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Remarks on hyperenergetic circulant graphs ☆

Dragan Stevanović *, Ivan Stanković

Faculty of Science and Mathematics, Višegradska 33, 18000 Niš, Serbia and Montenegro

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Abstract

We first settle an open problem of Balakrishnan from Linear Algebra Appl. 387 (2004) 287–295. Further, if $\overline{Ci}(n, k_1, k_2, \cdots, k_m)$, $n \in \mathbf{N}$, $k_1 < k_2 < \cdots < k_m < n/2$, $k_i \in \mathbf{N}$ for $i = 1, 2, \ldots, m$, denotes a circulant graph with the vertex set $V = \{0, 1, \ldots, n-1\}$ such that a vertex u is adjacent to all vertices of $V \setminus \{u\}$ except $u \pm k_i \pmod{n}$, $i = 1, 2, \ldots, m$, we show that for any given $k_1 < k_2 < \cdots < k_m$ almost all circulant graphs $\overline{Ci}(n, k_1, k_2, \ldots, k_m)$ are hyperenergetic.

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Here we consider only simple graphs. The eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ of a graph G with n vertices are the eigenvalues of its adjacency matrix $A(G)$. For other undefined notions, see [2]. The energy $E(G)$ of a graph G is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

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* Corresponding author.
E-mail address: dragance@pmf.ni.ac.yu (D. Stevanović), istankovic@pmf.ni.ac.yu (I. Stanković).

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Energy of a complete graph K_n is equal to $2(n-1)$. Earlier [4] it was conjectured that K_n has the largest energy among all n vertex graphs. After this conjecture has been disproved in [5], graphs for which $E(G) > 2(n-1)$ are called hyperenergetic graphs. (There is a typo in line 6 of [1, p. 288] in the definition of non-hyperenergetic graphs where it stands $E(G) \leqslant (2n-1)$ instead of $E(G) \leqslant 2(n-1)$.)

In [1] Balakrishnan considered graphs $K_n - H$, where H is a Hamilton cycle of K_n and, based on computations, posed an open problem that $K_n - H$ is not hyperenergetic for $n \geqslant 4$. We first solve this problem by showing that $K_n - H$ is indeed hyperenergetic for almost all $n \in \mathbf{N}$.

Graph $\overline{Ci}(n, k_1, k_2, \ldots, k_m)$, $n \in \mathbf{N}$, $k_1 < k_2 < \cdots < k_m < n/2$, $k_i \in \mathbf{N}$ for $i = 1, 2, \ldots, m$, is a circulant graph with the vertex set $V = \{0, 1, \ldots, n-1\}$ such that a vertex u is adjacent to all vertices of $V \setminus \{u\}$ except $u \pm k_i \pmod n$, $i = 1, 2, \ldots, m$. Note that $K_n - H$ is actually $\overline{Ci}(n, 1)$.

Adjacency matrix of $\overline{Ci}(n, k_1, k_2, \ldots, k_m)$ is a circulant matrix with first row having 0s on positions 0, $k_1, \ldots, k_m$, $n-k_1, \ldots, n-k_m$ and 1s elsewhere. Thus, if $\omega = \mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}$ is a primitive nth root of unity, eigenvalues of $\overline{Ci}(n, k_1, k_2, \ldots, k_m)$ are of the form

$$\sum \left\{ \omega^{jk} \colon 1 \leqslant k \leqslant n-1, k \neq k_i, n-k_i \text{ for } i = 1, 2, \ldots, m \right\},$$

for $j = 0, 1, \ldots, n-1$. For $j = 0$ we get an eigenvalue $n - 1 - 2m$, and for $j = 1, \ldots, n-1$ from $\sum_{k=1}^{n-1} \omega^{jk} = -1$ we get an eigenvalue $-1 - \sum_{i=1}^{m} 2\cos k_i \frac{2\pi j}{n}$.

Thus,

$$E(\overline{Ci}(n, 1)) = n - 3 + \sum_{j=1}^{n-1} \left| -1 - 2\cos\frac{2\pi j}{n} \right|.$$

Note that

$$\frac{2\pi}{n-1} \sum_{j=1}^{n-1} \left| -1 - 2\cos\frac{2\pi j}{n} \right|$$

is an integral sum which tends to

$$\int_0^{2\pi} |-1 - 2\cos x| \, \mathrm{d}x$$

for $n \to \infty$. So,

$$\lim_{n \mapsto \infty} \frac{E(\overline{Ci}(n, 1))}{n-1} = 1 + \frac{1}{2\pi} \int_0^{2\pi} |-1 - 2\cos x| \, \mathrm{d}x = \frac{4\sqrt{3}}{2\pi} + \frac{4}{3} \approx 2.43599,$$

which implies that for some $n_0 \in \mathbf{N}$ it holds that $E(\overline{Ci}(n, 1)) > 2(n-1)$ for each $n \geqslant n_0$. Our computations show that $n_0 = 10$.

Reason for Balakrishan's false computations and open problem that $K_n - H$ is not hyperenergetic lies in the fact that in line 8 of [1, p. 289] (s)he overlooked that for $j = 0$ the corresponding eigenvalue of $K_n - H$ is equal to $n - 3$, and not to -3.

Motivated by the above approach using integral sums, we show the following

Theorem 1. *Given $k_1 < k_2 < \cdots < k_m$ there exists $n_0 \in \mathbf{N}$ such that for each $n \geqslant n_0$ the graph $\overline{Ci}(n, k_1, k_2, \cdots, k_m)$ is hyperenergetic.*

Proof. In order to prove the theorem, it is enough to show that

$$c_{k_1,\ldots,k_m} = \lim_{n\to\infty} \frac{E(\overline{Ci}(n, k_1, \ldots, k_m))}{n-1} > 2.$$

Since

$$E(\overline{Ci}(n, k_1, \ldots, k_m)) = (n-1-2m) + \sum_{j=1}^{n-1} \left| -1 - \sum_{i=1}^{m} 2\cos k_i \frac{2\pi j}{n} \right|,$$

we have that

$$c_{k_1,\ldots,k_m} = 1 + \lim_{n\to\infty} \frac{1}{n-1} \sum_{j=1}^{n-1} \left| -1 - \sum_{i=1}^{m} 2\cos k_i \frac{2\pi j}{n} \right|.$$

As before, we note that

$$\frac{2\pi}{n-1} \sum_{j=1}^{n-1} \left| -1 - \sum_{i=1}^{m} 2\cos k_i \frac{2\pi j}{n} \right|$$

is an integral sum which for $n \to \infty$ tends to $\int_0^{2\pi} \left|-1 - \sum_{i=1}^{m} 2\cos k_i x\right| \mathrm{d}x$. Thus,

$$c_{k_1,\ldots,k_m} = 1 + \frac{1}{2\pi} \int_0^{2\pi} \left| -1 - \sum_{i=1}^{m} 2\cos k_i x \right| \mathrm{d}x.$$

It remains to show that

$$\int_0^{2\pi} \left| -1 - \sum_{i=1}^{m} 2\cos k_i x \right| \mathrm{d}x > 2\pi. \tag{1}$$

Since $|x| \geqslant -x$, one immediately has

$$\int_0^{2\pi} \left| -1 - \sum_{i=1}^{m} 2\cos k_i x \right| \mathrm{d}x \geqslant \int_0^{2\pi} 1 + \sum_{i=1}^{m} 2\cos k_i x \, \mathrm{d}x = 2\pi.$$

Since the function $-1 - \sum_{i=1}^{m} 2\cos k_i x$ is continuous, in order to prove strict inequality in (1) it is enough to show that

$$\max_{x\in[0,2\pi]} -1 - \sum_{i=1}^{m} 2\cos k_i x > 0. \tag{2}$$

We know of no elementary proof of this simple fact: in order to prove it, we shall go back to eigenvalues of circulant graphs.

Consider graph $G = \overline{Ci}(n, k_1, \ldots, k_m)$ for $n > 4k_m$. Its eigenvalues are $n-1-2m$ and $-1 - \sum_{i=1}^{m} 2\cos k_i \frac{2\pi j}{n}$ for $j = 1, 2, \ldots, n-1$, and the second largest eigenvalue of G is equal to $-1 - \sum_{i=1}^{m} 2\cos k_i \frac{2\pi j_0}{n}$ for some $j_0 \in \{1, 2, \ldots, n-1\}$. For each $u \in \{0, 1, \ldots, n-1\}$, among vertices u, $u + k_m$, $u + 2k_m$ and $u + 3k_m$ of G we have that $\{u, u + 2k_m\}$, $\{u, u + 3k_m\}$ and $\{u + k_m, u + 3k_m\}$ are pairs of adjacent vertices, while $\{u, u + k_m\}$, $\{u + k_m, u + 2k_m\}$ and $\{u + 2k_m, u + 3k_m\}$ are pairs of nonadjacent vertices. Thus, the subgraph of G induced by vertices u, $u + k_m$, $u + 2k_m$ and $u + 3k_m$ is isomorphic to P_4, which second largest eigenvalue is equal to $\frac{-1+\sqrt{5}}{2} \approx 0.618$. By the Interlacing theorem [3, p.19], we have that the second largest eigenvalue of G is at least the second largest eigenvalue of any induced subgraph of G, and thus

$$-1 - \sum_{i=1}^{m} 2 \cos k_i \frac{2\pi j_0}{n} \geqslant \frac{-1 + \sqrt{5}}{2},$$

which implies (2). □

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