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THE CATEGORY OF URYSOHN SPACES IS NOT COWELLPOWERED

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It is shown that the category of Urysohn spaces and continuous maps is not cowellpowered. To this end we will construct for each ordinal number β a Urysohn space Y_β with card $(Y_\beta) = \aleph_0 \cdot \text{card}\,(\beta)$ and a continuous map $e_\beta : \mathbb{Q} \to Y_\beta$ from the rationals into Y_β. It turns out that e_β is an extremal monomorphism in the category of Hausdorff spaces and an epimorphism in the category of Urysohn spaces.

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epimorphism	Urysohn space
cowellpowered	rationals

Introduction

Kannan and Rajagopalan [4], [5], Trnková [8] and Koubek [6] constructed a proper class of Hausdorff spaces with peculiar properties. Herrlich [3] used this class to define an unpleasant epireflective full subcategory $\mathcal{B}$ of the category of topological spaces and continuous maps. In $\mathcal{B}$ there are not enough morphisms to detect non-epimorphisms, hence it is not cowellpowered. Of course, it is not surprising that a strange category has strange properties. Unfortunately the situation is even worse. We will show that the category of Urysohn spaces and continuous maps is not cowellpowered. A topological space $(X, \mathcal{X})$ is called Urysohn space [9], if distinct points in X can be separated by disjoint closed neighbourhoods.

1. Notation. A category $\mathcal{C}$ is called *cowellpowered* if for every object X in $\mathcal{C}$ there is a set $\{e_i | e_i : X \to X_i \text{ epimorphism}\}$ such that for every epimorphism $e : X \to Y$ there is an epimorphism e_i and an isomorphism $h_e : X_i \to Y$ fulfilling $h_e \circ e_i = e$.

A monomorphism f in a category $\mathcal{C}$ is called *extremal monomorphism* if in every factorization $f = g \circ h$, where h is an epimorphism, h has to be an isomorphism.

2. Notation. Let **Haus** and **Ury** denote the full category of Hausdorff and Urysohn spaces, respectively. Let $\mathbb{R}$ and $\mathbb{Q}$ denote the reals and rationals, respectively. $[0, 1)$ denotes the halfclosed unit interval, $U(r, \varepsilon)$ denotes an open ε-neighbourhood of real numbers with center r; $r, \varepsilon \in \mathbb{R}$, $\varepsilon > 0$.

3. Definition. [7] Let $(X, \mathscr{X})$ be a topological space. If $A \subset X$, define $\hat{A} = \bigcap\{\mathrm{cl}(o) | o \in \mathscr{X}, A \subset o\}$, i.e. $\hat{A}$ is the set of cluster points of the neighbourhood filter of A.

4. Lemma. [7] *Let X, Y be Urysohn spaces. A morphism $f: X \to Y$ is epimorphism in* **Ury** *if and only if $f[X] \subset B \subset Y$ and $\hat{B} = B$ implies $B = Y$.*

5. Remark. The operator $\hat{}$ is not idempotent. Let $f: X \to Y$ be epimorphic in ***Ury***. Starting with $f[X]$, we can reach by transfinite application of $\hat{}$ the whole space Y. We see that epimorphisms in ***Ury*** have a dynamic property in contrast to ***Haus***, where epimorphisms are static.

This dynamic property is useful in the following construction. Because there are only a limited number of topologies on X, the only possible way to show that ***Ury*** is not cowellpowered is to find a space with no cardinality restriction on its epimorphic images.

6. Theorem. *for every ordinal number β there is a Urysohn space $(Y_\beta, \mathscr{Y}_\beta)$ with* $\mathrm{card}(Y_\beta) = \aleph_0 \cdot \mathrm{card}(\beta)$ *and an epimorphism $e_{\mathbb{Q}}: \mathbb{Q} \to Y_\beta$ in* ***Ury***. *In addition e_α is an extremal monomorphism in* ***Haus***.

Proof. We will first define the underlying set Y_β of the space $(Y_\beta, \mathscr{Y}_\beta)$. Let

$$X_0 = \mathbb{Q} \times \{0\} \times \{1\}, \text{ and}$$

$$X_\alpha = \mathbb{Q} \times (\mathbb{Q} \cap [0, 1)) \times \{\alpha\} \text{ for } \alpha > 0.$$

Then we define

$$Y_1 = X_0 \text{ and}$$

$$Y_\beta = \bigcup \{X_\alpha : \alpha < \beta\} \text{ for } \beta > 1.$$

Note that every element of Y_β can be written in the form (r, s, α) where $r, s \in Q$, $0 \leq s < 1$, and α is an ordinal with $\alpha < \beta$.

In order to define the topology on the set Y_β, we first define for any $\alpha > 0$ and any $r, s \in \mathbb{Q}$ a subset of $\mathbb{Q} \times \mathbb{Q} \times \{\alpha\}$ for every $0 < \varepsilon < 1$ by

$$K(r, s, \alpha, \varepsilon) = \{(u, v, \alpha) | v > 0 \quad \text{and} \quad d\left((u, v), \left(r - \frac{s}{\sqrt{3}}, 0\right)\right) < \varepsilon\}.$$

where d denotes the usual Euclidean distance. In some sense this is one half of Bing's triangular construction in [1], [2, p. 433] (see Fig. 1). It is absolutely important that $K(r, s, \alpha, \varepsilon)$ does not contain its bottom edge.

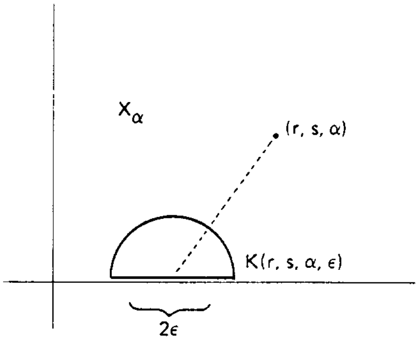

Fig. 1.

We now define the topology on Y_β by defining for each $p \in Y$ a neighborhood base $\mathcal{U}(p)$ for p by transfinite induction. For $p \in Y_1$, we take $\mathcal{U}(p)$ so that Y_1 is naturally homeomorphic to $\mathbb{Q}$. For $p \in Y_\alpha$ $(\alpha \geqslant 2)$ we proceed as follows:

(1) If $p = (r, 0, 1) \in X_1 \subset Y_\beta$, then

$$\mathcal{U}(p) = \{K(r, 0, 1, \varepsilon) \cup ((\mathbb{Q} \cap U(r, \varepsilon)) \times \{0\} \times \{1\}) | \varepsilon > 0\}.$$

If $p = (r, s, 1) \in X_1 \subset Y_\beta$, $s \neq 0$, then

$$\mathcal{U}(p) = \{K(r, s, 1, \varepsilon) \cup \{(r, s, 1)\} | \varepsilon > 0\}.$$

Y_1 is understood to carry the subspace topology inherited by Y_2.

(2) If $p = (r, 0, 2) \in X_2 \subset Y_\beta$, then

$$\mathcal{U}(p) = \{K(r, 1, 1, \varepsilon) \cup K(r, 0, 2, \varepsilon) \cup \{(r, 0, 2)\} | \varepsilon > 0\}.$$

If $p = (r, s, 2) \in X_2 \subset Y_\beta$, $s \neq 0$, then

$$\mathcal{U}(p) = \{K(r, s, 2, \varepsilon) \cup \{(r, s, 2)\} | \varepsilon > 0\}.$$

(3) Assume neighbourhood bases are defined for all points $p \in X_\alpha$, where $\alpha < \gamma$ and $\gamma < \beta$.

(a) If γ is a nonlimit ordinal: $\gamma = \gamma' + 1$, then for $p = (r, 0, \gamma) \in X_\gamma \subset Y_\beta$

$$\mathcal{U}(p) = \{K(r, 0, \gamma, \varepsilon) \cup K(r, 1, \gamma', \varepsilon) \cup \{(r, 0, \gamma)\} | \varepsilon > 0\},$$

If $p = (r, s, \gamma) \in X_\gamma \subset Y_\beta$, $s \neq 0$, then

$$\mathcal{U}(p) = \{K(r, s, \gamma, \varepsilon) \cup \{(r, s, \gamma)\} | \varepsilon > 0\},$$

(b) If γ is a limit ordinal, $p = (r, 0, \gamma) \in X_\gamma \subset Y_\beta$ then

$$\mathcal{U}(p) = \left\{ \bigcup \{K(r, 0, \delta, \varepsilon) | \tau \leqslant \delta \leqslant \gamma\} \cup \{(r, 0, \gamma)\} | 0 \leqslant \tau < \gamma \wedge \varepsilon > 0 \right\}.$$

If $p=(r,s,\gamma)\in X_\gamma\subset Y_\beta$, $s\neq 0$ then

$$\mathcal{U}(p)=\{K(r,s,\gamma,\varepsilon)\cup\{(r,s,\gamma)\}|\varepsilon>0\},$$

$\mathcal{U}(p)$ is a neighbourhood base for every $p\in Y_\beta$ (see Fig. 2): Let $U\in\mathcal{U}(p)$. U depends on $\varepsilon>0$. Choose $\varepsilon'<(\sqrt{3}/2)\cdot\varepsilon$, $\varepsilon'>0$. Replace all appearing $K(r,s,\alpha,\varepsilon)$ in U by $K(r,s,\alpha,\varepsilon')$. Denote the neighbourhood obtained in this way by V.

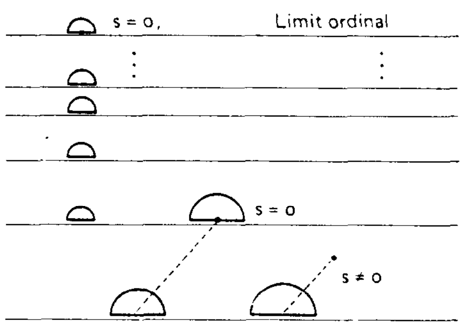

Fig. 2.

Of course $V\in\mathcal{U}(p)$. Further, for every $y\in V$ the set U is in the filter generated by $\mathcal{U}(y)$. The verification of the remaining axioms is left to the reader.

Let $\mathcal{Y}_\beta$ denote the topology defined by $\{\mathcal{U}(p)|p\in Y_\beta\}$. $(Y_\beta,\mathcal{Y}_\beta)$ is a Urysohn space: Let $p,q\in Y_\beta$, $p\neq q$, $p=(r,s,\alpha)$, $q=(r',s',\alpha')$, $\alpha\leqslant\alpha'$.

(1) $(r,s)\neq(r',s')$. Choose ε,ε' such that $\mathrm{cl}\,(U(r-s/\sqrt{3},\varepsilon))\cap\mathrm{cl}\,(U(r'-s'/\sqrt{3},\varepsilon'))=\varnothing$ in $\mathbb{R}$.

(2) $(r,s)=(r',s')$, hence $\alpha<\alpha'$. Assume $s=0$, the case $s\neq 0$ is always trivial.

(a) If neither α nor α' are limit ordinals choose arbitrary neighbourhoods with $\varepsilon,\varepsilon'<1/2\sqrt{3}$.

(b) α' is limit ordinal, α arbitrary. Choose a neighbourhood of q starting sufficiently high.

(c) α' is a nonlimit ordinal, α limit ordinal: proceed as in (a).

By construction, $\mathbb{Q}$ is isomorphic to the closed subspace Y_1 of Y_β, hence the injection $e_\beta:\mathbb{Q}\to Y_\beta$ onto Y_1 is an extremal monomorphism in ***Haus***. e_β is epimorphism:

We will proceed by transfinite induction. Assume there is $A\subset Y_\beta$ such that $\mathbb{Q}\cong Y_1\subset A$, $\hat{A}=A$ and $A\neq Y_\beta$. Let α be the smallest ordinal number such that there exists $(r,s,\alpha)\in Y_\beta$ fulfilling $(r,s,\alpha)\notin A$. It follows $Y_1\subset Y_\alpha\subset A$ and $\{(r,0,\alpha)/r\in\mathbb{Q}\}\cup Y_\alpha\subset\mathrm{cl}\,(Y_\alpha)\subset\mathrm{cl}\,(A)=A$, not depending on the type of α.

The closure of every open set containing $\{(r,0,\alpha)|r\in\mathbb{Q}\}\cup Y_\alpha$ also contains X_α, a contradiction. □

7. Corollary. *The category Ury is not cowellpowered.*

Proof. By Theorem 6, there is no representative set of epimorphisms in **Ury** for the rationals. □

8. Remark. (a) Observe that all morphisms $f: Y_\beta \to Z$ in ***Ury*** are defined by their values on the countable set $\mathbb{Q}$, as large as β may be.

(b) There are enough unpleasant objects in ***Ury***. Take an Urysohn space which can be mapped continuously onto $\mathbb{Q}$, e.g. an infinite discrete space or the product $\mathbb{Q} \times Z$, where Z is a Urysohn space. For each β, Y_β serves as an example, too. So it is not possible to improve ***Ury*** by deleting its unpleasant objects.

(c) Urysohn spaces do not differ very much from Hausdorff spaces, topologically. These small differences imply large consequences. Categorical topologists should be happy about the well behaved category ***Haus***.

(d) Maybe ***Ury*** is a suitable category to test categorical Theorems using cowellpoweredness.

(e) Only points in the field $\mathbb{Q}(\sqrt{3})$ are involved in our construction.

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