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Initial complex associated to a jet scheme of a determinantal variety

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ABSTRACT

We show in this paper that the principal component of the first-order jet scheme over the classical determinantal variety of $m \times n$ matrices of rank at most 1 is arithmetically Cohen–Macaulay, by showing that an associated Stanley–Reisner simplicial complex is shellable.

1. Introduction

Let F be an algebraically closed field and $\mathbf{A}_F^k$ the affine space of dimension k over F. By a *variety* in $\mathbf{A}_F^k$ we will mean the zero set of a collection of polynomials over F in k variables; in particular, our varieties are not assumed to be irreducible. In [4,5], Košir and Sethuraman had studied jet schemes over classical determinantal varieties, and had described their components in a large number of cases. In particular, they had shown that the variety of first-order jets, or loosely the "algebraic tangent bundle," over the determinantal variety of $m \times n$ matrices ($m \leq n$) of rank at most 1 has two components when $m \geq 3$. One component is simply the affine space $\mathbf{A}_F^{mn}$ supported over the origin. The other component, which is much more interesting, is the closure of the set of tangents at the nonsingular points of the base determinantal variety. We denote this component by Y, and refer to it as the *principal component.* (When $m = 2$, the variety of first-order jets is irreducible, and coincides with the principal component Y.) The goal of this paper is to show that Y is arithmetically Cohen–Macaulay, i.e., its coordinate ring is Cohen–Macaulay.

Consider the truncated polynomial ring $F[t]/(t^2)$, and let $X(t) = (f_{i,j}(t))_{i,j}$ be the generic $m \times n$ ($m \leq n$) matrix over this ring; thus, the (i, j) entry of $X(t)$ is of the form $f_{i,j}(t) = x_{i,j} + y_{i,j}t$, where $1 \leq i \leq m$, $1 \leq j \leq n$, and $x_{i,j}, y_{i,j}$ are variables. Let I be the ideal of $R = F[x_{i,j}, y_{i,j}]$, $1 \leq i \leq m$, $1 \leq j \leq n$, generated by the coefficients of powers of t in each 2×2 minor of the generic matrix $X(t)$. Then the variety of first-order jets over the $m \times n$ matrices ($m \leq n$) of rank at most 1 is precisely the zero set of I. Let J be the ideal of the principal component Y. In [5], Košir and Sethuraman showed that I is radical, and further, determined a Groebner basis for both I and J for the graded reverse lexicographical order using the following scheme: $y_{1,1} > y_{1,2} > \cdots > y_{1,n} > y_{2,1} > \cdots > y_{2,n} > \cdots > y_{m,n} > x_{1,1} > x_{1,2} > \cdots > x_{1,n} > x_{2,1} > \cdots > x_{2,n} > \cdots > x_{m,n}$ (see [5, Theorem 2.4], [5, Proposition 3.3], and also [5, Remark 2.2]).

It follows easily from the description in [5, Theorem 2.4] of the Groebner basis G of J that the leading term ideal of J, $LT(J) := \langle lm(g); g \in G \rangle$, is generated by the following family of monomials:

Proposition 1.1 (*Generators of $LT(J)$*). *The following families of monomials generate $LT(J)$: $A = \{x_{i,l}x_{j,k} \mid 1 \leq i < j \leq m,\ 1 \leq k < l \leq n\}$, $B = \{x_{i,k}y_{j,l} \mid 1 \leq i < j \leq m,\ 1 \leq k < l \leq n\}$, $C = \{x_{k,p}y_{j,q}y_{i,r} \mid 1 \leq i < j \leq k \leq m,\ 1 \leq p < q < r \leq n\}$, $D = \{x_{i,r}y_{j,q}y_{k,p} \mid 1 \leq i < j < k \leq m,\ 1 \leq p < q \leq r \leq n\}$, and $E = \{y_{i,r}y_{j,q}y_{k,p} \mid 1 \leq i < j < k \leq m,\ 1 \leq p < q < r \leq n\}$.*

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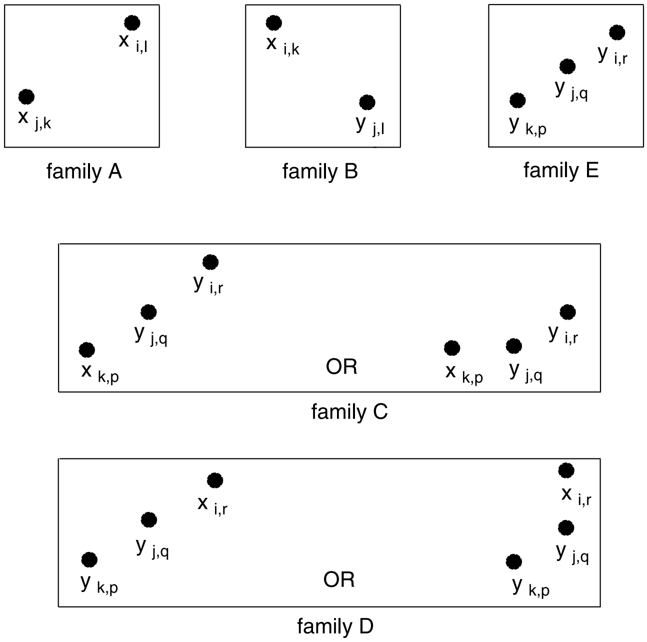

Fig. 1. Generators of $LT(J)$.

Since $LT(J)$ is generated by squarefree monomials we can construct the Stanley–Reisner complex $\Delta_{LT(J)}$ of $LT(J)$: this is the simplicial complex on vertices $\{x_{i,j}, y_{i,j} \mid 1 \le i \le m, 1 \le j \le n\}$ whose corresponding Stanley–Reisner ideal (see [7, Chap. 1] or [2, Chap. 5]) is $LT(J)$. The simplicial complex is defined by the relation $x_{i_1,j_1} \dots x_{i_k,j_k} \dots x_{i_s,j_s} y_{i'_1,j'_1} \dots y_{i'_l,j'_l} \dots y_{i'_t,j'_t}$, $1 \le i_k, i'_l \le m$, $1 \le j_k, j'_l \le n$, is a face of $\Delta_{LT(J)}$ if, as a monomial, it does not belong to $LT(J)$.

We will enumerate all the facets of $\Delta_{LT(J)}$ and we will describe an explicit ordering of the facets which will show that $\Delta_{LT(J)}$ is a shellable simplicial complex. By standard results, shellability of $\Delta_{LT(J)}$ allows us to conclude the following main result of the paper (a result that has been independently obtained by Smith and Weyman in [8] as well, using their geometric technique for computing syzygies):

Theorem 1.2. *The coordinate ring of Y, i.e., R/J, is Cohen–Macaulay.*

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2. Describing the facets of $\Delta_{LT(J)}$

It would be helpful in what follows to visualize the structure of the monomials in the families A, B, C, D, and E as described in Proposition 1.1. For this, see Fig. 1. In this paper, we will visualize a monomial as being positioned in a matrix, where each variable of the monomial is located in the matrix's entry corresponding to the index of the variable.

In this section, we will enumerate all facets of $\Delta_{LT(J)}$. First, some notation: we will denote a facet F of the simplicial complex $\Delta_{LT(J)}$ by $F = F_x F_y$, where F_x is a string composed of vertices $x_{i,j}$'s and F_y is a string composed of vertices $y_{i,j}$'s. We will view each of F_x and F_y as both strings of vertices or monomials, depending on the context. Note that $F_x F_y \in \Delta_{LT(J)}$ if, as a monomial, $F_x F_y$ does not belong in the ideal $LT(J)$ if and only if $F_x F_y$ is not divisible by the generators of $LT(J)$ (see Proposition 1.1).

We will start by showing a relation between the facets of $\Delta_{LT(J)}$ and those of the corresponding simplicial complexes arising from classical determinantal varieties. We refer to the excellent survey paper of Bruns and Conca [1]. In this paper, the authors consider the facets of Δ_t: the Stanley–Reisner complex attached to the ideal $LT(I_t)$, which is generated by the leading terms of the $t \times t$ minors of the generic $m \times n$ matrix $(w_{i,j})$. The order they use is one in which the leading term of a minor is the main diagonal, and it is known that the leading terms of the $t \times t$ minors generate the ideal of leading terms of I_t.

The key result for us is [1, Prop. 6.4], where they enumerate the facets of Δ_t. This is a purely combinatorial result that enumerates the maximal subsets of $V = \{w_{i,j} : i \le m, j \le n\}$ that intersect any t-subset of V arising from the diagonal of some $t \times t$ submatrix of $(w_{i,j})$ in at most $t - 1$ places, and can be applied by symmetry to enumerate the maximal subsets of V that intersect any t-subset of V arising from the antidiagonal of some $t \times t$ submatrix of $(w_{i,j})$ in at most $t - 1$ places. We quote this result as:

Proposition 2.1. *([1, Prop. 6.4]). Let I_t be the ideal of $F[\{w_{i,j}\}]$ generated the $t \times t$ minors of the generic $m \times n$ matrix $(w_{i,j})$. Write $LT(I_t)$ for the ideal generated by the lead terms of the $t \times t$ minors with respect to the graded reverse lexicographical order $w_{1,1} > w_{1,2} > \cdots > w_{1,n} > w_{2,1} > \cdots > w_{2,n} > \cdots > w_{m,n}$. Write Δ_t for the Stanley–Reisner complex of $LT(I_t)$. Then the facets of Δ_t correspond to all families of non-intersecting paths from $w_{1,1}, w_{2,1}, \ldots, w_{t-1,1}$ to $w_{m,n}, w_{m,n-1}, \ldots w_{m,n-t+2}$.*

Here, a path from $w_{a,b}$ to $w_{c,d}$, given $a \leq c$ and $b \leq d$, is a sequence of vertices starting at $w_{a,b}$ and ending at $w_{c,d}$ where each vertex in the sequence is either one step to the right or one step down from the previous vertex. A non-intersecting path of the kind described in the last line of the proposition above is a union of paths from $w_{i,1}$ to $w_{m,n-i+1}$ whose pairwise intersection is empty. (It is known that for the graded reverse lexicographic order as well, the leading terms of the $t \times t$ minors generate the ideal of leading terms of I_t.)

We observe that the monomials in A correspond to the generators of $LT(I_2)$ and the monomials in E correspond to the generators of $LT(I_3)$ (with the order specified in Proposition 2.1). So, for a facet $F = F_xF_y$ of $\Delta_{LT(J)}$, we have that F_x is not in $LT(I_2)$ and F_y is not in $LT(I_3)$. Therefore, by Proposition 2.1, we can state the following lemma:

Lemma 2.2. *F_x is a subset of a path from $x_{1,1}$ to $x_{m,n}$ and F_y is a subset of a pair of non-intersecting paths from $y_{1,1}, y_{2,1}$ to $y_{m,n}$, $y_{m,n-1}$.*

We will continue by showing that for each facet $F = F_xF_y$ of $\Delta_{LT(J)}$, F_x is a non-empty string that contains at least two x-vertices. It is straightforward to see that $x_{m,n}F$ cannot be divisible by any of the generators of $LT(J)$. Hence, maximality of F implies that $x_{m,n}$ is already in F. The next lemma shows that, in addition to $x_{m,n}$, F must contain another x-vertex.

Lemma 2.3. *Let F be a facet of the simplicial complex $\Delta_{LT(J)}$. Then F must contain at least two x-vertices, one of which is $x_{m,n}$.*

Proof. We already know that F must have $x_{m,n}$. Suppose that it is the only x-vertex that F has. Consider then $x_{m-1,n}F$. We can easily check then that $x_{m-1,n}F$ is not divisible by any of the monomials in A, B, C, D, or E . So, $x_{m-1,n}F \in \Delta_{LT(J)}$ and maximality of F implies that $x_{m-1,n}$ must already be in F, a contradiction to the assumption that the only x-vertex that F contains is $x_{m,n}$. □

Notation: Let $F = F_xF_y$ be any facet and recall that, by Lemma 2.2, F_x is a subset of a path from $x_{1,1}$ to $x_{m,n}$. Thus, for any two x-vertices in F_x, one is always to the north, west, or north-west of the other. Let $\mu(F)$ denote the x-vertex that is furthest to north and furthest to the west of all other x-vertices in F_x. Thus, $\mu(F) = x_{i,j}$ implies that $i \leq c$ and $j \leq d$ for all $x_{c,d}$ in F_x (see Fig. 3). Notice that $\mu(F) \neq x_{m,n}$ (Lemma 2.3).

The next lemma deals with the F_y part of a facet F and, in particular, the lemma lists some of the y-vertices that must be present in a given facet.

Lemma 2.4. *Let $F = F_xF_y$ be a facet of the simplicial complex $\Delta_{LT(J)}$ with $\mu(F) = x_{i,j}$. Then F must contain $y_{i,n}$ and $y_{m,j}$.*

Proof. To prove that F contains $y_{i,n}$, it suffices to show that $y_{i,n}F$ is not divisible by a monomial in A, B, C, D, or E. Maximality of F would then imply that $y_{i,n}$ must be in F.

Obviously, $y_{i,n}F$ cannot be divisible by a monomial in A. Also, $y_{i,n}F$ cannot be divisible by a monomial in B because otherwise it is easy to see that $y_{i,n}$ would have to be to the south-east of $\mu(F) = x_{i,j}$ — a contradiction. If $y_{i,n}F$ were divisible by a monomial in D, then another straightforward verification shows that $y_{i,n}$ must in a row below $\mu(F) = x_{i,j}$, which is impossible.

Suppose that $y_{i,n}F$ is divisible by a monomial in C. Then, there must be some $x_{c,d}$ and $y_{s,t}$ in F such that $x_{c,d}y_{s,t}y_{i,n}$ is in C (recall Fig. 1). But then the only possible location of $y_{s,t}$ is to the south-east of $x_{i,j}$. However, $x_{i,j}y_{s,t}$ is in B and in F — a contradiction.

Finally, suppose that $y_{i,n}F$ is divisible by a monomial in E. Then there must be some $y_{a,b}$ and $y_{c,d}$ in F such that $y_{i,n}y_{a,b}y_{c,d}$ is in E (recall Fig. 1). In particular, it must be the case that, say, $y_{c,d}$ is to the south-west of $y_{a,b}$ which, in turn, is to the south-west of $y_{i,n}$. But then either $x_{i,j}y_{a,b}$ is in B or $x_{i,j}y_{c,d}y_{a,b}$ is in D — a contradiction in both cases.

So, $y_{i,n}F$ is not divisible by a monomial in A, B, C, D, or E which implies, as argued above, that F must contain $y_{i,n}$. We can similarly show that F must contain $y_{m,j}$ as well. □

Now we are ready to describe the structure of all facets of $\Delta_{LT(J)}$. The following notation will be useful in the next theorem: for a given facet $F = F_xF_y$ with $\mu(F) = x_{i,j}$, consider the following partition of the y-vertices based on the index (i, j): $R_1 = \{y_{s,t} \mid s \leq i, j < t\}$, $R_2 = \{y_{s,t} \mid s \leq i, t \leq j\}$, $R_3 = \{y_{s,t} \mid i < s, t \leq j\}$, $R_4 = \{y_{s,t} \mid i < s, j < t\}$ (see Fig. 2) .

Theorem 2.5. *Let $F = F_xF_y$ be a facet of the simplicial complex $\Delta_{LT(J)}$ with $\mu(F) = x_{i,j}$. Then F_x is a path from $x_{i,j}$ to $x_{m,n}$ and F_y is a family of non-intersecting paths from $y_{1,1}, y_{2,1}$ to $y_{i,n}, y_{m,j}$.*

Proof. We will first show that $F = F_xF_y$ as described in the theorem is indeed a valid facet of $\Delta_{LT(J)}$. Then we will argue that any facet of $\Delta_{LT(J)}$ must have that form.

Let $F = F_xF_y$ with F_x a path from $x_{i,j}$ to $x_{m,n}$ and F_y a family of non-intersecting paths from $y_{1,1}, y_{2,1}$ to $y_{i,n}, y_{m,j}$ be given (see Fig. 3). We will first show that F is a facet of $\Delta_{LT(J)}$, i.e. F is not divisible by monomials in A, B, C, D, or E and F is maximal with respect to inclusion.

Obviously, F is not divisible by monomials in A and E. To see that F is not divisible by monomials in B, C, and D, it is enough to notice that F does not contain a y-variable in R_4 or variables of the form $y_{c,d}$ and $y_{e,f}$ such that one is to the south-west of the other and both are entirely in R_1 or R_3.

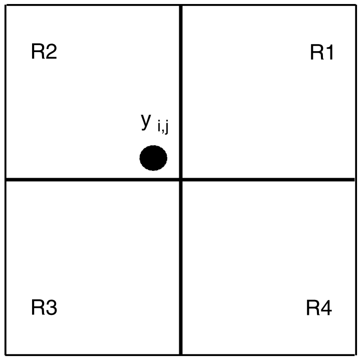

Fig. 2. Partition of the y-vertices.

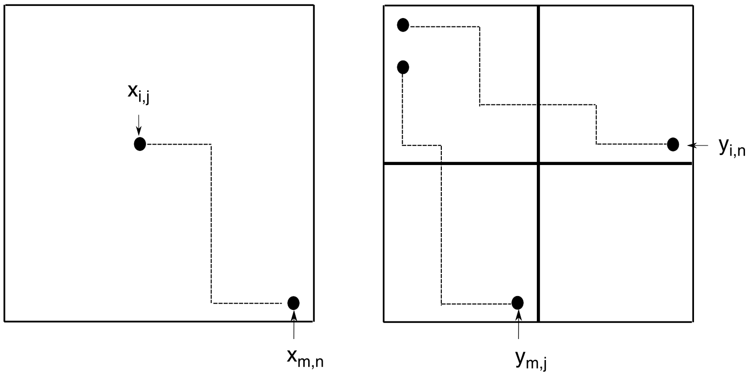

Fig. 3. A facet.

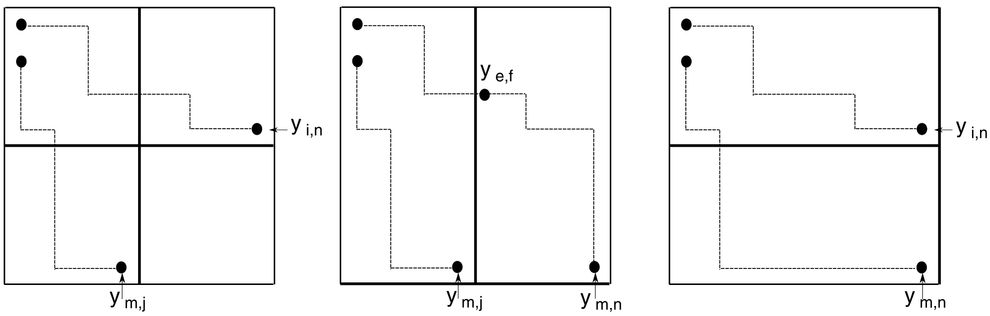

Fig. 4. The y-vertices of different facets.

Next, we will show that $F = F_x F_y$ is maximal with respect to inclusion by arguing that any vertex attached to F would make the resulting monomial divisible by some monomial in A, B, C, D, or E (i.e. that resulting monomial cannot be a face in $\Delta_{LT(J)}$). Recall that by Lemma 2.2 F_x is a subset of a path from $x_{1,1}$ to $x_{m,n}$. So, if we attach a vertex $x_{a,b}$ to F_x, it has to be to the north, west or north-west of $x_{i,j}$. But notice that in this case either $x_{a,b}y_{m,j}$ or $x_{a,b}y_{i,n}$, or both, would be a monomial in B when $i \neq m, j \neq n$ (in the cases $i = m$ or $j = n$, $x_{a,b}$ can also produce monomials in C and D). So, no x-vertex can be attached to F. Recall also that F_y is a subset of a pair non-intersecting paths from $y_{1,1}, y_{2,1}$ to $y_{m,n}, y_{m,n-1}$ (Lemma 2.2). So, if we attach a vertex $y_{c,d}$ to F_y, then $y_{c,d}$ must be in one of those two non-intersecting paths. If $i \neq m, j \neq n$ (see Fig. 4), then $y_{c,d}$ must be in R_4, but then $x_{i,j}y_{c,d}$ would be in B. If $i = m$, then $y_{c,d}$ must be in R_1 and in row m. But then $x_{i,j}y_{c,d}$ and some y-variable that is in the upper path of F_y and in R_1 would produce a monomial in C. Finally, if $j = n$, then $y_{c,d}$ must be in R_3 and in column n. But then $x_{i,j}y_{c,d}$ and some y-variable that is in the lower path of F_y and in R_3 would produce a monomial in D. So, no y-vertex can be attached to F either. Thus, F is maximal.

Finally, we will show that any facet $f = f_x f_y$ of $\Delta_{LT(J)}$ with $\mu(f) = x_{i,j}$ must be of the form described in the theorem. Since f_x is a subset of a path from $x_{1,1}$ to $x_{m,n}$ (by Lemma 2.2), and since $\mu(f) = x_{i,j}$, then it follows that f_x must actually be a subset of a path from $x_{i,j}$ to $x_{m,n}$.

Next, again by Lemma 2.2, f_y must be a subset of a pair of non-intersecting paths from $y_{1,1}$, $y_{2,1}$ to $y_{m,n}$, $y_{m,n-1}$. By Lemma 2.4, it follows that f_y must also contain $y_{i,n}$ and $y_{m,j}$.

Now, if $i \neq m, j \neq n$ (see Fig. 4), f_y cannot contain y-vertices in R_4, because $x_{i,j}$ and any vertex in that region is a monomial in B. So, f_y must be a subset of a family of two non-intersecting paths from $y_{1,1}, y_{2,1}$ to $y_{i,n}, y_{m,j}$. Next, suppose $i = m$ (see Fig. 4). Since f_y is a subset of a pair of non-intersecting paths from $y_{1,1}, y_{2,1}$ to $y_{m,n}, y_{m,n-1}$, it is straightforward to verify, using maximality of f, that f_y must contain the y-variable of the upper y-path that is furthest to north-west in R_1, call it $y_{e,f}$. Also, notice that there should be no y-vertices in $f_y \cap R_1$ such that one is to the south-west of the other (otherwise $x_{m,j}$ and those two y-vertices would produce a monomial in C). Therefore, $f_y \cap R_1$ must actually be a subset of a facet in Δ_2 on vertex set R_1, i.e. $f_y \cap R_1$ must be a subset of some path in R_1 starting in $y_{e,f}$ and ending at $y_{m,n}$ (recall Proposition 2.1). So, f_y must be a subset of a family of two non-intersecting paths from $y_{1,1}, y_{2,1}$ to $y_{i,n}, y_{m,j}$, $i = m$. Finally, we conclude the same result for the case $j = n$ (see Fig. 4) using similar arguments from the case $i = m$.

Finally, notice that $f = f_x f_y$ is actually a subset of a some facet F as described in the theorem. Maximality of f implies that it actually has to be one of those facets F. □

Knowing the structure of a facet $F = F_x F_y$ of the simplicial complex $\Delta_{LT(J)}$, we can easily count the number of vertices that F is composed of, so we can determine $\dim F = |F| - 1$. In particular, we see that the dimension of any facet F is $2(m+n) - 3$. Notice that the dimension of F depends only on the constants m and n. Thus, we can conclude that all facets of the simplicial complex $\Delta_{LT(J)}$ have the same dimension, i.e. $\Delta_{LT(J)}$ is a pure simplicial complex of dimension $2(m+n)-3$.

Corollary 2.6. *The dimension of R/J is $2(m+n) - 2$.*

Theorem 2.5 also allows to determine the total number of facets in $\Delta_{LT(J)}$. Thus, we can determine the multiplicity of R/J as well.

Corollary 2.7. *The multiplicity of R/J is given by*

$$\sum_{(i,j),(i,j)\neq(m,n)} \binom{m+n-i-j}{m-i} det \begin{pmatrix} \binom{i+n-2}{i-1} & \binom{m+j-2}{m-1} \\ \binom{i+n-3}{i-2} & \binom{m+j-3}{m-2} \end{pmatrix}. \tag{1}$$

Proof. The number of paths from $x_{i,j}$ to $x_{m,n}$ is $\binom{m+n-i-j}{m-i}$, while the number of non-intersecting paths from $y_{1,1}, y_{2,1}$ to $y_{i,n}$, $y_{m,j}$ is given by (see [6, Section 2.2])

$$\det \begin{pmatrix} \binom{i+n-2}{i-1} & \binom{m+j-2}{m-1} \\ \binom{i+n-3}{i-2} & \binom{m+j-3}{m-2} \end{pmatrix}. \quad \square$$

Remark 2.8. Professor Sudhir Ghorpade [3] has shown that the expression for the multiplicity of R/J above simplifies to $\binom{n+m-2}{m-1}^2$.

3. Shellability of $\Delta_{LT(J)}$

The main goal of this section is to prove that our simplicial complex $\Delta_{LT(J)}$ is shellable. Recall the following definition of shellability:

Definition 3.1. A simplicial complex Δ is *shellable* if it is pure and if its facets can be given a total order, say $F_1, F_2, \ldots, F_e$, so that the following condition holds: for all i and j with $1 \leq j < i \leq e$ there exists $v \in F_i \setminus F_j$ and an index k, $1 \leq k < i$, such that $F_i \setminus F_k = \{v\}$. A total order of the facets satisfying this condition is called *shelling* of Δ.

Theorem 3.2. *The simplicial complex $\Delta_{LT(J)}$ is shellable.*

Proof. Note that at the end of the previous section we have argued that $\Delta_{LT(J)}$ is pure. We will proceed by first giving a partial order to the facets of $\Delta_{LT(J)}$. Let $P = P_x P_y$ and $Q = Q_x Q_y$ be two facets of $\Delta_{LT(J)}$. If $\mu(P)$ is in a row below $\mu(Q)$, we set $P < Q$ (see Fig. 5). If $\mu(P)$ and $\mu(Q)$ are in the same row, but $\mu(P)$ is to the right of $\mu(Q)$, we set $P < Q$ (see Fig. 5). If $\mu(P) = \mu(Q)$ but P_x is to the right of Q_x as one goes from $\mu(P)$ to $x_{m,n}$, then $P < Q$ (see Fig. 5). If $P_x = Q_x$ and the upper y-path of P_y goes to the right of the upper y-path of Q_y, we set $P < Q$. Finally, if $P_x = Q_x$, the upper y-path of P_y is the same as the upper y-path of Q_y and the lower y-path of P_y goes to the right of the lower y-path of Q_y, we set $P < Q$. Now we arbitrarily extend this partial order on the facets of $\Delta_{LT(J)}$ to a total order.

Now we will prove that the selected total order is indeed a shelling of $\Delta_{LT(J)}$. Let $P = P_x P_y$ and $Q = Q_x Q_y$ be two facets of $\Delta_{LT(J)}$ such that $P < Q$. Our goal is to find $v \in Q \setminus P$ and a facet $R < Q$ such that $Q \setminus R = \{v\}$. Suppose that $\mu(P) \neq \mu(Q)$. Notice P cannot contain $\mu(Q) = x_{i,j}$ (otherwise $P < Q$ is contradicted). Take $v = x_{i,j}$. Take $R = R_x R_y$ to be the following: $R_x = Q_x \setminus x_{i,j}$ and $R_y = Q_y y_{m,j+1}$ if $\mu(R) = x_{i,j+1}$ or $R_y = Q_y y_{i+1,n}$ if $\mu(R) = x_{i+1,j}$. In the special case $\mu(Q) = x_{m-1,n}$, take $R_x = x_{m,n-1} x_{m,n}$, $R_y = Q_y$.

Next, suppose that $\mu(P) = \mu(Q)$, but $P_x \neq Q_x$. Then, there must be a right turn $H = x_{a,b}$ in Q_x that is not in P_x or else Q_x would be to the right of P_x, contradicting $P < Q$. So, in this case take $v = H = x_{a,b}$ and $R = R_x R_y$ where $R_x = Q_x$ with $H = x_{a,b}$ replaced by $x_{a+1,b-1}$ and $R_y = Q_y$.

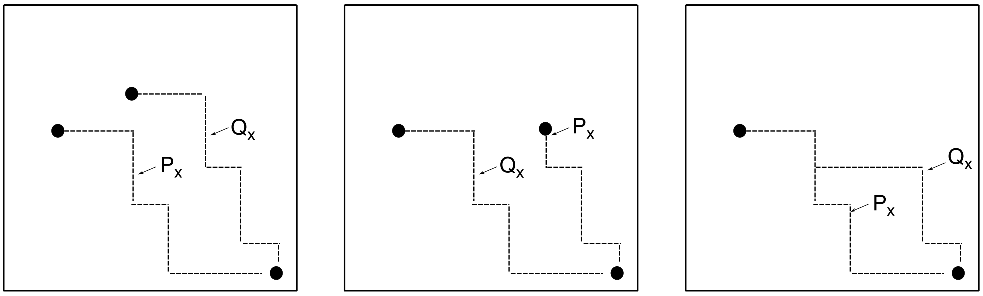

Fig. 5. Partial order of the facets.

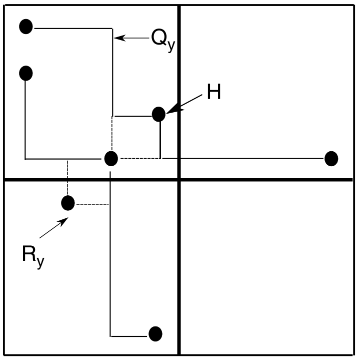

Fig. 6. Two right turns.

Next, suppose that $P_x = Q_x$ and the upper y-paths of the two facets are different. Notice that the upper path of Q_y cannot be strictly on the right of the upper path of P_y (otherwise $P < Q$ is contradicted). So, there must be a right turn $H = y_{c,d}$ of the upper path of Q_y strictly on the left of the upper path of P_y. Thus, $H = y_{c,d}$ cannot be in P_y. So, take $v = y_{c,d}$. If $y_{c+1,d-1}$ is not in the lower path of Q_y, let $R = R_xR_y$ be the following facet: $R_x = Q_x$ and $R_y = Q_y$ with $y_{c,d}$ replaced by $y_{c+1,d-1}$. If $y_{c+1,d-1}$ is in the lower path of Q_y (see Fig. 6), then notice that $y_{c+1,d-1}$ must be a right turn as well. Then take $R = R_xR_y$ to be the following facet: $R_x = Q_x$ and R_y is obtained from Q_y be removing $y_{c,d}$ and by adding $y_{c+2,d-2}$.

Finally, suppose that $P_x = Q_x$, the upper y-paths of the two facets are the same, but the lower y-paths are different. Similarly as in the previous paragraph, we see that there must be a right turn $H = y_{e,f}$ of the lower path of Q_y strictly on the left of the lower path of P_y. Notice that $H = y_{e,f}$ cannot be in the upper path of P_y because it is the same as the upper path of Q_y. So, take $v = y_{e,f}$. Let $R = R_xR_y$ be the facet: $R_x = Q_x$ and $R_y = Q_y$ with $y_{e,f}$ replaced by $y_{e+1,f-1}$. □

We are now in position to prove Theorem 1.2, the main result of the paper:

Proof of Theorem 1.2. By standard results, the ring R/J is Cohen–Macaulay if the ring $R/LT(J)$ is Cohen–Macaulay (see [7, Corollary 8.31]. By construction, $R/LT(J)$ is precisely the Stanley–Reisner ring associated to $\Delta_{LT(J)}$, and since $\Delta_{LT(J)}$ is shellable, $R/LT(J)$ will necessarily be Cohen–Macaulay (see [2, Theorem 5.1.13]). □

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