldots, n \), put \( b_{il} := d_{il} - c_{il} \) and
\[ B_{kl} := \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \quad \text{(← } i_k \text{th row)} \quad \text{and} \quad C_l := \begin{bmatrix} c_{1l} & \cdots & c_{ml} \\ \vdots & \ddots & \vdots \\ c_{1l} & \cdots & c_{ml} \end{bmatrix}. \]

Then by (19), we obtain
\[ TX = T_1X + T_2X, \quad (25) \]
where
\[ T_1X = (X, B_{11})E_{11} + \cdots + (X, B_{1m})E_{1m} + \cdots + (X, B_{n1})E_{n1} + \cdots + (X, B_{nm})E_{nm} \]
and
\[ T_2X = (X, C_{11})E_{11} + \cdots + (X, C_{m1})E_{1m} + \cdots + (X, C_{1n})E_{n1} + \cdots + (X, C_{nm})E_{nm}. \]

Taking \( B := [b_{ij}] \), we get
\[ T_1X = X_{i_1} \cdot B_{11}E_{11} + \cdots + X_{i_1} \cdot B_{1m}E_{1m} + \cdots + X_{i_n} \cdot B_{n1}E_{n1} + \cdots + X_{i_n} \cdot B_{nm}E_{nm}. \]

Therefore,
\[ T_1X = \begin{bmatrix} X_{i_1} \cdot B_{11} & \cdots & X_{i_1} \cdot B_{1m} \\ \vdots & \ddots & \vdots \\ X_{i_n} \cdot B_{n1} & \cdots & X_{i_n} \cdot B_{nm} \end{bmatrix} \]
\[ = \begin{bmatrix} x_{i_11} & \cdots & x_{i_1m} \\ \vdots & \ddots & \vdots \\ x_{i_n1} & \cdots & x_{i_nm} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mm} \end{bmatrix} \]
\[ = \begin{bmatrix} 0 & \cdots & 1 & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mm} \end{bmatrix} \]
\[ = PXB, \]
where $P = R(\pi)$ is the permutation matrix induced by $\pi$.

On the other hand,

$$T_2X = \left(\sum_{i=1}^{m} \langle X_{\bullet_i}, e \rangle c_i\right) E_{11} + \cdots + \left(\sum_{i=1}^{m} \langle X_{\bullet_i}, e \rangle c_{im}\right) E_{1m}$$

$$+ \cdots + \left(\sum_{i=1}^{m} \langle X_{\bullet_i}, e \rangle c_{i1}\right) E_{n1} + \cdots + \left(\sum_{i=1}^{m} \langle X_{\bullet_i}, e \rangle c_{im}\right) E_{nm},$$

and next

$$T_2X = \begin{bmatrix}
\sum_{i=1}^{m} \langle X_{\bullet_i}, e \rangle c_{i1} & \cdots & \sum_{i=1}^{m} \langle X_{\bullet_i}, e \rangle c_{im} \\
\vdots & \ddots & \vdots \\
\sum_{i=1}^{m} \langle X_{\bullet_i}, e \rangle c_{11} & \cdots & \sum_{i=1}^{m} \langle X_{\bullet_i}, e \rangle c_{1m}
\end{bmatrix}$$

$$= \begin{bmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1
\end{bmatrix} \begin{bmatrix}
\langle X_{\bullet 1}, e \rangle c_{11} & \cdots & \langle X_{\bullet 1}, e \rangle c_{1m} \\
\vdots & \ddots & \vdots \\
\langle X_{\bullet m}, e \rangle c_{11} & \cdots & \langle X_{\bullet m}, e \rangle c_{mm}
\end{bmatrix}$$

$$= \begin{bmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1
\end{bmatrix} \begin{bmatrix}
\langle X_{\bullet 1}, e \rangle & 0 \\
\vdots & \ddots & \vdots \\
0 & \langle X_{\bullet m}, e \rangle
\end{bmatrix} \begin{bmatrix}
c_{11} & \cdots & c_{1m} \\
\vdots & \ddots & \vdots \\
c_{m1} & \cdots & c_{mm}
\end{bmatrix}$$

$$= J_{n,m} D(X) C,$$

where $J_{n,m}$ is the $n \times m$ matrix of all ones. In consequence, by (25),

$$TX = PXB + J_{n,m} D(X) C, \quad X \in \mathbb{M}_{n,m}(\mathbb{R}),$$

(26)

for some matrices $B, C \in \mathbb{M}_{m,m}(\mathbb{R})$ and permutation matrix $P \in \mathcal{P}_n$.

Thus, we have proved that a linear map $T : \mathbb{M}_{n,m}(\mathbb{R}) \to \mathbb{M}_{n,m}(\mathbb{R})$ preserves (column) multivariate majorization on $\mathbb{M}_{n,m}(\mathbb{R})$ if and only if it has form either (21) or (26). To obtain analogous results of Beasley and Lee [3, Theorem 2.5] on (row) multivariate majorization, it is enough to note that if a linear map $L : \mathbb{M}_{m,n}(\mathbb{R}) \to \mathbb{M}_{m,n}(\mathbb{R})$ preserves (row) multivariate majorization on $\mathbb{M}_{m,n}(\mathbb{R})$, then the linear map $T : \mathbb{M}_{n,m}(\mathbb{R}) \to \mathbb{M}_{n,m}(\mathbb{R})$ defined by

$$TX = (LX^T)^T, \quad X \in \mathbb{M}_{n,m}(\mathbb{R}),$$

preserves (column) multivariate majorization on $\mathbb{M}_{n,m}(\mathbb{R})$.

References