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The H -covariant strong Picard groupoid

Stefan Jansen, Stefan Waldmann*

*Fakultät für Mathematik und Physik, Albert-Ludwigs-Universität Freiburg, Physikalisches Institut,
Hermann Herder Straße 3, D 79104 Freiburg, Germany*

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Abstract

The notion of H -covariant strong Morita equivalence is introduced for $*$ -algebras over $C = R(i)$ with an ordered ring R which are equipped with a $*$ -action of a Hopf $*$ -algebra H . This defines a corresponding H -covariant strong Picard groupoid which encodes the entire Morita theory. Dropping the positivity conditions one obtains H -covariant $*$ -Morita equivalence with its H -covariant $*$ -Picard groupoid. We discuss various groupoid morphisms between the corresponding notions of the Picard groupoids. Moreover, we realize several Morita invariants in this context as arising from actions of the H -covariant strong Picard groupoid. Crossed products and their Morita theory are investigated using a groupoid morphism from the H -covariant strong Picard groupoid into the strong Picard groupoid of the crossed products.

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1. Introduction

Morita equivalence is by now in many areas of mathematics an important tool to compare and relate objects beyond the notion of isomorphism: the general approach is to enhance a given category by allowing more general morphisms while keeping the objects. This way, more objects can become isomorphic in this enhanced category. The idea is that this ‘Morita

* Corresponding author.

E-mail addresses: stefan.jansen@physik.uni-freiburg.de (S. Jansen), stefan.waldmann@physik.uni-freiburg.de (S. Waldmann).

equivalence’ of objects implies that the objects have an equivalent ‘representation theory’. Each such enhanced category specifies its groupoid of invertible morphisms, which usually is called the corresponding Picard groupoid in this context. This (large) groupoid encodes then the entire Morita theory.

The list of examples is long, starting with Morita’s original version [37] where one considers the category of (unital) rings with certain bimodules between them as generalized morphisms, see e.g. [6,4,30]. Beside various algebraic refinements of the ring-theoretic notion, notions of Morita equivalence have been developed also in completely different contexts, notably for C^* -algebras by Rieffel [40,41] coining the notion of strong Morita equivalence which is now one of the crucial ingredients for Connes’ non-commutative geometry [17], for Poisson manifolds by Xu [47], see also [13], and for Lie groupoids, see e.g. [36] and references therein. We refer to [33] for a comparison of the latter three concepts.

Among the algebraic notions one has $*$ -Morita equivalence for involutive algebras by Ara [1,2] and strong Morita equivalence of $*$ -algebras over a ring $C = R(i)$, where R is an ordered ring and $i^2 = -1$, by Bursztyn and Waldmann [9,11]. These notions provide a bridge between the ring-theoretic notion and the C^* -algebraic framework and incorporate already many ideas of the later like positivity in a purely algebraic context. In particular, strong Morita equivalence of $*$ -algebras was used to study formal star products in deformation quantization, see [7,10,46] for a review. Here the $*$ -algebra in question is a formal associative deformation in the sense of Gerstenhaber [21] of the Poisson algebra of smooth functions $C^\infty(M)$ on a Poisson manifold M , which plays the role of the phase space of a classical mechanical system, see [5,19,22] for recent reviews. The deformed algebra is then interpreted as the observable algebra of the corresponding quantum system. Since the understanding of the representation theory is crucial for physical applications one is naturally interested in an adapted Morita theory in this context. Moreover, it is believed that Morita equivalence of star products is in some sense the quantum version of Morita equivalence of the underlying Poisson structures in the sense of Xu, while on the other hand, formal star products are seen to be a step into the direction of a C^* -algebraic description of the quantum observables. Thus one expects relations between all three types of Morita theory, see e.g. the discussions in [14,15,32–34].

As symmetries play a fundamental role in the understanding of classical and quantum mechanics, it is natural to ask for concepts of Rieffel induction and Morita equivalence which are compatible with given symmetries. In the C^* -algebraic framework such notions are well-established for C^* -dynamical systems, see e.g. [39, Chapter 7] and reference therein as well as [29,44] for more general constructions using locally compact quantum groups.

It is the purpose of this work to transfer these ideas from C^* -algebra theory to the more general and algebraic framework of $*$ -algebras over rings of the form $C = R(i)$. This framework still allows for the crucial notions of positivity but is wide enough to treat C^* -algebras and formal star products on equal footing. The notion of symmetry we are using is rather general, we consider $*$ -actions of a Hopf $*$ -algebra H on $*$ -algebras. After establishing an adapted notion of Morita equivalence, one main focus of this work is on the resulting notion of the Picard groupoid and the Morita invariants which are seen to arise from actions of the Picard groupoid.

The paper is organized as follows: in Section 2 we recall some well-known results on $*$ -algebras, their $*$ -representation theory on pre-Hilbert modules, and Hopf $*$ -algebras and

their $*$ -actions. This allows us to set up the basic notions of an H -covariant $*$ -representation theory. In Section 3, we define the tensor product of bimodules equipped with inner products and H -actions and discuss the definition of H -covariant strong Morita equivalence. We show that it is indeed an equivalence relation. Moreover, H -covariantly strongly Morita equivalent $*$ -algebras are shown to have equivalent H -covariant $*$ -representation theories on H -covariant pre-Hilbert modules. Section 4 is devoted to the definition of the H -covariant strong Picard groupoid $\text{Pic}_H^{\text{str}}$. We discuss several natural groupoid morphisms in this context, in particular the ‘forgetful’ groupoid morphism $\text{Pic}_H^{\text{str}} \rightarrow \text{Pic}^{\text{str}}$ into the strong Picard groupoid. We obtain a full characterization of its kernel. Section 5 illustrates the principle that Morita invariants are obtained from actions of the Picard groupoid. We discuss the representation theories, the H -invariant central elements, the H -equivariant strong K_0 -groups, the lattices of (\mathcal{D}, H) -closed ideals as well as the groups used to classify the (inequivalent) H -actions on equivalence bimodules. In Section 6, we investigate crossed products and establish a groupoid morphism from the H -covariant strong Picard groupoid into the strong Picard groupoid of the crossed products, proving thereby in particular that crossed products are strongly Morita equivalent if the underlying $*$ -algebras are H -covariantly strongly Morita equivalent, a theorem well-known in C^* -algebra theory. Finally, Appendix A contains the construction of the groups used in the characterization of H -actions on equivalence bimodules.

2. Preliminary results

In this section, we recall some basic definitions and results from representation theory of $*$ -algebras over ordered rings and Hopf algebra theory in order to make this work self-contained and to set up our notation, see [8,9,11,46] for details on $*$ -algebras over ordered rings, e.g. [26,32,42] for the representation theory of C^* -algebras and $*$ -algebras over \mathbb{C} and [27,28,35,43] for Hopf $*$ -algebras.

2.1. $*$ -Algebras over ordered rings

Let R be an *ordered ring* and let $C = R(i)$ with $i^2 = -1$. Motivated by deformation quantization, the main examples we have in mind are $R = \mathbb{R}$ and $R = \mathbb{R}[[\lambda]]$ with their natural ordering structures. Then a $*$ -algebra \mathcal{A} over C is an associative algebra over C with an involutive C -antilinear antiautomorphism, called the $*$ -involution, which we shall denote by $a \mapsto a^*$ for $a \in \mathcal{A}$.

A linear functional $\omega : \mathcal{A} \rightarrow C$ is called *positive* if $\omega(a^*a) \geq 0$ for all $a \in \mathcal{A}$. This allows to define *positive algebra elements* $a \in \mathcal{A}$ by the requirement $\omega(a) \geq 0$ for all positive linear functionals. The set of positive algebra elements is denoted by \mathcal{A}^+ . Clearly, elements of the form $\alpha_1 a_1^* a_1 + \dots + \alpha_n a_n^* a_n$ are positive, where $0 < \alpha_i \in \mathbb{R}$ and $a_i \in \mathcal{A}$. These elements will be denoted by \mathcal{A}^{++} . See [42] for more general positive wedges and [45] for a comparative discussion of these concepts of positive algebra elements.

A basic example of a $*$ -algebra is obtained as follows: a *pre-Hilbert space* \mathcal{H} is a C -module with a C -valued sesquilinear inner product (linear in the second argument) $\langle \cdot, \cdot \rangle :$

$\mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$ satisfying $\langle \phi, \psi \rangle = \overline{\langle \psi, \phi \rangle}$ for $\phi, \psi \in \mathcal{H}$ and $\langle \phi, \phi \rangle > 0$ for $\phi \neq 0$. Then a linear endomorphism $A \in \text{End}(\mathcal{H})$ is called *adjointable* if there is an *adjoint* A^* with $\langle A\phi, \psi \rangle = \langle \phi, A^*\psi \rangle$ for all $\phi, \psi \in \mathcal{H}$. It is easy to see that adjoints are unique (if they exist at all) and the set of all adjointable operators $\mathfrak{B}(\mathcal{H})$ becomes a unital $*$ -algebra in the obvious way. Similarly, one defines the adjointable maps $\mathfrak{B}(\mathcal{H}, \mathcal{H}')$ from \mathcal{H} to some other pre-Hilbert space \mathcal{H}' . By $\Theta_{\phi, \psi}$ we denote the *rank one operator* $\Theta_{\phi, \psi}\chi = \phi\langle \psi, \chi \rangle$ where $\phi, \psi, \chi \in \mathcal{H}$. The span of all rank one operators, i.e. the *finite rank operators*, is denoted by $\mathfrak{F}(\mathcal{H})$. Clearly, $\mathfrak{F}(\mathcal{H})$ is a $*$ -ideal in $\mathfrak{B}(\mathcal{H})$. Analogously, one defines $\mathfrak{F}(\mathcal{H}, \mathcal{H}')$.

2.2. Pre-Hilbert modules and $*$ -representation theory

Let \mathcal{A} be a $*$ -algebra and $\mathcal{E}_{\mathcal{A}}$ a right \mathcal{A} -module. We shall always assume that all occurring modules have an underlying compatible \mathbb{C} -module structure. Then an \mathcal{A} -valued inner product on $\mathcal{E}_{\mathcal{A}}$ is a \mathbb{C} -sesquilinear map $\langle \cdot, \cdot \rangle : \mathcal{E}_{\mathcal{A}} \times \mathcal{E}_{\mathcal{A}} \longrightarrow \mathcal{A}$, such that $\langle x, y \cdot a \rangle = \langle x, y \rangle a$ and $\langle x, y \rangle = \langle y, x \rangle^*$. Sometimes we indicate the dependence on the module and the algebra explicitly by $\langle \cdot, \cdot \rangle_{\mathcal{E}_{\mathcal{A}}}$. We call $\langle \cdot, \cdot \rangle$ *non-degenerate* if $\langle x, y \rangle = 0$ for all y implies $x = 0$, in which case $\mathcal{E}_{\mathcal{A}}$ is called a *inner product \mathcal{A} -module*. We also make use of left modules with inner products defined analogously, only linear to the left in the first argument. For an inner product \mathcal{A} -module one has the $*$ -algebra $\mathfrak{B}(\mathcal{E}_{\mathcal{A}})$ of adjointable (and necessarily right \mathcal{A} -linear) endomorphisms of $\mathcal{E}_{\mathcal{A}}$, whence \mathcal{E} becomes a $(\mathfrak{B}(\mathcal{E}_{\mathcal{A}}), \mathcal{A})$ -bimodule. Similarly, one defines $\mathfrak{B}(\mathcal{E}_{\mathcal{A}}, \mathcal{E}'_{\mathcal{A}})$ as well as the finite rank operators $\mathfrak{F}(\mathcal{E}_{\mathcal{A}})$ and $\mathfrak{F}(\mathcal{E}_{\mathcal{A}}, \mathcal{E}'_{\mathcal{A}})$.

An \mathcal{A} -valued inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{E}_{\mathcal{A}}$ is called *completely positive* if the matrix $(\langle x_i, x_j \rangle) \in M_n(\mathcal{A})$ is positive for all $x_1, \dots, x_n \in \mathcal{E}_{\mathcal{A}}$ and $n \in \mathbb{N}$. Here $M_n(\mathcal{A})$ is endowed with its canonical $*$ -algebra structure. An inner product \mathcal{A} -module with completely positive inner product is called a *pre-Hilbert \mathcal{A} -module*.

Let \mathcal{D} be another $*$ -algebra. A $*$ -representation of a $*$ -algebra \mathcal{A} on an inner product \mathcal{D} -module $\mathcal{H}_{\mathcal{D}}$ is a $*$ -homomorphism $\pi : \mathcal{A} \longrightarrow \mathfrak{B}(\mathcal{H}_{\mathcal{D}})$, generalizing thereby the usual notion of a $*$ -representation on a pre-Hilbert space, where $\mathcal{D} = \mathbb{C}$. An *intertwiner* T between two $*$ -representations $(\mathcal{H}_{\mathcal{D}}, \pi)$ and $(\mathcal{H}'_{\mathcal{D}}, \pi')$ of \mathcal{A} is an adjointable map $T \in \mathfrak{B}(\mathcal{H}_{\mathcal{D}}, \mathcal{H}'_{\mathcal{D}})$ with $T\pi(a) = \pi'(a)T$ for all $a \in \mathcal{A}$. It is easy to see that the $*$ -representations of \mathcal{A} on inner product \mathcal{D} -modules form a category, denoted by $*$ -mod $_{\mathcal{D}}(\mathcal{A})$, where morphisms are intertwiners. The subcategory of *strongly non-degenerate* $*$ -representations, i.e. those with $\pi(\mathcal{A})\mathcal{H}_{\mathcal{D}} = \mathcal{H}_{\mathcal{D}}$, is denoted by $*$ -Mod $_{\mathcal{D}}(\mathcal{A})$ and the subcategories of (strongly non-degenerate) $*$ -representations on pre-Hilbert \mathcal{D} -modules are denoted by $*$ -rep $_{\mathcal{D}}(\mathcal{A})$ and $*$ -Rep $_{\mathcal{D}}(\mathcal{A})$, respectively.

Remark 2.1. In the following we shall mainly be interested in *unital* $*$ -algebras, where we shall adopt the convention that $*$ -homomorphisms preserve units and units act as identities on modules. Thus for unital $*$ -algebras we have $*$ -mod= $*$ -Mod and $*$ -rep= $*$ -Rep by convention. In the non-unital case we still need some replacement for the units in order to obtain a reasonably good behavior. The right choices are *idempotent* and *non-degenerate* $*$ -algebras, see the discussion in [11].

From [11, Eq. (4.7)] one has a functorial tensor product of inner product modules

$$\widehat{\otimes}_{\mathcal{B}} \cdot^* \text{-mod}_{\mathcal{B}}(\mathcal{C}) \times \cdot^* \text{-mod}_{\mathcal{A}}(\mathcal{B}) \longrightarrow \cdot^* \text{-mod}_{\mathcal{A}}(\mathcal{C}) \tag{2.1}$$

for three \cdot^* -algebras $\mathcal{A}, \mathcal{B}, \mathcal{C}$, which is obtained as follows: For ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \in \cdot^* \text{-mod}_{\mathcal{B}}(\mathcal{C})$ and ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \in \cdot^* \text{-mod}_{\mathcal{A}}(\mathcal{B})$ one endows the algebraic tensor product ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ with the \mathcal{A} -valued inner product defined by

$$\langle y \otimes_{\mathcal{B}} x, y' \otimes_{\mathcal{B}} x' \rangle_{\mathcal{A}}^{\mathcal{F} \otimes \mathcal{E}} = \langle x, \langle y, y' \rangle_{\mathcal{B}} \cdot x' \rangle_{\mathcal{A}}^{\mathcal{E}}, \tag{2.2}$$

where $x, x' \in \mathcal{E}$ and $y, y' \in \mathcal{F}$. Then one divides by the (possibly non-empty) degeneracy space $({}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}})^{\perp}$ to obtain a non-degenerate \mathcal{A} -valued inner product on the quotient ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} = ({}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}) / ({}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}})^{\perp}$, which is then a \cdot^* -representation of \mathcal{C} . This construction generalizes Rieffel’s internal tensor product [40,41], which is a fundamental tool in C^* -algebra and Hilbert C^* -module theory, see e.g. [39,31]. The tensor product $\widehat{\otimes}$ is associative up to the usual canonical isomorphism. Moreover, if the inner products where both completely positive then the resulting inner product (2.2) is completely positive again, see [11, Theorem 4.7]. Thus $\widehat{\otimes}$ restricts to a functor

$$\widehat{\otimes}_{\mathcal{B}} \cdot^* \text{-rep}_{\mathcal{B}}(\mathcal{C}) \times \cdot^* \text{-rep}_{\mathcal{A}}(\mathcal{B}) \longrightarrow \cdot^* \text{-rep}_{\mathcal{A}}(\mathcal{C}). \tag{2.3}$$

Remark 2.2 (Complex conjugation of bimodules). Of course, we can also define \cdot^* -representations from the right on inner product left modules. Then the analogous statements are still true. Furthermore, we can pass from one to the other by *complex conjugation* of the bimodule. For ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \in \cdot^* \text{-mod}_{\mathcal{A}}(\mathcal{B})$ we define the $(\mathcal{A}, \mathcal{B})$ -bimodule ${}_{\mathcal{A}}\overline{\mathcal{E}}_{\mathcal{B}}$ by $\overline{\mathcal{E}} = \mathcal{E}$ as \mathcal{R} -module with \mathbb{C} -module structure given by $\alpha \overline{x} = \overline{\alpha x}$ for $\alpha \in \mathbb{C}$ and $x \in \mathcal{E}$, where $\mathcal{E} \ni x \mapsto \overline{x} \in \overline{\mathcal{E}}$ denotes the identity map of the underlying \mathcal{R} -module. Then the $(\mathcal{A}, \mathcal{B})$ -bimodule structure is defined by

$$a \cdot \overline{x} = \overline{x \cdot a^*} \quad \text{and} \quad \overline{x} \cdot b = \overline{b^* \cdot x}. \tag{2.4}$$

The \mathcal{A} -left linear \mathcal{A} -valued inner product is defined by

$${}_{\mathcal{A}}\langle \overline{x}, \overline{y} \rangle_{\overline{\mathcal{E}}} = \langle x, y \rangle_{\mathcal{E}}, \tag{2.5}$$

which is clearly compatible with the right \mathcal{B} -module structure. Then it is easily shown that ${}_{\mathcal{A}}\langle \cdot, \cdot \rangle_{\overline{\mathcal{E}}}$ is completely positive if and only if $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ is completely positive.

2.3. Hopf \cdot^* -algebras and \cdot^* -actions

Let H be a Hopf algebra over \mathbb{C} with comultiplication Δ , counit ε and antipode S . For Δ we use Sweedler’s notation, i.e. $\Delta(g) = g_{(1)} \otimes g_{(2)}$, etc. Now assume that H is in addition a \cdot^* -algebra. Then H is called a *Hopf \cdot^* -algebra* if Δ and ε are \cdot^* -homomorphisms and $S(S(g)^*)^* = g$, see e.g. [27, Section IV.8]. In particular, S is invertible with inverse $S^{-1}(g) = S(g^*)^*$. The basic examples are group algebras $\mathbb{C}[G]$ for a group G and complexified universal enveloping algebras $U_{\mathbb{C}}(\mathfrak{g}) = U(\mathfrak{g}) \otimes_{\mathbb{R}} \mathbb{C}$ for Lie algebras \mathfrak{g} over \mathbb{R} , each endowed with the canonical Hopf and \cdot^* -algebra structures. Both of them are *cocommutative*, i.e. $\Delta = \Delta^{\text{op}}$, where $\Delta^{\text{op}}(g) = g_{(2)} \otimes g_{(1)}$ denotes the opposite comultiplication.

Let \mathcal{A} be a $*$ -algebra over \mathbb{C} . A (left) $*$ -action of H on \mathcal{A} is a (left) H -module structure on \mathcal{A} , denoted by $(g, a) \mapsto g \triangleright a$ for $g \in H$ and $a \in \mathcal{A}$, such that in addition

$$g \triangleright (ab) = (g_{(1)} \triangleright a)(g_{(2)} \triangleright b), \tag{2.6}$$

$$(g \triangleright a)^* = S(g)^* \triangleright a^*. \tag{2.7}$$

We shall use resulting formulas like $(g \triangleright a)b = g_{(1)} \triangleright (aS(g_{(2)}) \triangleright b)$ for $a, b \in \mathcal{A}$ and $g \in H$ as well as $a(g \triangleright b) = g_{(2)} \triangleright ((S^{-1}(g_{(1)}) \triangleright a)b)$ in the sequel, see e.g. [27,28,35] for more details on the calculus with Hopf $*$ -algebras and $*$ -actions.

If \mathcal{A} is unital we require $g \triangleright \mathbb{1}_{\mathcal{A}} = \varepsilon(g)\mathbb{1}_{\mathcal{A}}$ and we always assume that $\mathbb{1}_H \triangleright a = a$. Recall that $a \in \mathcal{A}$ is called H -invariant if $g \triangleright a = \varepsilon(g)a$.

Similarly, one can define right $*$ -actions which we shall not need in the sequel. From now on, all $*$ -algebras are thought of being equipped with a particular $*$ -action of H .

Example 2.3 ($*$ -Actions of H). For later use we shall mention the following basic examples:

- (i) The *trivial $*$ -action* of H on \mathcal{A} is given by the counit ε , i.e.

$$g \triangleright a = \varepsilon(g)a. \tag{2.8}$$

It is easy to see that this is indeed a $*$ -action. The ring \mathbb{C} and the matrices $M_n(\mathbb{C})$ are always assumed to carry the trivial $*$ -action.

- (ii) The *adjoint action*, see e.g. [35, Example 1.6.3], of H on itself is given by

$$\text{Ad}_g h = g_{(1)} h S(g_{(2)}) \tag{2.9}$$

and it turns out to be a $*$ -action as well.

- (iii) If \mathcal{A} has a $*$ -action then the matrices $M_n(\mathcal{A})$ are endowed with a $*$ -action of H as well by applying $g \in H$ componentwise.

Let \mathcal{H} be a pre-Hilbert space over \mathbb{C} and let $\triangleright : H \rightarrow \text{End}_{\mathbb{C}}(\mathcal{H})$ be an action of H on \mathcal{H} . Then \triangleright is called *unitary* if

$$\varepsilon(g)\langle x, y \rangle = \langle S(g_{(1)})^* \triangleright x, g_{(2)} \triangleright y \rangle \tag{2.10}$$

for all $x, y \in \mathcal{H}$ and $g \in H$. Clearly, this gives unitary representations of groups and (anti-) Hermitian representations of real Lie algebras when applied to $\mathbb{C}[G]$ and $U_{\mathbb{C}}(\mathfrak{g})$, respectively.

We generalize (2.10) as follows: let $\mathcal{H}_{\mathcal{D}}$ be a right \mathcal{D} -module, where \mathcal{D} is an auxiliary $*$ -algebra over \mathbb{C} endowed with a $*$ -action of H . Moreover, let $\mathcal{H}_{\mathcal{D}}$ be endowed with an H -module structure and with a \mathcal{D} -valued inner product. Then the H -module structure is called *compatible with the right \mathcal{D} -module structure* if

$$g \triangleright (x \cdot d) = (g_{(1)} \triangleright x) \cdot (g_{(2)} \triangleright d) \tag{2.11}$$

and it is called *compatible with the inner product* if

$$g \triangleright \langle x, y \rangle = \langle S(g_{(1)})^* \triangleright x, g_{(2)} \triangleright y \rangle \quad (2.12)$$

for $x, y \in \mathcal{H}_{\mathcal{D}}$, $d \in \mathcal{D}$ and $g \in H$.

Lemma 2.4. *The covariance condition (2.12) is equivalent to the condition*

$$\langle x, g \triangleright y \rangle = g_{(2)} \triangleright \langle g_{(1)}^* \triangleright x, y \rangle \quad (2.13)$$

for $x, y \in \mathcal{H}_{\mathcal{D}}$ and $g \in H$. If the \mathcal{D} -valued inner product is non-degenerate then (2.12) implies (2.13).

Proof. The equivalence of the two conditions (2.13) and (2.12) is a simple computation. Moreover, applying (2.13) twice one obtains

$$\langle x, g \triangleright (y \cdot d) \rangle = \langle x, (g_{(1)} \triangleright y) \cdot (g_{(2)} \triangleright d) \rangle,$$

so the non-degeneracy of $\langle \cdot, \cdot \rangle$ implies (2.11). This generalizes [39, Remark 7.3]. \square

Definition 2.5. An inner product right \mathcal{D} -module is called *H-covariant* if the inner product satisfies (2.12) and hence also (2.13) and (2.11).

In case of a pre-Hilbert space (2.13) simply means that the action with $g \in H$ is *adjointable* and we have

$$\langle x, g \triangleright y \rangle = \langle g^* \triangleright x, y \rangle. \quad (2.14)$$

Remark 2.6. Note that in general the operator $x \mapsto g \triangleright x$ is *not* adjointable as endomorphism of $\mathcal{H}_{\mathcal{D}}$ since the action ‘outside’ the inner product can be non-trivial.

Remark 2.7. For a possibly degenerate inner product the condition (2.13) immediately ensures $H \triangleright \mathcal{H}_{\mathcal{D}}^{\perp} \subseteq \mathcal{H}_{\mathcal{D}}^{\perp}$. Thus if the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}_{\mathcal{D}}$ is degenerate the H -action passes to the quotient $\mathcal{H}_{\mathcal{D}} / \mathcal{H}_{\mathcal{D}}^{\perp}$ which then becomes an H -covariant inner product \mathcal{D} -module.

Proposition 2.8. *Let $\mathcal{H}_{\mathcal{D}}$ be an H-covariant inner product \mathcal{D} -module. Then*

$$(g \triangleright A)x = g_{(1)} \triangleright (AS(g_{(2)}) \triangleright x), \quad (2.15)$$

for $A \in \mathfrak{B}(\mathcal{H}_{\mathcal{D}})$ and $x \in \mathcal{H}_{\mathcal{D}}$, defines a $*$ -action of H on $\mathfrak{B}(\mathcal{H}_{\mathcal{D}})$ uniquely determined by the property

$$g \triangleright (Ax) = (g_{(1)} \triangleright A)(g_{(2)} \triangleright x). \quad (2.16)$$

Moreover, we have for the rank one operators

$$g \triangleright \Theta_{x,y} = \Theta_{g_{(1)} \triangleright x, S(g_{(2)})^* \triangleright y}, \quad (2.17)$$

whence we have $H \triangleright \mathfrak{K}(\mathcal{H}_{\mathcal{D}}) \subseteq \mathfrak{K}(\mathcal{H}_{\mathcal{D}})$.

Proof. Using the same kind of calculations as for the adjoint action of H on itself one shows that (2.15) defines an action of H on all endomorphisms $\text{End}_{\mathbb{C}}(\mathcal{H}_{\mathcal{D}})$, which is uniquely determined by the property (2.16). This part is fairly standard and well-known. It remains to show that for $A \in \mathfrak{B}(\mathcal{H}_{\mathcal{D}})$ the result $g \triangleright A$ is again adjointable with adjoint given by $S(g)^* \triangleright A^*$. Note that this is non-trivial according to Remark 2.6. One computes

$$\begin{aligned} \langle x, g_{(1)} \triangleright (AS(g_{(2)}) \triangleright y) \rangle &= g_{(2)} \triangleright S(g_{(3)})_{(2)} \triangleright \langle S(g_{(3)})_{(1)}^* \triangleright (A^* g_{(1)}^* \triangleright x), y \rangle \\ &= \langle S(g_{(2)})^* \triangleright (A^* g_{(1)}^* \triangleright x), y \rangle, \end{aligned}$$

using twice (2.13) and the fact that A is adjointable as well as $S \otimes S \circ \Delta^{\text{op}} = \Delta \circ S$. This shows that $g \triangleright A$ is indeed adjointable with adjoint given by

$$(g \triangleright A)^* x = S(g_{(2)})^* \triangleright A^* g_{(1)} \triangleright x = (S(g)^* \triangleright A^*) x,$$

whence the action (2.15) is a $*$ -action. The fact that $g \triangleright A$ is again right \mathcal{D} -linear follows from the existence of an adjoint. The statement about the rank one operator follows analogously. \square

For obvious reasons we call the action on $\mathfrak{B}(\mathcal{H}_{\mathcal{D}})$ the *adjoint action* induced by the action on $\mathcal{H}_{\mathcal{D}}$.

Let us now mention one of our motivating examples from geometry:

Example 2.9 (*Lie algebra action on a manifold*). Let \mathfrak{g} be a real finite-dimensional Lie algebra and M a smooth manifold and let $H = U_{\mathbb{C}}(\mathfrak{g})$ be the complexified universal enveloping algebra of \mathfrak{g} , viewed as Hopf $*$ -algebra. Then a $*$ -action of H on the complex-valued smooth functions $C^{\infty}(M)$ is equivalent to a Lie algebra action of \mathfrak{g} on M , i.e. an Lie algebra homomorphism $\mathfrak{g} \rightarrow \Gamma^{\infty}(TM)$. This follows from the fact that the condition (2.6) implies that $\zeta \in \mathfrak{g}$ acts as derivation on $C^{\infty}(M)$ together with the fact that any derivation of $C^{\infty}(M)$ is given by a vector field.

Example 2.10 (*Lie group action on a manifold*). If G is a Lie group and $\Phi : G \times M \rightarrow M$ a smooth Lie group action on a manifold M , then the pull-backs Φ_g^* act on $C^{\infty}(M)$ by $*$ -automorphisms. This yields a $*$ -action of the Hopf $*$ -algebra $\mathbb{C}[G]$. Note however, that in our (algebraic) definition of the group algebra $H = \mathbb{C}[G]$ no topological information about G is contained. Thus not every $*$ -action of H on $C^{\infty}(M)$ comes from a *smooth* action of G on M . Here one has to impose additional conditions which go beyond our purely algebraic treatment.

Remark 2.11. The above two examples provide the framework for symmetries in classical mechanics. In deformation quantization such symmetries are encoded in the notions of invariant star products, see e.g. [3,23,38] and references therein. Here, we have to pass from \mathbb{R} and \mathbb{C} to the ordered ring $\mathbb{R}[[\lambda]]$ and $\mathbb{C}[[\lambda]]$.

2.4. H -covariant representation theory

Let \mathcal{A} be a $*$ -algebra and let \mathcal{D} be an auxiliary $*$ -algebra as above, both endowed with a fixed $*$ -action of H . If $\mathcal{H}_{\mathcal{D}}$ is an H -covariant inner product right \mathcal{D} -module then

a $*$ -representation π of \mathcal{A} on $\mathcal{H}_{\mathcal{D}}$ is called *H-covariant* if

$$\pi(g \triangleright a)x = g_{(1)} \triangleright (\pi(a)S(g_{(2)}) \triangleright x) \quad (2.18)$$

holds for all $a \in \mathcal{A}$ and $x \in \mathcal{H}_{\mathcal{D}}$. Again, applied to pre-Hilbert spaces and for group algebras or complexified universal enveloping algebras one recovers the usual notion of a covariant $*$ -representation. Another way to view (2.18) is that the map $\pi : \mathcal{A} \rightarrow \mathfrak{B}(\mathcal{H}_{\mathcal{D}})$ is *H*-equivariant with respect to the adjoint action on $\mathfrak{B}(\mathcal{H}_{\mathcal{D}})$ induced by the action on $\mathcal{H}_{\mathcal{D}}$.

An intertwiner $T : \mathcal{H}_{\mathcal{D}} \rightarrow \mathcal{H}'_{\mathcal{D}}$ between two *H*-covariant $*$ -representations $(\mathcal{H}_{\mathcal{D}}, \pi)$ and $(\mathcal{H}'_{\mathcal{D}}, \pi')$ of \mathcal{A} is called *H-covariant* if T also intertwines the *H*-module structure, i.e.

$$T(g \triangleright x) = g \triangleright T(x). \quad (2.19)$$

Then one obtains the *category of H-covariant $*$ -representations* of \mathcal{A} on *H*-covariant inner product right \mathcal{D} -modules, denoted by $*\text{-mod}_{\mathcal{D}, H}(\mathcal{A})$, where *H*-covariant intertwiners are used as morphisms. Analogously, one defines the sub-categories $*\text{-Mod}_{\mathcal{D}, H}(\mathcal{A})$ as well as $*\text{-rep}_{\mathcal{D}, H}(\mathcal{A})$ and $*\text{-Rep}_{\mathcal{D}, H}(\mathcal{A})$.

Remark 2.12. Also in this framework we can pass from left to right $*$ -representations. For a left \mathcal{B} -representation on an *H*-covariant inner product right \mathcal{A} -module ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ we define the *H*-action $\overline{\triangleright}$ on ${}_{\mathcal{A}}\overline{\mathcal{E}}_{\mathcal{B}}$ by

$$g \overline{\triangleright} x = \overline{S(g)^* \triangleright x}, \quad (2.20)$$

which can be shown to be an *H*-action compatible with the complex conjugated bimodule structure as well as with the complex conjugated inner product ${}_{\mathcal{A}}\langle \cdot, \cdot \rangle_{\mathcal{B}}^{\overline{\mathcal{E}}}$. This is a straightforward computation. Moreover, $\overline{\overline{\mathcal{E}}} = \mathcal{E}$, including all its structures.

The prototype of an *H*-covariant $*$ -representation is obtained by the GNS representation with respect to an *H*-invariant positive linear functional on \mathcal{A} :

Example 2.13 (*H*-invariant GNS construction). The usual GNS construction of a $*$ -representation out of a positive linear functional can be generalized immediately to the *H*-covariant framework. Let $\omega : \mathcal{A} \rightarrow \mathbb{C}$ be an *H*-invariant positive linear functional, i.e. we have $\omega(a^*a) \geq 0$ and $\omega(g \triangleright a) = \varepsilon(g)\omega(a)$ for all $a \in \mathcal{A}$ and $g \in H$. Then we consider the inner product

$$\langle a, b \rangle_{\omega} = \omega(a^*b) \quad (2.21)$$

on \mathcal{A} , viewed as $(\mathcal{A}, \mathbb{C})$ -bimodule. We have

$$\langle g^* \triangleright a, b \rangle_{\omega} = \langle a, g \triangleright b \rangle_{\omega} \quad (2.22)$$

by a straightforward computation using the invariance of ω , whence $\langle \cdot, \cdot \rangle_{\omega}$ is compatible with the *H*-action. Thus we can apply Remark 2.7 and divide by the (possibly non-empty)

degeneracy space of $\langle \cdot, \cdot \rangle_\omega$ to obtain a pre-Hilbert module $\mathcal{H}_\omega = \mathcal{A}/\mathcal{A}^\perp$ over \mathbb{C} , i.e. a pre-Hilbert space. Note that

$$\mathcal{A}^\perp = \mathcal{I}_\omega = \{a \in \mathcal{A} \mid \omega(a^*a) = 0\} \quad (2.23)$$

is just the Gel'fand ideal of ω . Thus we recover the usual GNS representation π_ω of \mathcal{A} on \mathcal{H}_ω together with an H -action making the GNS representation H -covariant.

If in addition \mathcal{A} is unital then the class of $\mathbb{1}_{\mathcal{A}}$ in \mathcal{H}_ω is a cyclic H -invariant vector, the vacuum vector. Every other H -covariant $*$ -representation (\mathcal{H}, π) of \mathcal{A} with H -invariant cyclic vector $\Omega \in \mathcal{H}$ such that $\omega(a) = \langle \Omega, \pi(a)\Omega \rangle$ is unitarily equivalent to the GNS representation via the usual H -covariant intertwiner. Needless to say, this example is of fundamental importance for the understanding of the H -covariant $*$ -representation theory of \mathcal{A} .

2.5. The lattice of (\mathcal{D}, H) -closed $*$ -ideals

For a C^* -algebra a $*$ -ideal is topologically closed if and only if it is the kernel of a $*$ -representation, see e.g. [32, Chapter 1, Theorem 1.3.10]. This fact was the motivation to define a $*$ -ideal in a $*$ -algebra to be *closed* if it is the kernel of a $*$ -representation of \mathcal{A} on a pre-Hilbert space, see [8]. We extend this definition now in two directions, allowing for $*$ -representations on pre-Hilbert \mathcal{D} -modules instead of pre-Hilbert spaces and incorporating H -covariance.

For reasons which become clear in Section 5.4 we have to restrict the auxiliary $*$ -algebras \mathcal{D} . The problem is that for a pre-Hilbert \mathcal{D} -module $\mathcal{H}_\mathcal{D}$ the inner product $\langle \cdot, \cdot \rangle_\mathcal{D}$ is completely positive and non-degenerate *but* there may be elements $\phi \in \mathcal{H}_\mathcal{D}$ with $\langle \phi, \phi \rangle_\mathcal{D} = 0$ and $\phi \neq 0$. The Grassmann algebra $\Lambda(\mathbb{C}^n)$ provides a simple example, see [11, Example 3.5]. In order to avoid this we state the following definition: we call \mathcal{D} *admissible* if on any pre-Hilbert \mathcal{D} -module $\mathcal{H}_\mathcal{D}$ the inner product $\langle \cdot, \cdot \rangle_\mathcal{D}$ is in addition *positive definite*, i.e. $\langle \phi, \phi \rangle_\mathcal{D} = 0$ implies $\phi = 0$. This is the case if e.g. \mathcal{D} has sufficiently many positive linear functionals in the sense that for any Hermitian element $d = d^* \neq 0$ we find a positive linear functional $\omega : \mathcal{D} \rightarrow \mathbb{C}$ with $\omega(d) \neq 0$ and if $d + d = 0$ implies $d = 0$ for all $d \in \mathcal{D}$, see [11, Example 3.6]. Then we can state the following definition:

Definition 2.14. Let \mathcal{D} be admissible. Then $\mathcal{I} \subseteq \mathcal{A}$ is called a (\mathcal{D}, H) -closed ideal if $\mathcal{I} = \ker \pi$ for some $*$ -representation $(\mathcal{H}_\mathcal{D}, \pi) \in \text{rep}_{\mathcal{D}, H}(\mathcal{A})$. The set of all (\mathcal{D}, H) -closed ideals is denoted by $\mathfrak{Q}_{\mathcal{D}, H}(\mathcal{A})$.

Clearly, if $\mathcal{D} = \mathbb{C}$ the non-covariant version gives the lattice of closed ideals $\mathfrak{Q}(\mathcal{A})$ as in [8, Section 4]. We collect a few first properties of $\mathfrak{Q}_{\mathcal{D}, H}(\mathcal{A})$ which can be obtained completely analogously as for $\mathfrak{Q}(\mathcal{A})$.

Lemma 2.15. *Let \mathcal{D} be admissible and let \mathcal{A} be idempotent.*

- (i) *If $\mathcal{I} \in \mathfrak{Q}_{\mathcal{D}, H}(\mathcal{A})$ then \mathcal{I} is an H -invariant $*$ -ideal of \mathcal{A} .*
- (ii) *If $\mathcal{I} \in \mathfrak{Q}_{\mathcal{D}, H}(\mathcal{A})$ then there exists a strongly non-degenerate $*$ -representation $(\mathcal{H}_\mathcal{D}, \pi) \in \text{rep}_{\mathcal{D}, H}(\mathcal{A})$ with $\mathcal{I} = \ker \pi$.*

- (iii) Let I be a set and $\mathcal{I}_\alpha \in \mathfrak{L}_{\mathcal{D}, H}(\mathcal{A})$ for $\alpha \in I$. Then $\bigcap_{\alpha \in I} \mathcal{I}_\alpha \in \mathfrak{L}_{\mathcal{D}, H}(\mathcal{A})$.
 (iv) For an arbitrary subset $\mathcal{J} \subseteq \mathcal{A}$ let

$$\mathcal{J}^{\text{cl}} = \bigcap_{\mathcal{I} \subseteq \mathcal{J}, \mathcal{I} \in \mathfrak{L}_{\mathcal{D}, H}(\mathcal{A})} \mathcal{I}. \quad (2.24)$$

Then $\mathcal{J}^{\text{cl}} \in \mathfrak{L}_{\mathcal{D}, H}(\mathcal{A})$ is the smallest (\mathcal{D}, H) -closed ideal containing \mathcal{J} and $(\mathcal{J}^{\text{cl}})^{\text{cl}} = \mathcal{J}^{\text{cl}}$.

- (v) The operations $\mathcal{I} \wedge \mathcal{J} = \mathcal{I} \cap \mathcal{J}$ and $\mathcal{I} \vee \mathcal{J} = (\mathcal{I} \cup \mathcal{J})^{\text{cl}}$ define the structure of a lattice on $\mathfrak{L}_{\mathcal{D}, H}(\mathcal{A})$ such that $\mathcal{I} \leq \mathcal{J}$ if and only if $\mathcal{I} \subseteq \mathcal{J}$.

Proof. The proof is completely analogous to the corresponding ones in [8] since inner products are always positive definite. \square

We call $\mathfrak{L}_{\mathcal{D}, H}(\mathcal{A})$ the lattice of (\mathcal{D}, H) -closed ideals of the $*$ -algebra \mathcal{A} . Note that only for the second part of the lemma one needs that \mathcal{D} is admissible.

3. H -covariant strong Morita equivalence

In this section we adapt the tensor product $\widehat{\otimes}$ to the H -covariant situation and obtain this way an H -covariant version of Rieffel induction. This tensor product will allow a definition of H -covariant strong Morita equivalence which implies the usual strong Morita equivalence, see e.g. [39] for the analogous construction for G -covariant strong Morita equivalence of C^* -algebras.

3.1. H -covariant tensor products

First we show how the functor $\widehat{\otimes}$ from (2.3) restricts to H -covariant $*$ -representations. For a right \mathcal{B} -module $\mathcal{F}_{\mathcal{B}}$ with H -covariant \mathcal{B} -valued inner product $\langle \cdot, \cdot \rangle_{\mathcal{B}}^{\mathcal{F}}$ and an H -covariant $(\mathcal{B}, \mathcal{A})$ -bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ with H -covariant \mathcal{A} -valued inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}}^{\mathcal{E}}$ compatible with the \mathcal{B} -action in the sense that $\langle b \cdot x, y \rangle_{\mathcal{A}}^{\mathcal{E}} = \langle x, b^* \cdot y \rangle_{\mathcal{A}}^{\mathcal{E}}$, we have the inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}}^{\mathcal{F} \otimes \mathcal{E}}$ on $\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ given by (2.2). On the tensor product we also have canonically an action of H defined as usual by

$$g \triangleright (x \otimes_{\mathcal{B}} y) = g_{(1)} \triangleright x \otimes_{\mathcal{B}} g_{(2)} \triangleright y, \quad (3.1)$$

which is indeed easily shown to be well-defined over $\otimes_{\mathcal{B}}$ and an action of H .

Lemma 3.1. *The canonical H -action on $\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ given by the tensor product of the action on $\mathcal{F}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ makes $\langle \cdot, \cdot \rangle_{\mathcal{A}}^{\mathcal{F} \otimes \mathcal{E}}$ an H -covariant inner product. Moreover, the H -action passes to the quotient $\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} / (\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}})^{\perp}$ which becomes an H -covariant inner product \mathcal{A} -module.*

Proof. Let $x, x' \in \mathcal{F}$ and $y, y' \in \mathcal{E}$ as well as $g \in H$. From the H -covariance of the inner products $\langle \cdot, \cdot \rangle_{\mathcal{A}}^{\mathcal{E}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{B}}^{\mathcal{F}}$ we conclude that

$$\begin{aligned} g \triangleright \langle x \otimes y, x' \otimes y' \rangle_{\mathcal{A}}^{\mathcal{E}} &= g \triangleright \left\langle y, \langle x, x' \rangle_{\mathcal{B}}^{\mathcal{F}} \cdot y' \right\rangle_{\mathcal{A}}^{\mathcal{E}} \\ &= \left\langle S(g(1))^* \triangleright y, g(2) \triangleright \left(\langle x, x' \rangle_{\mathcal{B}}^{\mathcal{F}} \cdot y' \right) \right\rangle_{\mathcal{A}}^{\mathcal{E}} \\ &= \left\langle S(g(1))^* \triangleright y, \langle S(g(2))^* \triangleright x, g(3) \triangleright x' \rangle_{\mathcal{B}}^{\mathcal{F}} \cdot (g(4) \triangleright y') \right\rangle_{\mathcal{A}}^{\mathcal{E}} \\ &= \langle S(g(2))^* \triangleright x \otimes S(g(1))^* \triangleright y, g(3) \triangleright x' \otimes g(4) \triangleright y' \rangle_{\mathcal{A}}^{\mathcal{F} \otimes \mathcal{E}} \\ &= \langle S(g(1))^* \triangleright (x \otimes y), g(2) \triangleright (x' \otimes y') \rangle_{\mathcal{A}}^{\mathcal{F} \otimes \mathcal{E}}, \end{aligned}$$

using $S \otimes S \circ \Delta^{\text{op}} = \Delta \circ S$ in the last step. This proves the compatibility of the inner product with the H -action. The passage to the quotient follows immediately from Remark 2.7. \square

Thus, we can define the H -covariant internal tensor product of $\mathcal{F}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ to be the right \mathcal{A} -module $\mathcal{F}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} = \mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} / (\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}})^{\perp}$ endowed with its H -action and its H -covariant \mathcal{A} -valued inner product. If $\mathcal{F}_{\mathcal{B}}$ carries in addition an H -covariant $*$ -representation of some $*$ -algebra \mathcal{C} then the induced $*$ -representation of \mathcal{C} on $\mathcal{F}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is again H -covariant. The functoriality of the tensor product of H -actions, i.e. tensor products of intertwiners give intertwiners, together with the functoriality of the internal tensor product of inner products as in [11, Lemma 4.16] finally gives a functor

$$\widehat{\otimes}_{\mathcal{B}}: *\text{-mod}_{\mathcal{B}, H}(\mathcal{C}) \times *\text{-mod}_{\mathcal{A}, H}(\mathcal{B}) \longrightarrow *\text{-mod}_{\mathcal{A}, H}(\mathcal{C}). \tag{3.2}$$

It is easy to see that the usual associativity of the tensor product gives associativity of $\widehat{\otimes}$ up to the usual canonical isomorphism, i.e.

$$(\mathcal{G}_{\mathcal{C}} \widehat{\otimes}_{\mathcal{C}} \mathcal{F}_{\mathcal{B}}) \widehat{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \cong \mathcal{G}_{\mathcal{C}} \widehat{\otimes}_{\mathcal{C}} (\mathcal{F}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}), \tag{3.3}$$

see [11, Lemma 4.5] for the non-covariant case. Since $\widehat{\otimes}$ is compatible with complete positivity of inner products [11, Theorem 4.7] the functor (3.2) restricts to a functor

$$\widehat{\otimes}_{\mathcal{B}}: *\text{-rep}_{\mathcal{B}, H}(\mathcal{C}) \times *\text{-rep}_{\mathcal{A}, H}(\mathcal{B}) \longrightarrow *\text{-rep}_{\mathcal{A}, H}(\mathcal{C}). \tag{3.4}$$

Fixing one of the two arguments of $\widehat{\otimes}$ we get the H -covariant versions of Rieffel induction and the change of the base ring as in [11, Examples 4.9 and 4.10]. The H -covariant Rieffel induction with some ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \in *\text{-rep}_{\mathcal{A}, H}(\mathcal{B})$ is denoted by

$$R_{\mathcal{E}} = {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \widehat{\otimes} \cdot: *\text{-rep}_{\mathcal{D}, H}(\mathcal{A}) \longrightarrow *\text{-rep}_{\mathcal{D}, H}(\mathcal{B}) \tag{3.5}$$

and the H -covariant change of the base ring with some ${}_{\mathcal{D}}\mathcal{G}_{\mathcal{D}'} \in *\text{-rep}_{\mathcal{D}', H}(\mathcal{D})$ is denoted by

$$S_{\mathcal{G}} = \widehat{\otimes}_{\mathcal{D}} \mathcal{G}_{\mathcal{D}'}: *\text{-rep}_{\mathcal{D}, H}(\mathcal{A}) \longrightarrow *\text{-rep}_{\mathcal{D}', H}(\mathcal{A}). \tag{3.6}$$

The functors $R_{\mathcal{E}}$ and $S_{\mathcal{G}}$ commute up to the usual natural transformation induced by (3.3).

3.2. H -covariant strong Morita equivalence

We are now able to adapt the notions of Ara's $*$ -Morita equivalence [1] and strong Morita equivalence [11] to the H -covariant framework. Recall that an inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}}^{\mathcal{E}}$ is *full* if the C -span of the elements $\langle x, y \rangle_{\mathcal{A}}^{\mathcal{E}} \in \mathcal{A}$ gives the whole $*$ -algebra \mathcal{A} and analogously for ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}$.

Definition 3.2. A $(\mathcal{B}, \mathcal{A})$ -bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ with inner products ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{A}}^{\mathcal{E}}$ is called an H -covariant $*$ -equivalence bimodule if it is a $*$ -equivalence bimodule in the sense of Ara [1, Definition 5.1] together with an action of H , such that

$$g \triangleright \langle x, y \rangle_{\mathcal{A}}^{\mathcal{E}} = \langle S(g_{(1)})^* \triangleright x, g_{(2)} \triangleright y \rangle_{\mathcal{A}}^{\mathcal{E}} \quad (3.7)$$

and

$$g \triangleright {}_{\mathcal{B}}\langle x, y \rangle^{\mathcal{E}} = {}_{\mathcal{B}}\langle g_{(1)} \triangleright x, S(g_{(2)})^* \triangleright y \rangle^{\mathcal{E}} \quad (3.8)$$

for all $x, y \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ and $g \in H$. It is called an H -covariant strong equivalence bimodule if in addition the underlying $*$ -equivalence bimodule is a strong equivalence bimodule in the sense of Bursztyn and Waldmann [11, Definition 5.1], i.e. the inner products are both completely positive.

Recall that ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is a $*$ -equivalence bimodule in Ara's sense if both inner products are non-degenerate, full and satisfy the compatibility conditions

$$\begin{aligned} \langle b \cdot x, y \rangle_{\mathcal{A}}^{\mathcal{E}} &= \langle x, b^* \cdot y \rangle_{\mathcal{A}}^{\mathcal{E}}, & {}_{\mathcal{B}}\langle x \cdot a, y \rangle^{\mathcal{E}} &= {}_{\mathcal{B}}\langle x, y \cdot a^* \rangle^{\mathcal{E}} \quad \text{and} \\ {}_{\mathcal{B}}\langle x, y \rangle^{\mathcal{E}} \cdot z &= x \cdot \langle y, z \rangle_{\mathcal{A}}^{\mathcal{E}} \end{aligned} \quad (3.9)$$

and if ${}_{\mathcal{B}} \cdot \mathcal{E} = \mathcal{E} = \mathcal{E} \cdot \mathcal{A}$, see [1, Section 5.1] for details. The compatibility (3.9) can also be interpreted as ${}_{\mathcal{B}}\langle x, y \rangle^{\mathcal{E}} \cdot z = \Theta_{x,y}(z)$. Moreover, (3.7) and (3.8) imply by Lemma 2.4 and the non-degeneracy of the inner products that the bimodule structure is compatible with the H -action, i.e.

$$g \triangleright (b \cdot x) = (g_{(1)} \triangleright b) \cdot (g_{(2)} \triangleright x) \quad \text{and} \quad g \triangleright (x \cdot a) = (g_{(1)} \triangleright x) \cdot (g_{(2)} \triangleright a). \quad (3.10)$$

Thus $({}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}, \langle \cdot, \cdot \rangle_{\mathcal{A}}^{\mathcal{E}})$ is an H -covariant $*$ -representation of \mathcal{B} on an H -covariant inner product right \mathcal{A} -module and analogously for exchanged roles of \mathcal{A} and \mathcal{B} .

Definition 3.3. Two $*$ -algebras \mathcal{A} and \mathcal{B} with $*$ -action of H are called H -covariantly $*$ -Morita equivalent (resp., H -covariantly strongly Morita equivalent) if there exists an H -covariant $*$ -Morita (resp., strong Morita) equivalence bimodule for them.

Clearly, H -covariant $*$ - or strong Morita equivalence implies $*$ - or strong Morita equivalence, respectively, and H -covariant strong Morita equivalence implies H -covariant $*$ -Morita equivalence. Moreover, as expected, H -covariant $*$ - as well as strong Morita equivalence turn out to be equivalence relations when applied to non-degenerate and idempotent $*$ -algebras. This restriction is necessary according to [1, 11].

Theorem 3.4. *Within the class of idempotent and non-degenerate $*$ -algebras with $*$ -actions of H , H -covariant $*$ - or strong Morita equivalence are both equivalence relations. Moreover, H -equivariantly $*$ -isomorphic $*$ -algebras are H -covariantly strongly Morita equivalent and hence also H -covariantly $*$ -Morita equivalent.*

Proof. We already know that the underlying $*$ - or strong Morita equivalence is an equivalence relation where the bimodule ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ with the canonical inner products

$$\langle a, b \rangle_{\mathcal{A}} = a^*b \quad \text{and} \quad {}_{\mathcal{A}}\langle a, b \rangle = ab^* \tag{3.11}$$

gives reflexivity. The complex conjugate bimodule ${}_{\mathcal{A}}\overline{\mathcal{E}}_{\mathcal{B}}$, see Remark 2.2, gives symmetry. Finally, the internal tensor product $\widehat{\otimes}$ gives transitivity, see [1,11]. Thus it remains to show that the three constructions are compatible with the H -covariance. Clearly, the bimodule structure on ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ is H -covariant and we have

$$\begin{aligned} g \triangleright \langle a, b \rangle_{\mathcal{A}} &= g \triangleright (a^*b) = (g_{(1)} \triangleright a^*)(g_{(2)} \triangleright b) = (S(g_{(1)})^* \triangleright a)^*(g_{(2)} \triangleright b) \\ &= \langle S(g_{(1)})^* \triangleright a, g_{(2)} \triangleright b \rangle_{\mathcal{A}} \end{aligned}$$

and similarly for ${}_{\mathcal{A}}\langle \cdot, \cdot \rangle$. Thus the inner products on ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ are H -covariant which proves reflexivity. On ${}_{\mathcal{A}}\overline{\mathcal{E}}_{\mathcal{B}}$ we have already constructed the candidate for the H -action in Remark 2.12. A simple computation shows that \triangleright is compatible with the \mathcal{B} -valued inner product as well. This follows immediately from the compatibility (3.9) or from a straightforward direct computation. Finally, transitivity follows from our considerations in Lemma 3.1 where we have already shown that $\widehat{\otimes}$ is compatible with H -actions. Note however, that now we have to check the compatibility with two inner products, which can be done in a completely analogous way as for one. Thus, H -covariant $*$ - or strong Morita equivalence is an equivalence relation. Let us finally consider a $*$ -isomorphism $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ such that Φ is H -equivariant, i.e. $\Phi(g \triangleright a) = g \triangleright \Phi(a)$ for all $a \in \mathcal{A}$ and $g \in H$. Then we claim that ${}_{\mathcal{B}}\mathcal{B}_{\mathcal{A}}^{\Phi}$ is an H -covariant strong Morita equivalence bimodule, where the right \mathcal{A} -module structure on \mathcal{B} is defined by $b \cdot_{\Phi} a = b\Phi(a)$ and the \mathcal{A} -valued inner product is $\langle b_1, b_2 \rangle_{\mathcal{A}}^{\Phi} = \Phi^{-1}(b_1^*b_2)$. Again, we only have to check the H -covariance which is a simple computation. \square

Remark 3.5. From now on we shall always assume that the $*$ -algebras in question are idempotent and non-degenerate since otherwise Morita theory becomes somewhat pathological as Morita equivalence no longer defines a reflexive relation.

As we shall need the tensor product of equivalence bimodules throughout this article, we introduce a new notation: For two equivalence bimodules (either H -covariant $*$ - or strong Morita equivalence) ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ we denote their internal tensor product by ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ to stress that now *two* inner products are involved. From [11, Lemma 5.7] we know that the degeneracy spaces of the two inner products on the algebraic tensor product ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ coincide if each of the bimodules is an equivalence bimodule. This is a simple consequence of (3.9). Thus dividing by the degeneracy space is non-ambiguous. It is clear that $\widehat{\otimes}$ enjoys analogous functoriality properties as $\widehat{\otimes}$.

Let us now discuss some basic consequences of H -covariant $*$ - or strong Morita equivalence:

Proposition 3.6. *Let \mathcal{A}, \mathcal{B} be non-degenerate and idempotent $*$ -algebras over \mathbb{C} and let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be an H -covariant $*$ -Morita equivalence bimodule. Then \mathcal{B} is canonically $*$ -isomorphic to $\mathfrak{F}(\mathcal{E}_{\mathcal{A}})$ via the action map*

$$\mathcal{B} \ni b \mapsto (x \mapsto b \cdot x) \in \mathfrak{F}(\mathcal{E}_{\mathcal{A}}) \quad (3.12)$$

and the $*$ -action of H on \mathcal{B} corresponds under (3.12) to the adjoint action on $\mathfrak{F}(\mathcal{E}_{\mathcal{A}})$ induced by the action on \mathcal{E} . In particular, if \mathcal{B} is unital then $\mathcal{B} \cong \mathfrak{F}(\mathcal{E}_{\mathcal{A}}) = \mathfrak{B}(\mathcal{E}_{\mathcal{A}})$. Conversely, if $\mathcal{E}_{\mathcal{A}}$ is a right \mathcal{A} -module with H -action and compatible full inner product $\langle \cdot, \cdot \rangle_{\mathcal{E}_{\mathcal{A}}}$, such that $\mathcal{E}_{\mathcal{A}} = \mathcal{E}_{\mathcal{A}} \cdot \mathcal{A}$ then the $*$ -algebra $\mathfrak{F}(\mathcal{E}_{\mathcal{A}})$, equipped with the adjoint action of H , is H -covariantly $*$ -Morita equivalent to \mathcal{A} via $\mathfrak{F}(\mathcal{E}_{\mathcal{A}})\mathcal{E}_{\mathcal{A}}$.

Proof. The non-covariant part of this proposition is well-known, see Ara's work [1] as well as the discussion in [11]. Thus, we only have to determine the H -action induced on $\mathfrak{F}(\mathcal{E}_{\mathcal{A}})$ by the isomorphism (3.12). Since

$$g \triangleright (b \cdot x) = (g_{(1)} \triangleright b) \cdot (g_{(2)} \triangleright x)$$

by compatibility, we see by Proposition 2.8 that this is precisely the defining property of the adjoint action. The other direction also follows directly from this observation. \square

Remark 3.7. It follows from the proposition that the maps $b \mapsto (x \mapsto b \cdot x)$ as well as $a \mapsto (x \mapsto x \cdot a)$ are *injective* for an equivalence bimodule.

Remark 3.8. The case of H -covariant strong Morita equivalence is analogous with the only additional requirement that both inner products are completely positive.

Example 3.9. As usual the standard example is the Morita equivalence of \mathcal{A} and $M_n(\mathcal{A})$ via the bimodule \mathcal{A}^n where \mathcal{A} acts componentwisely from the right and $M_n(\mathcal{A})$ acts by matrix multiplication from the left. The canonical, completely positive, full and non-degenerate inner product is

$$\langle x, y \rangle_{\mathcal{A}} = \sum_{i=1}^n x_i^* y_i, \quad (3.13)$$

which determines ${}_{M_n(\mathcal{A})}\langle \cdot, \cdot \rangle$ by compatibility (3.9). The H -action on \mathcal{A}^n is componentwise and the induced $*$ -action on $M_n(\mathcal{A}) = \mathfrak{F}(\mathcal{A}^n_{\mathcal{A}})$ is just the one from Example 2.3(iii). Thus we get the H -covariant strong Morita equivalence of \mathcal{A} and $M_n(\mathcal{A})$.

One of the original aims of Morita theory is to establish the equivalence of representation theories. In our case this is based on the following observation inspired by Bursztyn and Waldmann [11, Lemmas 5.13 and 5.14]:

Proposition 3.10. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be idempotent and non-degenerate $*$ -algebras with $*$ -actions of H . Let ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be H -covariant $*$ -equivalence bimodules and let ${}_{\mathcal{A}}\mathcal{H}_{\mathcal{D}} \in {}^* \text{-Mod}_{\mathcal{D}, H}(\mathcal{A})$ be a strongly non-degenerate $*$ -representation of \mathcal{A} , such that in addition ${}_{\mathcal{A}}\mathcal{H}_{\mathcal{D}} \cdot \mathcal{D} = {}_{\mathcal{A}}\mathcal{H}_{\mathcal{D}}$.*

(i) *One has*

$$({}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \widetilde{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}) \widehat{\otimes}_{\mathcal{A}} {}_{\mathcal{A}}\mathcal{H}_{\mathcal{D}} \cong {}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} ({}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \widehat{\otimes}_{\mathcal{A}} {}_{\mathcal{A}}\mathcal{H}_{\mathcal{D}}) \tag{3.14}$$

via the usual natural H -covariant isometric isomorphism.

(ii) *One has*

$${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \widehat{\otimes}_{\mathcal{A}} {}_{\mathcal{A}}\mathcal{H}_{\mathcal{D}} \cong {}_{\mathcal{A}}\mathcal{H}_{\mathcal{D}} \cong {}_{\mathcal{A}}\mathcal{H}_{\mathcal{D}} \widehat{\otimes}_{\mathcal{D}} \mathcal{D}_{\mathcal{D}} \tag{3.15}$$

via the canonical H -covariant isometric isomorphisms $a \otimes x \mapsto a \cdot x$ and $x \otimes d \mapsto x \cdot d$, respectively.

(iii) *One has*

$${}_{\mathcal{A}}\overline{\mathcal{E}}_{\mathcal{B}} \widetilde{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \cong {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \quad \text{and} \quad {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \widetilde{\otimes}_{\mathcal{A}} \overline{\mathcal{E}}_{\mathcal{B}} \cong {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}} \tag{3.16}$$

via the natural H -covariant isometric isomorphisms $\bar{x} \otimes y \mapsto \langle x, y \rangle_{\mathcal{A}}^{\mathcal{E}}$ and $y \otimes \bar{x} \mapsto {}_{\mathcal{B}}\langle y, x \rangle^{\mathcal{E}}$, respectively.

Proof. The only thing to be checked is that the isomorphisms are compatible with the H -actions. The remaining properties were already shown in [11, Lemmas 5.13, 5.14]. The compatibility for the first part is contained in (3.3). The action on $\mathcal{A} \otimes \mathcal{H}$ is by definition $g \triangleright (a \otimes x) = g_{(1)} \triangleright a \otimes g_{(2)} \triangleright x$ whence $g \triangleright (a \otimes x)$ is mapped to $(g_{(1)} \triangleright a) \cdot (g_{(2)} \triangleright x) = g \triangleright (a \cdot x)$ under the isomorphism (3.15). This shows the second part as the argument for ${}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}$ is analogous. For the third part recall that the action on the complex conjugate bimodule is $g \triangleright \bar{x} = \overline{S(g)^* \triangleright x}$ whence the action on $\overline{\mathcal{E}} \otimes \mathcal{E}$ is given by $g \triangleright (\bar{x} \otimes y) = \overline{S(g_{(1)})^* \triangleright x} \otimes g_{(2)} \triangleright y$. Thus $g \triangleright (\bar{x} \otimes y)$ is mapped under (3.16) to $\langle S(g_1)^* \triangleright x, g_{(2)} \triangleright y \rangle_{\mathcal{A}}^{\mathcal{E}} = g \triangleright \langle x, y \rangle_{\mathcal{A}}^{\mathcal{E}}$ by H -covariance of the inner product showing the H -covariance of the first isomorphism. The H -covariance of the second isomorphism in (3.16) is analogous. \square

Corollary 3.11. *For equivalence bimodules ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ there is a natural equivalence*

$$R_{\mathcal{F}} \circ R_{\mathcal{E}} \cong R_{\mathcal{F} \widetilde{\otimes}_{\mathcal{B}} \mathcal{E}} \tag{3.17}$$

for the H -covariant Rieffel induction functors. Furthermore, when restricted to $$ -Mod (or $*$ -Rep in the completely positive case, respectively) there are natural equivalences*

$$R_{\mathcal{A}} \cong \text{id}_{* \text{-Mod}(\mathcal{A})}, \tag{3.18}$$

$$R_{\mathcal{E}} \circ R_{\overline{\mathcal{E}}} \cong \text{id}_{* \text{-Mod}(\mathcal{B})} \quad \text{and} \quad R_{\overline{\mathcal{E}}} \circ R_{\mathcal{E}} \cong \text{id}_{* \text{-Mod}(\mathcal{A})} \tag{3.19}$$

for the H -covariant Rieffel induction functors. Analogous statements hold for the functor $S_{\mathcal{E}}$.

Corollary 3.12. *Let \mathcal{A}, \mathcal{B} be H -covariantly $*$ -Morita equivalent via ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$. Then*

$$R_{\mathcal{E}}: {}^* \text{-Mod}_{\mathcal{D}, H}(\mathcal{A}) \xrightarrow{\cong} {}^* \text{-Mod}_{\mathcal{D}, H}(\mathcal{B}) \quad (3.20)$$

is an equivalence of categories with ‘inverse’ $R_{\overline{\mathcal{E}}}$. If in addition ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is even an H -covariant strong Morita equivalence bimodule, then $R_{\mathcal{E}}$ restricts to an equivalence

$$R_{\mathcal{E}}: {}^* \text{-Rep}_{\mathcal{D}, H}(\mathcal{A}) \xrightarrow{\cong} {}^* \text{-Rep}_{\mathcal{D}, H}(\mathcal{B}). \quad (3.21)$$

This is the H -covariant version of [11, Corollary 5.15] which itself is the algebraic generalization of Rieffel’s theorem on equivalent $*$ -representation theories of C^* -algebras [41]. In the case of C^* -algebras and strongly continuous group actions of locally compact groups analogous statements are well-known, see e.g. the discussion in [39, Section 7.2]. It is an interesting problem whether and how one can use our purely algebraic approach to obtain those results. We will address these questions in future projects.

4. The H -covariant Picard groupoid and Morita invariants

As already mentioned, Morita theory can be seen as resulting from an extended notion of morphisms between algebras: one considers isomorphism classes of bimodules as morphisms and obtains a new category with the same underlying class of objects but bigger classes of morphisms. Isomorphism in this category is then precisely Morita equivalence. This point of view is classical for ring-theoretic Morita equivalence, see e.g. [4,6], and it was discussed in detail for the $*$ - and strong Morita equivalence of $*$ -algebras in [11, Section 6]. See also Landsman’s work [33,34] in the context of C^* -algebras. Alternatively, one could use a bicategorical approach by not identifying the bimodules up to isomorphism in a first step.

Important for us is that any such enlarged category defines its groupoid of invertible arrows, the corresponding *Picard groupoid*. Strictly speaking, this is not an honest groupoid for two reasons: first the class of units (here the class of $*$ -algebras) is not a set, so it can not be a *small* category. Second, the class of invertible arrows between two units is, a priori, not known to be a set either. This is more severe, but in the case of unital $*$ -algebras one actually can show that the space of arrows between two units in the Picard groupoid forms a set as it is given by equivalence classes of certain *finitely generated projective modules*. Thus, we shall ignore these subtleties in the following and focus mainly on the unital case. In any case, throughout this section all algebras will be idempotent and non-degenerate.

4.1. The H -covariant Picard groupoid

Instead of defining the Picard groupoid in the above described way, we give a more direct definition using the equivalence bimodules directly. Both approaches are completely equivalent which can easily be obtained from an H -covariant version of Bursztyn and Waldmann [11, Theorem 6.1] using Proposition 3.6.

Definition 4.1. Let \mathcal{A}, \mathcal{B} be $*$ -algebras over \mathbb{C} and define $\text{Pic}_H^*(\mathcal{B}, \mathcal{A})$ to be the class of isomorphism classes of H -covariant $*$ -Morita equivalence bimodules ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ and set $\text{Pic}_H^*(\mathcal{A}) = \text{Pic}_H^*(\mathcal{A}, \mathcal{A})$. Similarly, we define $\text{Pic}_H^{\text{str}}(\mathcal{B}, \mathcal{A})$ to be the class of isomorphism classes of H -covariant strong Morita equivalence bimodules ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ and set $\text{Pic}_H^{\text{str}}(\mathcal{A}) = \text{Pic}_H^{\text{str}}(\mathcal{A}, \mathcal{A})$.

Here and in the following ‘isomorphism’ of equivalence bimodules includes all relevant structures, i.e. the H -action, the bimodule structure as well as the inner products.

Theorem 4.2. Viewing $\text{Pic}_H^*(\mathcal{B}, \mathcal{A})$ as space of arrows $\mathcal{A} \rightarrow \mathcal{B}$ one obtains the H -covariant $*$ -Picard groupoid Pic_H^* , where the composition law is $\tilde{\otimes}$, the units are the $*$ -algebras themselves with the classes of the canonical bimodules $[{}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}]$ as unit arrows. The inverse arrows are the classes of the complex conjugated bimodules. Similarly, one obtains the H -covariant strong Picard groupoid $\text{Pic}_H^{\text{str}}$.

The proof is obvious by use of Proposition 3.10 and the fact that $\tilde{\otimes}$ is functorial and hence well-defined on isomorphism classes.

Remark 4.3. For unital $*$ -algebras $\text{Pic}_H^*(\mathcal{B}, \mathcal{A})$ as well as $\text{Pic}_H^{\text{str}}(\mathcal{B}, \mathcal{A})$ are in bijection to certain finitely generated projective modules and hence they are sets. Thus Pic_H^* as well as $\text{Pic}_H^{\text{str}}$ become ‘large’ groupoids in this case. For non-unital $*$ -algebras this is a priori not clear. Dropping the information about the inner products one obtains the ring-theoretic notions of the Picard groupoid which we denote by Pic_H and Pic , respectively, see also [4,6].

The isotropy groups $\text{Pic}_H^{\text{str}}(\mathcal{A})$ and $\text{Pic}_H^*(\mathcal{A})$, respectively, of the Picard groupoids are called the H -covariant strong (resp., $*$ -) Picard groups of \mathcal{A} .

By successively forgetting the additional structures one obtains groupoid morphisms:

$$\begin{array}{ccccc}
 \text{Pic}_H^{\text{str}} & \xrightarrow{\quad} & & \xrightarrow{\quad} & \text{Pic}_H^* \\
 & \searrow & & \swarrow & \\
 & & \text{Pic}_H & & \\
 & \swarrow & & \searrow & \\
 \text{Pic}^{\text{str}} & \xrightarrow{\quad} & & \xrightarrow{\quad} & \text{Pic}^* \\
 & \searrow & & \swarrow & \\
 & & \text{Pic} & &
 \end{array}
 \tag{4.1}$$

Each of them induces the identity on the units of the groupoids. Clearly, all combinations of possible compositions commute in this diagram. Thus an interesting program will be to investigate under which reasonable restrictions and conditions on the $*$ -algebras and the H -actions one can say something on the images and kernels of these groupoid morphisms. In the situation without H -action the lower commuting triangle in (4.1) has been investigated in some detail in [11] for a large class of unital $*$ -algebras.

Before investigating (4.1) we relate the Picard groupoid to the *isomorphism groupoid* as we want to interpret the elements in $\text{Pic}_H^*(\mathcal{B}, \mathcal{A})$ and $\text{Pic}_H^{\text{str}}(\mathcal{B}, \mathcal{A})$ as generalized isomorphisms of $*$ -algebras. We denote by $\text{Iso}_H^*(\mathcal{B}, \mathcal{A})$ the H -equivariant $*$ -isomorphisms from \mathcal{A} to \mathcal{B} and set $\text{Aut}_H^*(\mathcal{A}) = \text{Iso}_H^*(\mathcal{A}, \mathcal{A})$ for the H -equivariant $*$ -automorphisms of \mathcal{A} . The non-equivariant case is denoted by $\text{Iso}^*(\mathcal{B}, \mathcal{A})$ and $\text{Aut}^*(\mathcal{A})$, respectively.

Remark 4.4. Viewing $\text{Iso}_H^*(\mathcal{B}, \mathcal{A})$ as space of arrows from \mathcal{A} to \mathcal{B} one obtains the usual (large) groupoid of H -equivariant $*$ -isomorphisms with the H -equivariant $*$ -automorphism groups as isotropy groups.

Let us first recall and adapt some results and definitions from [12, Section 2]. Let $\Phi \in \text{Iso}^*(\mathcal{B}, \mathcal{A})$ be given and let ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}$ be a representative for a class $[{}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}] \in \text{Pic}^*(\mathcal{C}, \mathcal{B})$ (or in $\text{Pic}^{\text{str}}(\mathcal{C}, \mathcal{B})$, respectively), Then we can twist \mathcal{F} into a right \mathcal{A} -module by setting

$$x \cdot \phi a = x \cdot \Phi(a) \tag{4.2}$$

for $x \in \mathcal{F}$ and $a \in \mathcal{A}$. Clearly, this gives a $(\mathcal{C}, \mathcal{A})$ -bimodule, denoted by ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{A}}^{\Phi}$. Moreover, we define

$$\langle x, y \rangle_{\mathcal{A}}^{\mathcal{F}^{\Phi}} = \Phi^{-1} \left(\langle x, y \rangle_{\mathcal{B}}^{\mathcal{F}} \right). \tag{4.3}$$

Since Φ is a $*$ -isomorphism, this gives a full and non-degenerate \mathcal{A} -valued inner product on ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{A}}^{\Phi}$ (completely positive in the case of a strong equivalence bimodule) which is compatible with the \mathcal{C} -module structure and with the \mathcal{C} -valued inner product on \mathcal{F} . Thus we obtain a $*$ -, respectively, strong Morita equivalence bimodule ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{A}}^{\Phi}$. A last simple check ensures that the class $[{}_{\mathcal{C}}\mathcal{F}_{\mathcal{A}}^{\Phi}]$ only depends on the class $[{}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}]$. Similarly, we can twist equivalence bimodules ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ from the left with some $\Psi \in \text{Iso}^*(\mathcal{C}, \mathcal{B})$ by setting

$$c \cdot \psi x = \Psi^{-1}(c) \cdot x \quad \text{and} \quad {}_{\mathcal{C}}\langle x, y \rangle^{\Psi \mathcal{E}} = \Psi({}_{\mathcal{B}}\langle x, y \rangle^{\mathcal{E}}) \tag{4.4}$$

and obtain an equivalence bimodule ${}_{\mathcal{C}}\mathcal{E}_{\mathcal{A}}^{\Psi}$. Again, this works either for $*$ -equivalence or strong equivalence bimodules. The H -covariant situation is as follows:

Lemma 4.5. *Let $\Phi \in \text{Iso}^*(\mathcal{B}, \mathcal{A})$ and let $[{}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}] \in \text{Pic}_H^*(\mathcal{C}, \mathcal{B})$ (or in $\text{Pic}_H^{\text{str}}(\mathcal{B}, \mathcal{A})$, respectively). Then ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{A}}^{\Phi}$ is an H -covariant $*$ - (or strong, respectively) equivalence bimodule if and only if Φ is H -equivariant. In this case $[{}_{\mathcal{C}}\mathcal{F}_{\mathcal{A}}^{\Phi}] \in \text{Pic}_H^*(\mathcal{C}, \mathcal{A})$ (or in $\text{Pic}_H^{\text{str}}(\mathcal{C}, \mathcal{A})$, respectively) is well-defined.*

Proof. If Φ is H -equivariant, then it follows from a simple computation that the twisted bimodule is an H -covariant equivalence bimodule as well. For the converse direction, we note that if the twisted bimodule is still H -covariant then

$$\begin{aligned} x \cdot \Phi(g \triangleright a) &= x \cdot \phi(g \triangleright a) = g_{(2)} \triangleright ((S^{-1}(g_{(2)}) \triangleright x) \cdot \phi a) \\ &= g_{(2)} \triangleright ((S^{-1}(g_{(2)}) \triangleright x) \cdot \Phi(a)) = x \cdot (g \triangleright \Phi(a)). \end{aligned}$$

Thanks to Remark 3.7 the H -equivariance of Φ follows. The well-definedness of the class is clear. \square

The analogous statement holds if we twist with an H -equivariant $*$ -isomorphism from the left. In particular, we can twist the ‘unit’ bimodules (as we have already done implicitly in the proof of Theorem 3.4) continuing the discussion of Bursztyn and Waldmann [12] as well as the well-known ring-theoretic case.

Proposition 4.6. *Let $\Phi \in \text{Iso}_H^*(\mathcal{B}, \mathcal{A})$ and denote by $\ell(\Phi) \in \text{Pic}_H^{\text{str}}(\mathcal{B}, \mathcal{A})$ the class of the bimodule ${}_{\mathcal{B}}\mathcal{B}_{\mathcal{A}}^{\Phi}$.*

- (i) *We have $[{}_{\mathcal{B}}\mathcal{B}_{\mathcal{A}}^{\Phi}] = [{}_{\mathcal{B}}\mathcal{A}_{\mathcal{A}}^{\Phi}]$.*
- (ii) *The map*

$$\ell : \text{Iso}_H^* \longrightarrow \text{Pic}_H^{\text{str}} \tag{4.5}$$

is a groupoid morphism inducing the identity on the units.

- (iii) *For $[{}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}] \in \text{Pic}_H^{\text{str}}(\mathcal{C}, \mathcal{B})$ we have*

$$[{}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}] \widetilde{\otimes}_{\mathcal{B}} \ell(\Phi) = [{}_{\mathcal{C}}\mathcal{F}_{\mathcal{A}}^{\Phi}]. \tag{4.6}$$

*We can replace strong by *-Picard groupoids as well.*

Proof. The bimodule isomorphism for the first part is simply given by $b \mapsto \Phi^{-1}(b)$. Now let $\Phi \in \text{Iso}_H^*(\mathcal{B}, \mathcal{A})$ and $\Psi \in \text{Iso}_H^*(\mathcal{C}, \mathcal{B})$ be given. Then we consider $\ell(\Psi \circ \Phi)$ and compare it with $\ell(\Psi) \widetilde{\otimes}_{\mathcal{B}} \ell(\Phi)$. We consider the map defined by

$${}_{\mathcal{C}}\mathcal{C}_{\mathcal{B}}^{\Psi} \widetilde{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{B}_{\mathcal{A}}^{\Phi} \ni c \otimes b \mapsto c\Psi(b) \in {}_{\mathcal{C}}\mathcal{C}_{\mathcal{A}}^{\Psi \circ \Phi}.$$

On the level of $\otimes_{\mathcal{B}}$ rather than $\widetilde{\otimes}_{\mathcal{B}}$ it is easy to see that this map is well-defined over $\otimes_{\mathcal{B}}$. Moreover, it is surjective since \mathcal{C} is idempotent. A straightforward check shows that it is a $(\mathcal{C}, \mathcal{A})$ -bimodule morphism isometric with respect to both inner products. Thus the quotient by the degeneracy spaces yields an injective map, well-defined over $\widetilde{\otimes}_{\mathcal{B}}$. Hence we end up with a bimodule isomorphism. A last simple computation using the H -equivariance of Ψ shows that it is even an H -equivariant isomorphism as wanted. This proves the second part as (4.5) clearly maps the unit $\text{id}_{\mathcal{A}}$ to the unit $[{}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}]$. For the last part we check that

$${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \widetilde{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{B}_{\mathcal{A}}^{\Phi} \ni x \otimes b \mapsto x \cdot b \in {}_{\mathcal{C}}\mathcal{F}_{\mathcal{A}}^{\Phi}$$

is the desired isomorphism. This is again a straightforward computation. In the proof the positivity of the inner products was not essential. \square

At least for unital *-algebras one can describe the *kernel* of the groupoid morphism (4.5) rather explicitly. We slightly extend and specialize the arguments from [12, Proposition 2.3] for our purposes. First, we denote by $\text{InnAut}_H^*(\mathcal{A})$ those *inner *-automorphisms* $a \mapsto uau^{-1}$ where $u^* = u^{-1}$ is unitary and H -invariant $g \triangleright u = \varepsilon(g)u$. Clearly, $\text{InnAut}_H^*(\mathcal{A}) \subseteq \text{Aut}_H^*(\mathcal{A})$ is a normal subgroup.

Proposition 4.7. *Let \mathcal{A}, \mathcal{B} be unital *-algebras.*

- (i) *For $\Phi \in \text{Aut}_H^*(\mathcal{B})$ and $[{}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}] \in \text{Pic}_H^{\text{str}}(\mathcal{B}, \mathcal{A})$ we have $[{}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}^{\Phi}] = [{}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}]$ if and only if $\Phi \in \text{InnAut}_H^*(\mathcal{B})$.*

(ii) We have the exact sequence of groups

$$1 \longrightarrow \text{InnAut}_H^*(\mathcal{A}) \longrightarrow \text{Aut}_H^*(\mathcal{A}) \xrightarrow{\ell} \text{Pic}_H^{\text{str}}(\mathcal{A}) \tag{4.7}$$

Again, strong can be replaced by *-Picard groupoids.

Proof. Assume $U : \mathcal{B}^{\mathcal{E}}_{\mathcal{A}} \rightarrow \mathcal{B}^{\mathcal{E}}_{\mathcal{A}}$ is an isomorphism. Then $U(x \cdot a) = U(x) \cdot a$ implies that there exists an invertible element $u \in \mathcal{B}$ with $U(x) = u \cdot x$ thanks to Proposition 3.6 and since \mathcal{B} is unital. Then $b \cdot u \cdot x = b \cdot U(x) = U(b \cdot_{\Phi} x) = u \cdot \Phi^{-1}(b) \cdot x$ implies $\Phi(b) = ubu^{-1}$ thanks to Remark 3.7. Note that Φ being a *-automorphism does not necessarily imply that u is unitary. Nevertheless, we have by isometry of U

$$u_{\mathcal{B}}\langle x, y \rangle^{\mathcal{E}} u^* = \mathcal{B}\langle u \cdot x, u \cdot y \rangle^{\mathcal{E}} = \mathcal{B}\langle U(x), U(y) \rangle^{\mathcal{E}} = \mathcal{B}\langle x, y \rangle^{\Phi \mathcal{E}} = \Phi(\mathcal{B}\langle x, y \rangle^{\mathcal{E}}),$$

whence by fullness $\Phi(b) = ubu^*$. Thus $u^* = u^{-1}$ turns out to be unitary. Finally, we have from $g \triangleright U(x) = U(g \triangleright x)$ the relation

$$\begin{aligned} (g \triangleright u) \cdot x &= g_{(1)} \triangleright (u \cdot S(g_{(2)}) \triangleright x) = g_{(1)} \triangleright (U(S(g_{(2)}) \triangleright x)) \\ &= U(g_{(1)} \triangleright (S(g_{(2)}) \triangleright x)) = \varepsilon(g)u \cdot x. \end{aligned}$$

Again by Remark 3.7 we see $g \triangleright u = \varepsilon(g)u$. This proves the first statement as the converse is a trivial computation. The second part is then an easy consequence if we apply the first part to $\mathcal{B} = \mathcal{A}$ and $\mathcal{B}^{\mathcal{E}}_{\mathcal{A}} = \mathcal{A} \mathcal{A} \mathcal{A}$. \square

From this proposition we see that $\text{Pic}_H^{\text{str}}$ as well as Pic_H^* indeed generalize Iso_H^* in a very precise way. Though the kernel of ℓ in (4.5) can be described by this proposition explicitly, the lack of surjectivity usually depends very much on the example.

4.2. The groupoid morphism $\text{Pic}_H^{\text{str}} \rightarrow \text{Pic}^{\text{str}}$

We shall now discuss the canonical groupoid morphism of ‘forgetting’ the H -covariance

$$\text{Pic}_H^{\text{str}} \rightarrow \text{Pic}^{\text{str}}, \tag{4.8}$$

where we treat the case of the strong Picard groupoids. For the *-Picard groupoids and the ring-theoretic Picard groupoids the results will be analogous.

In general, the question whether $\text{Pic}_H^{\text{str}}(\mathcal{B}, \mathcal{A}) \rightarrow \text{Pic}^{\text{str}}(\mathcal{B}, \mathcal{A})$ is surjective for two given *-algebras with *-action of H is very difficult and depends very much on the example. The problem is to ‘lift’ the action of H from the algebras to an equivalence bimodule $\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}$. In general, there will be obstructions for this lifting.

Example 4.8. Coming back to the Examples 2.9 and 2.10, we can see that for $\mathcal{A} = C^\infty(M)$ any equivalence bimodule is of the form $\Gamma^\infty(E)$, where $E \rightarrow M$ is a complex vector bundle and $C^\infty(M)$ acts by pointwise multiplication from the right. Then the Morita equivalent algebra \mathcal{B} is isomorphic to $\Gamma^\infty(\text{End}(E))$, where the action is by pointwise application of the endomorphism. Now if a Lie algebra action of \mathfrak{g} on M is given, then the question is whether one can lift this action to an action on the sections of E . In general, there are obstructions:

consider a Lie group G with action of its Lie algebra \mathfrak{g} by left-invariant vector fields. Suppose $E \rightarrow G$ is a complex vector bundle admitting a lifted action $\xi \mapsto \mathcal{L}_\xi$ of \mathfrak{g} , i.e. $\mathcal{L}_\xi : \Gamma^\infty(E) \rightarrow \Gamma^\infty(E)$ is a representation of \mathfrak{g} and satisfies $\mathcal{L}_\xi(sf) = \mathcal{L}_\xi(s)f + sX_\xi(f)$, for all $\xi \in \mathfrak{g}$, where X_ξ is the corresponding left-invariant vector field on G . Since for a basis e_1, \dots, e_n of \mathfrak{g} we obtain a module basis X_{e_1}, \dots, X_{e_n} of all vector fields $\Gamma^\infty(TG)$ on G , we define by

$$\nabla_Y = \sum_{i=1}^n Y^i \mathcal{L}_{X_{e_i}} \quad \text{with} \quad Y = \sum_{i=1}^n Y^i X_{e_i} \quad \text{and} \quad Y^i \in C^\infty(G) \tag{4.9}$$

a covariant derivative, which is easily shown to be *flat*. In general, the existence of a flat covariant derivative is a cohomological obstruction on E , unless E is a trivial vector bundle.

The question about injectivity is in how many ways such a lifting can be done. Surprisingly, there is a general answer to this question which is even independent on the particular bimodule but universal for all bimodules $\mathcal{B}^{\mathcal{E}} \mathcal{A}$ as long as they allow for lifting at all.

In the following we fix a strong Morita equivalence bimodule $\mathcal{B}^{\mathcal{E}} \mathcal{A}$ and assume that there is at least one H -action \triangleright on $\mathcal{B}^{\mathcal{E}} \mathcal{A}$ such that it becomes an H -covariant strong Morita equivalence bimodule. If \triangleright' is another such H -action then we define

$$u_g(x) = g_{(1)} \triangleright (S(g_{(2)}) \triangleright' x) \tag{4.10}$$

to ‘measure’ the difference between the two actions, where $x \in \mathcal{B}^{\mathcal{E}} \mathcal{A}$. Knowing \triangleright and all the maps $g \mapsto u_g \in \text{End}_{\mathbb{C}}(\mathcal{B}^{\mathcal{E}} \mathcal{A})$ allows to reconstruct \triangleright' by

$$g \triangleright' x = g_{(2)} \triangleright (u_{S^{-1}(g_{(1)})}(x)) \tag{4.11}$$

and conversely \triangleright is determined by

$$g \triangleright x = u_{g_{(1)}}(g_{(2)} \triangleright' x). \tag{4.12}$$

Thus we have to investigate the maps u_g and find conditions, such that for a given H -action, say \triangleright' , the formula (4.12) defines again an H -action with the same properties.

Lemma 4.9. *Let $(\mathcal{B}^{\mathcal{E}} \mathcal{A}, \triangleright')$ be an H -covariant strong Morita equivalence bimodule and let $H \ni g \mapsto u_g \in \text{End}_{\mathbb{C}}(\mathcal{B}^{\mathcal{E}} \mathcal{A})$ be a linear map. Then \triangleright defined by (4.12) satisfies $g \triangleright (x \cdot a) = (g_{(1)} \triangleright x) \cdot (g_{(2)} \triangleright a)$ if and only if u_g is right \mathcal{A} -linear for all $g \in H$.*

Proof. First we assume that u_g is right \mathcal{A} -linear. Then

$$\begin{aligned} g \triangleright (x \cdot a) &= u_{g_{(1)}}(g_{(2)} \triangleright' (x \cdot a)) = u_{g_{(1)}}((g_{(2)} \triangleright' x) \cdot (g_{(3)} \triangleright a)) \\ &= (u_{g_{(1)}}(g_{(2)} \triangleright' x)) \cdot (g_{(3)} \triangleright a) = (g_{(1)} \triangleright x) \cdot (g_{(2)} \triangleright a). \end{aligned}$$

For the converse, note first that $g_{(1)} \triangleright (S(g_{(2)}) \triangleright' x) = u_g(x)$ since \triangleright' is an action (whether \triangleright is an action or not). If \triangleright is an action,

$$\begin{aligned} u_g(x \cdot a) &= g_{(1)} \triangleright ((S(g_{(3)}) \triangleright' x) \cdot (S(g_{(2)}) \triangleright a)) = (g_{(1)} \triangleright (S(g_{(3)}) \triangleright' x)) \cdot (\varepsilon(g_{(2)})a) \\ &= u_g(x) \cdot a. \quad \square \end{aligned}$$

Thus, we have to investigate right \mathcal{A} -linear endomorphisms of ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$. Now the crucial observation is that in the *unital* case any right \mathcal{A} -linear endomorphism is a left multiplication by a unique element in \mathcal{B} . To make use of this drastic simplification we shall assume that in this section all $*$ -algebras are unital.

Thus we can rephrase Lemma 4.9 in the following way: if we want to pass from \triangleright to \triangleright' then it will be necessary and sufficient to consider a map u_g of the form

$$u_g(x) = b(g) \cdot x = g_{(1)} \triangleright (S(g_{(2)}) \triangleright' x), \quad (4.13)$$

if we want to keep the compatibility with the right \mathcal{A} -module structure. Here $b \in \text{Hom}_{\mathbb{C}}(H, \mathcal{B})$.

The following proposition clarifies under which conditions on b we stay in the class of H -covariant strong Morita equivalence bimodules.

Proposition 4.10. *Let $({}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}, \triangleright')$ be an H -covariant strong Morita equivalence bimodule and let $b \in \text{Hom}_{\mathbb{C}}(H, \mathcal{B})$. Then for \triangleright defined by*

$$g \triangleright x = b(g_{(1)}) \cdot (g_{(2)} \triangleright' x) \quad (4.14)$$

one has the following properties:

- (i) $\mathbb{1}_H \triangleright x = x$ if and only if $b(\mathbb{1}_H) = \mathbb{1}_{\mathcal{B}}$.
- (ii) \triangleright is an H -action if and only if for all $g, h \in H$

$$b(gh) = b(g_{(1)})(g_{(2)} \triangleright b(h)). \quad (4.15)$$

- (iii) \triangleright is compatible with the left \mathcal{B} -module structure if and only if for all $g \in H$ and $b \in \mathcal{B}$

$$(g_{(1)} \triangleright b)b(g_{(2)}) = b(g_{(1)})(g_{(2)} \triangleright b). \quad (4.16)$$

- (iv) \triangleright is compatible with the inner product ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}$ if and only if b fulfills (4.16) and for all $g \in H$

$$b(g_{(1)})(b(S(g_{(2)}))^*)^* = \varepsilon(g)\mathbb{1}_{\mathcal{B}}. \quad (4.17)$$

If \triangleright fulfills (i)–(iv) then \triangleright is compatible with the inner product $\langle \cdot, \cdot \rangle^{\mathcal{E}}$, too.

Proof. The first part is trivial. For the second we compute under assumption of (4.15)

$$\begin{aligned} g \triangleright (h \triangleright x) &= b(g_{(1)}) \cdot (g_{(2)} \triangleright' (b(h_{(1)}) \cdot (h_{(2)} \triangleright' x))) \\ &= (b(g_{(1)})(g_{(2)} \triangleright b(h_{(1)}))) \cdot (g_{(3)} \triangleright' (h_{(2)} \triangleright' x)) \\ &= b(g_{(1)}h_{(1)}) \cdot ((g_{(2)}h_{(2)}) \triangleright' x) \\ &= (gh) \triangleright x, \end{aligned}$$

using the compatibility of \triangleright' with the module structure as well as (4.15) and that \triangleright' is an action. Conversely, we have $b(g) \cdot x = g_{(1)} \triangleright (S(g_{(2)}) \triangleright' x)$ whether \triangleright is an action or not.

Now, if \triangleright is an action, too, then

$$\begin{aligned} \mathbf{b}(gh) \cdot x &= (g_{(1)}h_{(1)})\triangleright(S(g_{(2)}h_{(2)})\triangleright'x) \\ &= g_{(1)}\triangleright(h_{(1)}\triangleright(S(g_{(2)}h_{(2)})\triangleright'x)) \\ &= \mathbf{b}(g_{(1)}) \cdot (g_{(2)}\triangleright'(b(h_{(1)}) \cdot (h_{(2)}\triangleright'(S(g_{(3)}h_{(3)})\triangleright'x)))) \\ &= (\mathbf{b}(g_{(1)})(g_{(2)}\triangleright\mathbf{b}(h_{(1)}))) \cdot ((g_{(3)}h_{(2)}S(g_{(4)}h_{(3)}))\triangleright'x) \\ &= (\mathbf{b}(g_{(1)})(g_{(2)}\triangleright\mathbf{b}(h))) \cdot x, \end{aligned}$$

whence by Remark 3.7 the second part follows. For the third part we assume (4.16) and compute

$$\begin{aligned} g\triangleright(b \cdot x) &= \mathbf{b}(g_{(1)}) \cdot (g_{(2)}\triangleright'(b \cdot x)) \\ &= (\mathbf{b}(g_{(1)})(g_{(2)}\triangleright b)) \cdot (g_{(3)}\triangleright'x) \\ &= ((g_{(1)}\triangleright b)\mathbf{b}(g_{(2)})) \cdot (g_{(3)}\triangleright'x) \\ &= (g_{(1)}\triangleright b) \cdot (g_{(2)}\triangleright x). \end{aligned}$$

Conversely, assuming \triangleright is compatible with the module structure gives by a similar computation

$$((g_{(1)}\triangleright b)\mathbf{b}(g_{(2)})) \cdot x = (\mathbf{b}(g_{(1)})(g_{(2)}\triangleright b)) \cdot x,$$

whence again by Remark 3.7 the third part follows. For the fourth part, note that (4.16) is necessary by Lemma 2.4 anyway whence we assume (4.16). Then we have

$$\begin{aligned} \mathcal{B}\langle g_{(1)}\triangleright x, S(g_{(2)})^*\triangleright y \rangle^{\mathcal{E}} &= \mathcal{B}\langle \mathbf{b}(g_{(1)}) \cdot (g_{(2)}\triangleright'x), \mathbf{b}(S(g_{(4)}))^* \cdot (S(g_{(3)}))^*\triangleright' y \rangle^{\mathcal{E}} \\ &= \mathbf{b}(g_{(1)})\mathcal{B}\langle g_{(2)}\triangleright'x, S(g_{(3)})^*\triangleright' y \rangle^{\mathcal{E}} (\mathbf{b}(S(g_{(4)}))^*)^* \\ &= \mathbf{b}(g_{(1)})(g_{(2)}\triangleright_{\mathcal{B}}\langle x, y \rangle^{\mathcal{E}}) (\mathbf{b}(S(g_{(3)}))^*)^* \\ &= (g_{(1)}\triangleright_{\mathcal{B}}\langle x, y \rangle^{\mathcal{E}}) \mathbf{b}(g_{(2)}) (\mathbf{b}(S(g_{(3)}))^*)^*. \end{aligned}$$

Now if (4.17) is fulfilled, then the last line gives $g\triangleright_{\mathcal{B}}\langle x, y \rangle^{\mathcal{E}}$ whence \triangleright is compatible with the inner product. Conversely, if \triangleright is compatible, then we obtain from this computation

$$g\triangleright_{\mathcal{B}}\langle x, y \rangle^{\mathcal{E}} = (g_{(1)}\triangleright_{\mathcal{B}}\langle x, y \rangle^{\mathcal{E}}) \mathbf{b}(g_{(2)}) (\mathbf{b}(S(g_{(3)}))^*)^*.$$

Since the inner product is full we can take linear combinations in x and y to get $g\triangleright\mathbb{1}_{\mathcal{B}} = \varepsilon(g)\mathbb{1}_{\mathcal{B}}$ on the left-hand side. Then the right-hand side gives $\varepsilon(g_{(1)})\mathbf{b}(g_{(2)}) (\mathbf{b}(S(g_{(3)}))^*)^*$ whence (4.17) follows. From the compatibility of the two inner products as in (3.9) and the compatibility of one of them with the H -action \triangleright the compatibility of the other with the H -action follows in general. \square

This proposition has now the following easy interpretation in terms of the group $U(H, \mathcal{B})$ as defined in Definition A.1. Clearly, we can exchange the roles of \triangleright and \triangleright' again (only for aesthetic reasons) as we have a bijective correspondence.

Corollary 4.11. *Let $(\mathcal{B}^{\mathcal{E}}, \mathcal{A}, \triangleright)$ be an H -covariant strong Morita equivalence bimodule. Then all other compatible H -actions on \mathcal{E} are parameterized in a unique way by*

elements $b \in U(H, \mathcal{B})$ by

$$g \triangleright^b x = b(g_{(1)}) \cdot (g_{(2)} \triangleright x). \tag{4.18}$$

Proof. The four conditions in Proposition 4.10 are precisely the defining relations for elements in $U(H, \mathcal{B})$, thereby explaining the names of the conditions in Definition A.1. \square

We want to understand in which case two given actions give an isomorphic bimodule and hence the same element in the Picard groupoid. We use some notation from Appendix A.

Lemma 4.12. *Let $\mathcal{B} \mathcal{E} \mathcal{A}$ be a strong Morita equivalence bimodule.*

- (i) *The group of isometric bimodule automorphisms $\text{Aut}(\mathcal{B} \mathcal{E} \mathcal{A})$ of $\mathcal{B} \mathcal{E} \mathcal{A}$ is canonically isomorphic to $U(\mathcal{Z}(\mathcal{B}))$ via*

$$U(\mathcal{Z}(\mathcal{B})) \ni c \mapsto (\Phi_c : x \mapsto c \cdot x) \in \text{Aut}(\mathcal{B} \mathcal{E} \mathcal{A}). \tag{4.19}$$

- (ii) *Assume that $\mathcal{B} \mathcal{E} \mathcal{A}$ allows for compatible H -actions such that it becomes an H -covariant strong Morita equivalence bimodule. Then $\text{Aut}(\mathcal{B} \mathcal{E} \mathcal{A}) = U(\mathcal{Z}(\mathcal{B}))$ acts on the set of such compatible H -actions by*

$$(\Phi, \triangleright) \mapsto \triangleright^\Phi \quad \text{where } g \triangleright^\Phi x = \Phi(g \triangleright \Phi^{-1}(x)). \tag{4.20}$$

Two H -actions $\triangleright, \triangleright'$ define isomorphic H -covariant strong Morita equivalence bimodules if and only if \triangleright and \triangleright' lie in the same $\text{Aut}(\mathcal{B} \mathcal{E} \mathcal{A})$ -orbit.

- (iii) *For $c \in U(\mathcal{Z}(\mathcal{B}))$ we have $\triangleright^{\Phi_c} = \triangleright$ if and only if $c \in U(\mathcal{Z}(\mathcal{B}))^H$.*
- (iv) *Let $b \in U(H, \mathcal{B})$. Then \triangleright and \triangleright^b define isomorphic H -covariant strong Morita equivalence bimodules if and only if $b = \hat{c}$ for some $c \in U(\mathcal{Z}(\mathcal{B}))$ where $\hat{c}(g) = c(g \triangleright c^{-1})$.*

Proof. The first part is obvious as any bimodule homomorphism can be written as the left multiplication with a unique central element of \mathcal{B} . Then the isometry condition implies immediately $c^* = c^{-1}$. For the second part a straightforward computation shows that \triangleright^Φ is indeed a compatible H -action again. The remaining statements are obvious. The third part is clear. The fourth part is then a simple consequence as

$$g \triangleright^{\Phi_c} x = c \cdot (g \triangleright (c^{-1} \cdot x)) = (c(g_{(1)} \triangleright c^{-1})) \cdot (g_{(2)} \triangleright x) = \hat{c}(g_{(1)}) \cdot (g_{(2)} \triangleright x). \quad \square$$

The last ingredient we need to describe the kernel of the groupoid morphism (4.8) is the following statement:

Lemma 4.13. *Let $\mathcal{B} \mathcal{E} \mathcal{A}$ be an H -covariant strong Morita equivalence bimodule.*

- (i) *The group $U(H, \mathcal{B})$ acts transitively and freely on the set of all H -actions which make $\mathcal{B} \mathcal{E} \mathcal{A}$ an H -covariant strong Morita equivalence bimodule by*

$$(b, \triangleright) \mapsto \triangleright^b. \tag{4.21}$$

- (ii) The action of $U(H, \mathcal{B})$ and $U(\mathcal{L}(\mathcal{B}))$ are compatible with the group morphism $U(\mathcal{L}(\mathcal{B})) \rightarrow U(H, \mathcal{B})$, i.e. we have

$$\triangleright^{\hat{c}} = \triangleright^{\Phi_c} \quad (4.22)$$

for all $c \in U(\mathcal{L}(\mathcal{B}))$.

Proof. For the first part we already know that $U(H, \mathcal{B})$ parameterizes the H -actions in a one-to-one correspondence. Thus, we only have to show that (4.21) with \triangleright^b as in (4.18) defines a group action. Let $b, \tilde{b} \in U(H, \mathcal{B})$ be given. Then a straightforward computation shows $\triangleright^{b*\tilde{b}} = (\triangleright^{\tilde{b}})^b$ and $\triangleright^e = \triangleright$, whence the first part follows. The second is obvious from the preceding lemma. \square

According to Lemma 4.12 (iv) the interesting twists \triangleright^b of the action \triangleright are described by the quotient $U_0(H, \mathcal{B}) = U(H, \mathcal{B})/U(\widehat{\mathcal{L}(\mathcal{B})})$, see also (A.8). Thus, we can determine the kernel of the groupoid morphism completely.

Theorem 4.14. For unital $*$ -algebras the kernel of the groupoid morphism (4.8) can be described as follows: let \mathcal{A}, \mathcal{B} be unital $*$ -algebras with $*$ -action of H such that $\text{Pic}^{\text{str}}(\mathcal{B}, \mathcal{A}) \neq \emptyset$. Then we have the following alternatives:

- (i) $\text{Pic}_H^{\text{str}}(\mathcal{B}, \mathcal{A}) = \emptyset$.
- (ii) $\text{Pic}_H^{\text{str}}(\mathcal{B}, \mathcal{A}) \rightarrow \text{im}(\text{Pic}_H^{\text{str}}(\mathcal{B}, \mathcal{A})) \subseteq \text{Pic}^{\text{str}}(\mathcal{B}, \mathcal{A})$ is a principal $U_0(H, \mathcal{B})$ -bundle over the image $\text{im}(\text{Pic}_H^{\text{str}}(\mathcal{B}, \mathcal{A}))$, i.e. the group $U_0(H, \mathcal{B})$ acts freely and transitively (from the left) on the fibers of the projection.

Thus, it is of major importance to understand the group $U_0(H, \mathcal{B})$ for a given $*$ -algebra \mathcal{B} . As shown in the appendix, this group can be quite non-trivial. Note that the first alternative in the theorem may well happen and note also that the image in the second case may not exhaust the whole set $\text{Pic}^{\text{str}}(\mathcal{B}, \mathcal{A})$, see Example 4.8.

The symmetry of the relation ‘ H -covariant strong Morita equivalence’ already suggests that if $\text{Pic}_H^{\text{str}}(\mathcal{B}, \mathcal{A})$ is non-empty then $U_0(H, \mathcal{B}) \cong U_0(H, \mathcal{A})$. This is indeed the case and will be investigated more systematically in Section 5.5.

Finally, note that the Theorem is literally the same for $\text{Pic}_H^{\text{str}}$ and Pic^{str} being replaced by Pic_H^* and Pic^* , respectively, as we have never used the positivity of the inner products. It is also valid in the ring-theoretic situation if one replaces $U_0(H, \mathcal{B})$ by $\text{GL}_0(H, \mathcal{B})$.

Remark 4.15. One of our original motivations was to understand the covariant Morita theory for star products. Using the techniques developed in this section one would like to proceed analogously to [12] in order to understand the covariant strong Picard groupoid of deformed algebras in terms of their classical limits. We address these topics in a project together with Nikolai Neumaier [25].

5. Morita invariants and actions of the Picard groupoid

We shall now use the Picard groupoid in the spirit of [45] to obtain *Morita invariants* (most of which are well-known) as arising from *actions* of $\text{Pic}_H^{\text{str}}$ (or Pic_H^*) on ‘something’. Here an H -covariant, strong Morita invariant is a property \mathcal{P} of $*$ -algebras with a $*$ -action of H such that if \mathcal{A} has this property then any algebra \mathcal{B} which is H -covariantly and strongly Morita equivalent to \mathcal{A} has this property \mathcal{P} as well, see also [30, Definition 18.4] for the ring-theoretic definition.

From this point of view, the Picard groups are the most fundamental Morita invariant as they arise from the Picard groupoid acting on itself by multiplication. Hence (as for any groupoid) the isotropy groups are all isomorphic along an orbit.

5.1. The representation theories

The statements of Corollary 3.11 can be rephrased in the following way, specializing the discussion in [11,45] to the H -covariant situation. Up to natural unitary equivalence the Picard groupoids ‘act’ on the representation theories by Rieffel induction

$$“ R : \text{Pic}_H^*(\mathcal{B}, \mathcal{A}) \times {}^* \text{-Mod}_{\mathcal{D}, H}(\mathcal{A}) \longrightarrow {}^* \text{-Mod}_{\mathcal{D}, H}(\mathcal{B}) ” \quad (5.1)$$

and

$$“ R : \text{Pic}_H^{\text{str}}(\mathcal{B}, \mathcal{A}) \times {}^* \text{-Rep}_{\mathcal{D}, H}(\mathcal{A}) \longrightarrow {}^* \text{-Rep}_{\mathcal{D}, H}(\mathcal{B}) ”, \quad (5.2)$$

where we have of course *not* an honest action as the Rieffel induction functor $R_{\mathcal{E}}$ depends on \mathcal{E} and not only on its class in Pic_H^* (or $\text{Pic}_H^{\text{str}}$, respectively) and the action properties $R_{\mathcal{F}} \circ R_{\mathcal{E}} \cong R_{\mathcal{F} \otimes \mathcal{E}}$ and $R_{\mathcal{A}} \cong \text{id}$ are only fulfilled up to a natural transformation.

Thus (5.1) and (5.2) become actions once we pass to unitary equivalence classes of H -covariant $*$ -representations. Alternatively, one should view (5.1) and (5.2) as an action of the Picard *bigroupoids*, where we have not yet identified isomorphic bimodules. We shall not give a precise definition of an action of a bigroupoid on a collection of categories (though in principle this could be done) but leave this as a suggestive picture. In any case, this gives a conceptually clear picture why H -covariantly strongly (or $*$ -) Morita equivalent $*$ -algebras have equivalent H -covariant representation theories. Moreover, we see that the Picard groups $\text{Pic}_H^*(\mathcal{A})$ and $\text{Pic}_H^{\text{str}}(\mathcal{A})$, respectively, act on the unitary equivalence classes of H -covariant $*$ -representations. In more physical terms, these are just the *super-selection rules* of the $*$ -algebra \mathcal{A} .

5.2. The H -invariant central elements

We consider unital $*$ -algebras in this subsection. Clearly, any H -equivariant $*$ -homomorphism $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ restricts to a $*$ -homomorphism $\mathcal{Z}(\mathcal{A})^H \longrightarrow \mathcal{Z}(\mathcal{B})^H$ of the H -invariant central elements. In particular, this gives a groupoid action of the isomorphism groupoid

$$\text{Iso}_H^*(\mathcal{B}, \mathcal{A}) \times \mathcal{Z}(\mathcal{A})^H \longrightarrow \mathcal{Z}(\mathcal{B})^H. \quad (5.3)$$

We shall now extend this to an action of Pic_H^* in the following way: first, we recall some standard results from Morita theory, see e.g. [1,9,12]. If $\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}$ is a $*$ -equivalence bimodule then for any central element $a \in \mathcal{Z}(\mathcal{A})$ there exists a unique central element $h_{\mathcal{E}}(a) \in \mathcal{Z}(\mathcal{B})$, such that

$$h_{\mathcal{E}}(a) \cdot x = x \cdot a \tag{5.4}$$

for all $x \in \mathcal{B}^{\mathcal{E}}_{\mathcal{A}}$ and the map $h_{\mathcal{E}} : \mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{Z}(\mathcal{B})$ is a $*$ -isomorphism. Moreover, $h_{\mathcal{E}} = h_{\mathcal{E}'}$ if $\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}$ and $\mathcal{B}^{\mathcal{E}'}$ are isomorphic $*$ -Morita equivalence bimodules and we have $h_{\mathcal{E}} \circ h_{\mathcal{E}'} = h_{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}'}$ as well as $h_{\mathcal{A}} = \text{id}_{\mathcal{Z}(\mathcal{A})}$, see [12, Section 2.3] and [9, Proposition 7.6]. This can be rephrased as an action of the $*$ -Picard groupoid on centers

$$h : \text{Pic}^*(\mathcal{B}, \mathcal{A}) \times \mathcal{Z}(\mathcal{A}) \ni ([\mathcal{E}], a) \mapsto h_{\mathcal{E}}(a) \in \mathcal{Z}(\mathcal{B}) \tag{5.5}$$

by $*$ -isomorphisms. In particular, centers are invariant as $*$ -algebras under $*$ -Morita equivalence [1].

Lemma 5.1. *Let $\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}$ be an H -covariant $*$ -Morita equivalence bimodule. Then $h_{\mathcal{E}}$ restricts to a $*$ -isomorphism $h_{\mathcal{E}} : \mathcal{Z}(\mathcal{A})^H \rightarrow \mathcal{Z}(\mathcal{B})^H$. Thus we have an action of the H -covariant $*$ -Picard groupoid on H -invariant central elements*

$$h : \text{Pic}_H^*(\mathcal{B}, \mathcal{A}) \times \mathcal{Z}(\mathcal{A})^H \rightarrow \mathcal{Z}(\mathcal{B})^H. \tag{5.6}$$

Proof. We only have to check that $h_{\mathcal{E}}$ maps H -invariant elements to H -invariant ones. For $x \in \mathcal{B}^{\mathcal{E}}_{\mathcal{A}}$ we have

$$\begin{aligned} (g \triangleright h_{\mathcal{E}}(a)) \cdot x &= g_{(1)} \triangleright (h_{\mathcal{E}}(a) \cdot (S(g_{(2)}) \triangleright x)) = g_{(1)} \triangleright ((S(g_{(2)}) \triangleright x) \cdot a) \\ &= \varepsilon(g)x \cdot a = \varepsilon(g)h_{\mathcal{E}}(a) \cdot x, \end{aligned}$$

since a is invariant. By Remark 3.7 we get $g \triangleright h_{\mathcal{E}}(a) = \varepsilon(g)h_{\mathcal{E}}(a)$. Then the action properties for (5.6) follow immediately from those of (5.5) and (4.1). \square

Corollary 5.2. *The H -covariant $*$ -Picard group $\text{Pic}_H^*(\mathcal{A})$ acts on $\mathcal{Z}(\mathcal{A})^H$ by $*$ -isomorphisms whence $\mathcal{Z}(\mathcal{A})^H$ as $\text{Pic}_H^*(\mathcal{A})$ -space is invariant under H -covariant $*$ -Morita equivalence.*

Moreover, we have a compatibility between the canonical groupoid action (5.3) and the action h , adapting [12, Proposition 2.4] to this situation:

Lemma 5.3. *The actions (5.3) and (5.6) are compatible in the sense that the diagram*

$$\begin{array}{ccc} \text{Iso}_H^*(\mathcal{B}, \mathcal{A}) \times \mathcal{Z}(\mathcal{A})^H & & \\ \ell \times \text{id} \downarrow & \searrow & \\ \text{Pic}_H^*(\mathcal{B}, \mathcal{A}) \times \mathcal{Z}(\mathcal{A})^H & \xrightarrow{h} & \mathcal{Z}(\mathcal{B})^H \end{array} \tag{5.7}$$

commutes.

In general, the center $\mathcal{Z}(\mathcal{A})$ needs not to be preserved by the action of H . However, if H is *cocommutative* then this is the case whence $\mathcal{Z}(\mathcal{A})$ inherits a $*$ -action of H . In this case, the action h of Pic^* on centers (5.5) restricts to an action, also denoted by h , of Pic_H^* on the centers

$$h : \text{Pic}_H^*(\mathcal{B}, \mathcal{A}) \times \mathcal{Z}(\mathcal{A}) \longrightarrow \mathcal{Z}(\mathcal{B}) \tag{5.8}$$

by H -equivariant $*$ -isomorphisms as a simple computation shows. Hence we have

Lemma 5.4. *Let H be cocommutative. Then Pic_H^* acts on the centers by H -equivariant $*$ -isomorphisms whence $\mathcal{Z}(\mathcal{A})$ as a $\text{Pic}_H^*(\mathcal{A})$ -space is invariant under H -covariant $*$ -Morita equivalence.*

5.3. Equivariant K -theory

Again we shall restrict ourselves to unital $*$ -algebras in this subsection for simplicity. There are many notions of equivariant K -theory, we shall use a rather naive definition taking care of the inner products as well.

We consider H -covariant pre-Hilbert (right) \mathcal{A} -modules $\mathcal{P}_{\mathcal{A}}$ with the following additional properties: The inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}}^{\mathcal{P}}$ is *strongly non-degenerate*, i.e. the map $x \mapsto \langle x, \cdot \rangle_{\mathcal{A}}^{\mathcal{P}} \in \text{Hom}_{\mathcal{A}}(\mathcal{P}_{\mathcal{A}}, \mathcal{A})$ is bijective. Moreover, we want $\mathcal{P}_{\mathcal{A}}$ to be *finitely generated* and *projective*. The subcategory of all H -covariant pre-Hilbert \mathcal{A} -modules with these two additional properties is denoted by $\underline{\text{Proj}}_H^{\text{str}}(\mathcal{A})$, where the morphisms are adjointable module morphisms as before. By $\text{Proj}_H^{\text{str}}(\mathcal{A})$ we denote the set of isometric isomorphism classes of $\underline{\text{Proj}}_H^{\text{str}}(\mathcal{A})$. Then $\text{Proj}_H^{\text{str}}(\mathcal{A})$ becomes an abelian semigroup where the addition \oplus is induced by the direct orthogonal sum of elements in $\underline{\text{Proj}}_H^{\text{str}}(\mathcal{A})$. The H -equivariant strong K_0 -group $K_{0,H}^{\text{str}}(\mathcal{A})$ of \mathcal{A} is then by definition the Grothendieck group associated to $\text{Proj}_H^{\text{str}}(\mathcal{A})$. Similarly, dropping the complete positivity of the inner product (but keeping the strong non-degeneracy) we obtain $*$ -versions $\underline{\text{Proj}}_H^*(\mathcal{A})$, $\text{Proj}_H^*(\mathcal{A})$ and $K_{0,H}^*(\mathcal{A})$, respectively.

A H -covariant pre-Hilbert module $\mathcal{P}_{\mathcal{A}}$ is in $\underline{\text{Proj}}_H^{\text{str}}(\mathcal{A})$ if and only if there exist $x_i, y_i \in \mathcal{P}_{\mathcal{A}}$ with $i = 1, \dots, n$ such that

$$x = \sum_i x_i \cdot \langle y_i, x \rangle_{\mathcal{A}}^{\mathcal{P}} \tag{5.9}$$

for all $x \in \mathcal{P}_{\mathcal{A}}$. This is an easy adaption of the dual basis lemma for projective modules, see e.g. [30, Lemma 2.9]. We shall call such vectors x_i, y_i a *Hermitian dual basis*. Then we have the following lemma:

Lemma 5.5. *Let $\mathcal{P}_{\mathcal{B}} \in \underline{\text{Proj}}_H^{\text{str}}(\mathcal{B})$ and let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be a H -covariant strong Morita equivalence bimodule. Then ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ as right \mathcal{A} -module is in $\underline{\text{Proj}}_H^{\text{str}}(\mathcal{A})$ and $\mathcal{P}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \in \underline{\text{Proj}}_H^{\text{str}}(\mathcal{A})$, too.*

Proof. The first statement is well-known and follows directly from the fullness of ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}$ and the compatibility (3.9). For the second statement, let $\{x_i, y_i\}_{i=1, \dots, n}$ be a Hermitian dual basis for $\mathcal{P}_{\mathcal{B}}$ and let $\{\zeta_{\alpha}, \eta_{\alpha}\}_{\alpha=1, \dots, m}$ be a Hermitian dual basis for ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ viewed as right \mathcal{A} -module. Then $\{x_i \otimes_{\mathcal{B}} \zeta_{\alpha}, y_i \otimes_{\mathcal{B}} \eta_{\alpha}\}_{i, \alpha}$ is easily shown to be a Hermitian dual basis for $\mathcal{P}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$. In particular, the inner product on $\mathcal{P}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is already non-degenerate whence the usual quotient procedure for $\widehat{\otimes}$ is not needed here. \square

From this and the associativity properties of $\widehat{\otimes}$ and $\widetilde{\otimes}$ as in Proposition 3.10 we immediately obtain the following result:

Proposition 5.6. *The H -covariant strong Picard groupoid acts on $\text{Proj}_H^{\text{str}}$ by semi-group isomorphisms from the right, i.e.*

$$\begin{aligned} S : \text{Proj}_H^{\text{str}}(\mathcal{B}) \times \text{Pic}_H^{\text{str}}(\mathcal{B}, \mathcal{A}) &\ni ([\mathcal{P}_{\mathcal{B}}], [{}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}]) \\ \mapsto [S_{\mathcal{E}}(\mathcal{P}_{\mathcal{B}})] &= [{}_{\mathcal{A}}\widehat{\otimes}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}] \in \text{Proj}_H^{\text{str}}(\mathcal{A}) \end{aligned} \tag{5.10}$$

and hence it also acts on the H -equivariant strong K_0 -groups by group isomorphisms

$$S : K_{0,H}^{\text{str}}(\mathcal{B}) \times \text{Pic}_H^{\text{str}}(\mathcal{B}, \mathcal{A}) \longrightarrow K_{0,H}^{\text{str}}(\mathcal{A}). \tag{5.11}$$

The analogous result holds for Pic_H^* , Proj_H^* and $K_{0,H}^*$.

Corollary 5.7. *The H -covariant strong Picard group $\text{Pic}_H^{\text{str}}(\mathcal{A})$ acts on $K_{0,H}^{\text{str}}(\mathcal{A})$ by group automorphisms and $K_{0,H}^{\text{str}}(\mathcal{A})$ is invariant as $\text{Pic}_H^{\text{str}}(\mathcal{A})$ -space under H -covariant strong Morita equivalence.*

Note that this result corresponds to the ‘action’ by Rieffel induction R on representation theories, where we have replaced the action from the left via R by an action from the right via the change of base ring functors S .

Again the H -equivariant isomorphisms Iso_H^* act on $\text{Proj}_H^{\text{str}}$ and hence on $K_{0,H}^{\text{str}}$ as well and the above actions (5.10) and (5.11) restrict to this via the groupoid morphism ℓ from Proposition 4.6.

5.4. The lattice $\mathcal{L}_{\mathcal{D},H}(\mathcal{A})$

Let \mathcal{D} be admissible and all other $*$ -algebras are idempotent and non-degenerate as before. Then we can act with $\text{Pic}_H^{\text{str}}$ on the lattices of (\mathcal{D}, H) -closed ideals by the following construction. Let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be an H -covariant strong Morita equivalence bimodule and let $\mathcal{J} \subseteq \mathcal{A}$ be a subset. Then we define

$$\Phi_{\mathcal{E}}(\mathcal{J}) = \left\{ b \in \mathcal{B} \mid \langle x, b \cdot y \rangle_{\mathcal{A}}^{\mathcal{E}} \in \mathcal{J} \text{ for all } x, y \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \right\}. \tag{5.12}$$

We have the following properties of the map $\Phi_{\mathcal{E}}$ generalizing the results of Bursztyn and Waldmann [8]:

Lemma 5.8. *Let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be an H -covariant strong Morita equivalence bimodule and let \mathcal{D} be admissible.*

- (i) *If $\mathcal{J} = \ker \pi$ for $(\mathcal{H}_{\mathcal{D}}, \pi) \in {}^*\text{-Rep}_{\mathcal{D}, H}(\mathcal{A})$ then $\Phi_{\mathcal{E}}(\mathcal{J}) = \ker R_{\mathcal{E}}\pi$ whence in particular $\Phi_{\mathcal{E}}(\mathcal{J}) \in \mathfrak{L}_{\mathcal{D}, H}(\mathcal{B})$ for any $\mathcal{J} \in \mathfrak{L}_{\mathcal{D}, H}(\mathcal{A})$.*
- (ii) *If ${}_{\mathcal{B}}\mathcal{E}'_{\mathcal{A}}$ is another H -covariant strong Morita equivalence bimodule isomorphic to ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ then $\Phi_{\mathcal{E}} = \Phi_{\mathcal{E}'}$.*
- (iii) *If ${}_{\mathcal{F}}\mathcal{F}_{\mathcal{B}}$ is another H -covariant strong Morita equivalence bimodule then $\Phi_{\mathcal{F}} \circ \Phi_{\mathcal{E}} = \Phi_{\mathcal{F} \otimes_{\mathcal{E}}} \mathcal{E}$ and $\Phi_{\mathcal{A}} = \text{id}_{\mathfrak{L}_{\mathcal{D}, H}(\mathcal{A})}$.*

Proof. The first part is analogous to [8, Proposition 5.1]. The second part follows as $R_{\mathcal{E}}(\pi)$ and $R_{\mathcal{E}'}(\pi)$ are unitarily equivalent $*$ -representations which therefore have the same kernel. The same Rieffel induction argument can be used for the third part since we can restrict to strongly non-degenerate $*$ -representations by Lemma 2.15. \square

From this lemma we easily conclude the following statement generalizing Rieffel's correspondence from the theory of C^* -algebras, see e.g. [39, Theorem 3.24], as well as [8, Theorem 5.4]:

Theorem 5.9. *Let \mathcal{D} be admissible. Then the map*

$$\Phi : \text{Pic}_H^{\text{str}}(\mathcal{B}, \mathcal{A}) \times \mathfrak{L}_{\mathcal{D}, H}(\mathcal{A}) \longrightarrow \mathfrak{L}_{\mathcal{D}, H}(\mathcal{B}) \quad (5.13)$$

defines an action of the H -covariant strong Picard groupoid on the lattices of (\mathcal{D}, H) -closed ideals by lattice isomorphisms.

Proof. The only thing to be checked is that $\Phi_{\mathcal{E}}$ is a lattice homomorphism as the well-definedness and the action properties follow from Lemma 5.8. Clearly, we have $\Phi_{\mathcal{E}}(\mathcal{J}) \leq \Phi_{\mathcal{E}}(\mathcal{I})$ for $\mathcal{J} \leq \mathcal{I}$ and $\Phi_{\mathcal{E}}(\mathcal{I} \wedge \mathcal{J}) = \Phi_{\mathcal{E}}(\mathcal{I}) \wedge \Phi_{\mathcal{E}}(\mathcal{J})$. From these properties and the bijectivity of $\Phi_{\mathcal{E}}$ it follows that $\Phi_{\mathcal{E}}(\mathcal{I} \vee \mathcal{J}) = \Phi_{\mathcal{E}}(\mathcal{I}) \vee \Phi_{\mathcal{E}}(\mathcal{J})$. \square

Corollary 5.10. *Let \mathcal{D} be admissible. Then $\text{Pic}_H^{\text{str}}(\mathcal{A})$ acts on the lattice $\mathfrak{L}_{\mathcal{D}, H}(\mathcal{A})$ by lattice automorphisms and $\mathfrak{L}_{\mathcal{D}, H}(\mathcal{A})$ as $\text{Pic}_H^{\text{str}}(\mathcal{A})$ -space is invariant under H -covariant strong Morita equivalence.*

5.5. The groups $U(H, \mathcal{A})$ and $U_0(H, \mathcal{A})$

Also in this subsection the $*$ -algebras are required to be unital. In the characterization of the kernel of the canonical groupoid morphism $\text{Pic}_H^{\text{str}} \longrightarrow \text{Pic}_H^*$ as well as for $\text{Pic}_H^* \longrightarrow \text{Pic}^*$ the groups $U(H, \mathcal{A})$ and $U_0(H, \mathcal{A})$ play the dominant role which already suggests that they are a Morita invariant.

As we have outlined in the appendix, the H -equivariant $*$ -isomorphisms Iso_H^* act not only on $U(H, \mathcal{A})$ and $U_0(H, \mathcal{A})$ in a canonical way but also on the whole exact sequence (A.10). We shall now extend this to an action of Pic_H^* extending thereby the action h of Pic_H^* on the centers.

Lemma 5.11. *Let $\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}$ be an H -covariant $*$ -Morita equivalence bimodule and let $\mathfrak{a} \in \text{Hom}_{\mathbb{C}}(H, \mathcal{A})$.*

(i) *The definition*

$$g \triangleright_{\mathfrak{a}} x = (g_{(1)} \triangleright x) \cdot \mathfrak{a}(g_{(2)}) \tag{5.14}$$

gives another compatible H -action on $\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}$ such that $(\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}, \triangleright_{\mathfrak{a}})$ is an H -covariant $$ -Morita equivalence bimodule if and only if $\mathfrak{a} \in U(H, \mathcal{A})$ and any such action is of this form for a uniquely determined $\mathfrak{a} \in U(H, \mathcal{A})$.*

(ii) *The group $U(H, \mathcal{A})$ acts freely and transitively from the right on the set of all compatible H -actions on $\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}$ by $(\triangleright, \mathfrak{a}) \mapsto \triangleright_{\mathfrak{a}}$.*

(iii) *For $\mathfrak{b} \in U(H, \mathcal{B})$ and $\mathfrak{a} \in U(H, \mathcal{A})$ we have $(\triangleright_{\mathfrak{a}})^{\mathfrak{b}} = (\triangleright^{\mathfrak{b}})_{\mathfrak{a}}$ and there exists a unique $h_{\mathcal{E}}(\mathfrak{a}) \in U(H, \mathcal{B})$ such that*

$$\triangleright_{\mathfrak{a}} = \triangleright^{h_{\mathcal{E}}(\mathfrak{a})}. \tag{5.15}$$

(iv) *The map*

$$h_{\mathcal{E}} : U(H, \mathcal{A}) \ni \mathfrak{a} \mapsto h_{\mathcal{E}}(\mathfrak{a}) \in U(H, \mathcal{B}) \tag{5.16}$$

is a group isomorphism.

Proof. The first part is lengthy computation but completely analogous to Proposition 4.10. The second part is in the same spirit as Proposition 4.10 as well. For the third part we have

$$\begin{aligned} g(\triangleright_{\mathfrak{a}})^{\mathfrak{b}} x &= \mathfrak{b}(g_{(1)}) \cdot (g_{(2)} \triangleright_{\mathfrak{a}} x) = \mathfrak{b}(g_{(1)}) \cdot (g_{(2)} \triangleright x) \cdot \mathfrak{a}(g_{(3)}) = (g_{(1)} \triangleright^{\mathfrak{b}} x) \cdot \mathfrak{a}(g_{(2)}) \\ &= g(\triangleright^{\mathfrak{b}})_{\mathfrak{a}} x, \end{aligned}$$

since $\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}$ is a bimodule. Then the remaining statements are general facts on commuting free and transitive group actions. \square

The next lemma investigates the dependence of the isomorphism $h_{\mathcal{E}}$ on the bimodule \mathcal{E} :

Lemma 5.12. *Let $\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}$ and $\mathcal{B}^{\mathcal{E}'}$ be isomorphic H -covariant $*$ -Morita equivalence bimodules. Then $h_{\mathcal{E}} = h_{\mathcal{E}'}$.*

Proof. Let $U : \mathcal{E} \rightarrow \mathcal{E}'$ be an isomorphism. Then on one hand

$$\begin{aligned} U(g \triangleright^{h_{\mathcal{E}}(\mathfrak{a})} x) &= U(h_{\mathcal{E}}(\mathfrak{a})(g_{(1)}) \cdot (g_{(2)} \triangleright x)) = h_{\mathcal{E}}(\mathfrak{a})(g_{(1)}) \cdot (g_{(2)} \triangleright' U(x)) \\ &= g(\triangleright')^{h_{\mathcal{E}}(\mathfrak{a})} U(x) \end{aligned}$$

and on the other hand

$$\begin{aligned} U(g \triangleright^{h_{\mathcal{E}}(\mathfrak{a})} x) &= U(g \triangleright_{\mathfrak{a}} x) = U((g_{(1)} \triangleright x) \cdot \mathfrak{a}(g_{(2)})) = (g_{(1)} \triangleright' U(x)) \cdot \mathfrak{a}(g_{(2)}) \\ &= g(\triangleright')^{h_{\mathcal{E}'(\mathfrak{a})}} U(x). \end{aligned}$$

This implies $h_{\mathcal{E}}(\mathfrak{a}) = h_{\mathcal{E}'(\mathfrak{a})}$ by the uniqueness from Lemma 5.11. \square

Lemma 5.13. Let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ and ${}_{\mathcal{A}}\mathcal{F}_{\mathcal{B}}$ be H -covariant $*$ -Morita equivalence bimodules. Then

$$h_{\mathcal{F}} \circ h_{\mathcal{E}} = h_{\mathcal{F} \otimes \mathcal{E}} \quad \text{and} \quad h_{\mathcal{A}} = \text{id}_{U(H, \mathcal{A})}. \quad (5.17)$$

For $\Phi \in \text{Iso}_H^*(\mathcal{A})$ we have

$$h_{\ell(\Phi)} = \Phi_*, \quad (5.18)$$

where $\Phi_* : U(H, \mathcal{A}) \rightarrow U(H, \mathcal{B})$ as in Proposition A.5.

Proof. Let $x \in \mathcal{F}$, $\phi \in \mathcal{E}$ and $\mathfrak{a} \in U(H, \mathcal{A})$. Then

$$\begin{aligned} g \triangleright^{h_{\mathcal{F} \otimes \mathcal{E}}(\mathfrak{a})} (x \otimes \phi) &= (g_{(1)} \triangleright (x \otimes \phi)) \cdot \mathfrak{a}(g_{(2)}) \\ &= (g_{(1)} \triangleright x) \otimes (g_{(2)} \triangleright \phi) \cdot \mathfrak{a}(g_{(3)}) \\ &= (g_{(1)} \triangleright x) \otimes (h_{\mathcal{E}}(\mathfrak{a})(g_{(2)})) \cdot (g_{(3)} \triangleright \phi) \\ &= ((g_{(1)} \triangleright x) \cdot h_{\mathcal{E}}(\mathfrak{a})(g_{(2)})) \otimes (g_{(3)} \triangleright \phi) \\ &= (h_{\mathcal{F}}(h_{\mathcal{E}}(\mathfrak{a}))(g_{(1)})) \cdot (g_{(2)} \triangleright x) \otimes (g_{(3)} \triangleright \phi) \\ &= h_{\mathcal{F}}(h_{\mathcal{E}}(\mathfrak{a}))(g_{(1)}) \cdot (g_{(2)} \triangleright (x \otimes \phi)) \\ &= g \triangleright^{h_{\mathcal{F}}(h_{\mathcal{E}}(\mathfrak{a}))} (x \otimes \phi) \end{aligned}$$

proves the first part. The second statement in (5.17) is trivial using the ‘module condition’ for \mathfrak{a} . The last statement (5.18) is also a straightforward computation. \square

Collecting these results, we get a generalization of the action h of the Picard groupoid on centers and a generalization of the action of Iso_H^* on the exact sequence (A.14).

Theorem 5.14. The map

$$h : \text{Pic}_H^*(\mathcal{B}, \mathcal{A}) \times U(H, \mathcal{A}) \ni ([\mathcal{E}], \mathfrak{a}) \mapsto h_{\mathcal{E}}(\mathfrak{a}) \in U(H, \mathcal{B}) \quad (5.19)$$

determines an action of Pic_H^* on the exact sequence (A.10), i.e.

$$\begin{array}{ccccccc} 1 \longrightarrow & U(\mathcal{Z}(\mathcal{A}))^H & \longrightarrow & U(\mathcal{Z}(\mathcal{A})) & \longrightarrow & U(H, \mathcal{A}) & \longrightarrow & U_0(H, \mathcal{A}) & \longrightarrow & 1 \\ & \downarrow h_{\mathcal{E}} & & \downarrow h_{\mathcal{E}} & & \downarrow h_{\mathcal{E}} & & \downarrow h_{\mathcal{E}} & & \\ 1 \longrightarrow & U(\mathcal{Z}(\mathcal{B}))^H & \longrightarrow & U(\mathcal{Z}(\mathcal{B})) & \longrightarrow & U(H, \mathcal{B}) & \longrightarrow & U_0(H, \mathcal{B}) & \longrightarrow & 1. \end{array} \quad (5.20)$$

commutes and all $h_{\mathcal{E}}$ are group isomorphisms. Moreover, this groupoid action is compatible with the groupoid morphism $\ell : \text{Iso}_H^* \rightarrow \text{Pic}_H^*$ and the canonical action of Iso_H^* on the exact sequence as in Corollary A.6.

Proof. The only thing left to show is the commutativity of the box in the middle of (5.20) since then the last vertical arrow is defined in such a way, that (5.20) commutes. Thus let

$c \in U(\mathcal{L}(\mathcal{A}))$ be given. Then for $x \in \mathcal{B}^{\mathcal{E}}_{\mathcal{A}}$ we have

$$\begin{aligned} g \triangleright^{h_{\mathcal{E}}(\hat{c})} x &= (g_{(1)} \triangleright x) \cdot \hat{c}(g_{(2)}) \\ &= g_{(1)} \triangleright x \cdot (c g_{(2)} \triangleright c^{-1}) \\ &= h_{\mathcal{E}}(c) \cdot (g_{(1)} \triangleright (x \cdot c^{-1})) \\ &= h_{\mathcal{E}}(c) \cdot (g_{(1)} \triangleright (h_{\mathcal{E}}(c)^{-1} \cdot x)) \\ &= \widehat{h_{\mathcal{E}}(c)}(g_{(1)}) \cdot g_{(2)} \triangleright x \\ &= g \triangleright \widehat{h_{\mathcal{E}}(c)} x, \end{aligned}$$

whence $h_{\mathcal{E}}(\hat{c}) = \widehat{h_{\mathcal{E}}(c)}$. \square

We leave it to the reader to draw the appropriate big commutative diagram expressing all compatibilities relating ℓ and h stated in this theorem.

Corollary 5.15. *The H -covariant *-Picard group $\text{Pic}_H^*(\mathcal{A})$ acts on the exact sequence (A.10) by isomorphisms whence (A.10) as a $\text{Pic}_H^*(\mathcal{A})$ -space is invariant under H -covariant *-Morita equivalence. In particular, each of the groups $U(\mathcal{L}(\mathcal{A}))^H$, $U(\mathcal{L}(\mathcal{A}))$, $U(H, \mathcal{A})$, and $U_0(H, \mathcal{A})$ carries a canonical $\text{Pic}_H^*(\mathcal{A})$ -action by group automorphisms. They are invariant under H -covariant *-Morita equivalence.*

We can interpret the result of Lemma 5.13 also in another way. According to Theorem 4.14 the group $U_0(H, \mathcal{B})$ acts on $\text{Pic}_H^{\text{str}}(\mathcal{B}, \mathcal{A})$ freely by twisting the H -action

$$[b] \cdot [\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}, \triangleright] = [\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}, \triangleright^b]. \tag{5.21}$$

Similarly, $U_0(H, \mathcal{A})$ acts from the right by

$$[\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}, \triangleright] \cdot [a] = [\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}, \triangleright_a]. \tag{5.22}$$

Then from the proof of Lemma 5.13 we see that we have the following compatibilities between these two actions and the tensor product of bimodules, namely

$$[c] \cdot ([{}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}] \tilde{\otimes} [\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}]) = ([c] \cdot [{}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}]) \tilde{\otimes} [\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}], \tag{5.23}$$

$$[b] \cdot [\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}] = [\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}] \cdot [h_{\mathcal{E}}^{-1}(b)], \tag{5.24}$$

$$([{}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}] \cdot [b]) \tilde{\otimes} [\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}] = [{}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}] \tilde{\otimes} ([b] \cdot [\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}]) \tag{5.25}$$

and

$$([{}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}] \tilde{\otimes} [\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}]) \cdot [a] = [{}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}] \tilde{\otimes} ([\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}] \cdot [a]). \tag{5.26}$$

for $c \in U(H, \mathcal{C})$, $b \in U(H, \mathcal{B})$ and $a \in U(H, \mathcal{A})$. From this we conclude the following statement:

Proposition 5.16. *The map*

$$U_0(H, \mathcal{A}) \ni [a] \mapsto [a] \cdot [{}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}] = [{}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}, \triangleright^a] \in \text{Pic}_H^{\text{str}}(\mathcal{A}) \tag{5.27}$$

is an injective group homomorphism, such that

$$1 \longrightarrow U_0(H, \mathcal{A}) \longrightarrow \text{Pic}_H^{\text{str}}(\mathcal{A}) \longrightarrow \text{Pic}^{\text{str}}(\mathcal{A}) \quad (5.28)$$

is exact.

Proof. It follows from a straightforward computation using (5.23), (5.24), (5.25), and (5.26) that (5.27) is a well-defined group homomorphism. The exactness of (5.28) is then a consequence of Theorem 4.14. \square

Though this observation helps to understand the H -covariant strong Picard group one should not overestimate its importance as the group morphism $\text{Pic}_H^{\text{str}}(\mathcal{A}) \longrightarrow \text{Pic}^{\text{str}}(\mathcal{A})$ is only in the very simplest cases surjective.

6. Crossed products

In this section we shall investigate the crossed product algebras $\mathcal{A} \rtimes H$ and relate their Picard groupoids with the H -covariant Picard groupoids of the underlying algebras \mathcal{A} .

6.1. Definitions and preliminary results

Let \mathcal{A} be a $*$ -algebra over \mathbb{C} with a $*$ -action of a Hopf $*$ -algebra H . Recall that the crossed product $*$ -algebra $\mathcal{A} \rtimes H$ is $\mathcal{A} \otimes H$ as a \mathbb{C} -module with multiplication defined by

$$(a \otimes g)(b \otimes h) = (a(g_{(1)} \triangleright b)) \otimes g_{(2)}h \quad (6.1)$$

and $*$ -involution

$$(a \otimes g)^* = g_{(1)}^* \triangleright a^* \otimes g_{(2)}^*. \quad (6.2)$$

Then it is well-known that $\mathcal{A} \rtimes H$ is a $*$ -algebra over \mathbb{C} , sometimes also called the smash product of \mathcal{A} and H , see e.g. [27,35] for this and more general crossed product constructions and e.g. [43] for their representation theory.

For later use we note the following simple and well-known fact expressing the functoriality of the crossed product construction:

Lemma 6.1. *If $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ is a H -equivariant $*$ -homomorphism then*

$$\Phi \otimes \text{id} : \mathcal{A} \rtimes H \longrightarrow \mathcal{B} \rtimes H \quad (6.3)$$

is a $*$ -homomorphism. In particular, this induces a groupoid morphism

$$\cdot \rtimes H : \text{Iso}_H^* \longrightarrow \text{Iso}^*, \quad (6.4)$$

such that the identities \mathcal{A} in Iso_H^* are mapped to their crossed products $\mathcal{A} \rtimes H$ with H and arrows Φ are mapped to $\Phi \otimes \text{id}$.

On $\mathcal{A} \rtimes H$ one has a canonical $*$ -action of H defined by

$$g \triangleright (a \otimes h) = (g_{(1)} \triangleright a) \otimes g_{(2)} h S(g_{(3)}) \tag{6.5}$$

and there is a canonical $*$ -homomorphism

$$\iota : \mathcal{A} \ni a \mapsto \iota(a) = a \otimes \mathbb{1}_H \in \mathcal{A} \rtimes H, \tag{6.6}$$

which, up to possible torsion-effects due to $\otimes_{\mathbb{C}}$, is injective. Furthermore, ι is H -equivariant, i.e. $g \triangleright \iota(a) = \iota(g \triangleright a)$.

If \mathcal{A} is unital then $\mathcal{A} \rtimes H$ is unital with unit $\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_H$ and we have a canonical $*$ -homomorphism

$$j : H \ni g \mapsto \mathbb{1}_{\mathcal{A}} \otimes g \in \mathcal{A} \rtimes H, \tag{6.7}$$

such that under this inclusion the action (6.5) becomes ‘inner’ in the sense that

$$g \triangleright (a \otimes h) = j(g_{(1)})(a \otimes h)j(S(g_{(2)})), \tag{6.8}$$

see the adjoint action (2.9) of H on itself. Finally, in the unital case the crossed product is *universal* with respect to these properties, i.e. if \mathcal{B} is another unital $*$ -algebra with two unital $*$ -homomorphisms $\iota_{\mathcal{B}} : \mathcal{A} \rightarrow \mathcal{B}$ and $j_{\mathcal{B}} : H \rightarrow \mathcal{B}$ such that $\iota_{\mathcal{B}}(g \triangleright a) = j_{\mathcal{B}}(g_{(1)})\iota_{\mathcal{B}}(a)j_{\mathcal{B}}(S(g_{(2)}))$ then there exists a unique unital $*$ -homomorphism $\phi : \mathcal{A} \rtimes H \rightarrow \mathcal{B}$ such that $\iota_{\mathcal{B}} = \phi \circ \iota$ and $j_{\mathcal{B}} = \phi \circ j$. In fact, $\phi(a \otimes g) = \iota_{\mathcal{B}}(a)j_{\mathcal{B}}(g)$.

This observation immediately implies the following crucial property of $\mathcal{A} \rtimes H$ which is one of the motivations to study crossed products. This statement is well-known in various contexts.

Lemma 6.2. *The categories $*\text{-mod}_H(\mathcal{A})$ and $*\text{-mod}(\mathcal{A} \rtimes H)$ are equivalent, where the equivalence on objects is given by*

$$*\text{-mod}_H(\mathcal{A}) \ni (\mathcal{H}, \pi) \mapsto (\hat{\mathcal{H}}, \hat{\pi}) \in *\text{-mod}(\mathcal{A} \rtimes H), \tag{6.9}$$

where $\hat{\mathcal{H}} = \mathcal{H}$ as pre-Hilbert spaces and $\hat{\pi}(a \otimes g)\phi = \pi(a)g \triangleright \phi$, and on morphisms $T : (\mathcal{H}_1, \pi_1) \rightarrow (\mathcal{H}_2, \pi_2)$ it is the identity. The same statement holds for $*\text{-Mod}$, $*\text{-rep}$ and $*\text{-Rep}$ instead of $*\text{-mod}$, too.

The following proposition should be well-known and allows to construct positive functionals for $\mathcal{A} \rtimes H$ and hence $*$ -representations via the GNS construction.

Proposition 6.3. *Let $\omega : \mathcal{A} \rightarrow \mathbb{C}$ be a H -invariant positive linear functional and let $\mu : H \rightarrow \mathbb{C}$ be a positive linear functional. Then $\omega \otimes \mu : \mathcal{A} \rtimes H \rightarrow \mathbb{C}$ is again positive.*

Proof. Let $\sum_i a_i \otimes g_i$ be given. Then a straightforward computation using the invariance of ω gives

$$(\omega \otimes \mu) \left(\left(\sum_i a_i \otimes g_i \right)^* \left(\sum_j a_j \otimes g_j \right) \right) = \sum_{i,j} \omega(a_i^* a_j) \mu(g_i^* g_j) \geq 0,$$

since both ω and μ are positive linear functionals and hence completely positive, see e.g. [9, Lemma 4.3]. \square

In particular, $\omega \otimes \varepsilon$ is always a positive linear functional on $\mathcal{A} \rtimes H$ whence we can embed the H -invariant positive functionals of \mathcal{A} into the positive linear functionals of $\mathcal{A} \rtimes H$. More generally, if $\chi : H \rightarrow \mathbb{C}$ is a *unitary character*, i.e. a unital $*$ -homomorphism, then $\omega \otimes \chi$ is positive, the counit is an example. We have the following invariance with respect to the $*$ -action (6.8)

$$(\omega \otimes \chi)(g \triangleright (a \otimes h)) = \varepsilon(g)(\omega \otimes \chi)(a \otimes h). \quad (6.10)$$

Remark 6.4. If $\omega : \mathcal{A} \rightarrow \mathbb{C}$ is H -invariant then the H -covariant GNS representation $(\mathcal{H}_\omega, \pi_\omega)$ of \mathcal{A} corresponds to the GNS representation $(\mathcal{H}_{\omega \otimes \varepsilon}, \pi_{\omega \otimes \varepsilon})$ of $\mathcal{A} \rtimes H$ under the functor (6.9). In fact, the map

$$U : (\hat{\mathcal{H}}_\omega, \hat{\pi}_\omega) \ni \psi_a \mapsto \psi_{a \otimes \mathbb{1}_H} \in (\mathcal{H}_{\omega \otimes \varepsilon}, \pi_{\omega \otimes \varepsilon}) \quad (6.11)$$

is a unitary intertwiner which can be verified easily. Here ψ_a and $\psi_{a \otimes \mathbb{1}_H}$ denote the equivalence classes of a and $a \otimes \mathbb{1}_H$, respectively.

6.2. Crossed products of $*$ -representations

We shall now extend the crossed product construction to modules and $*$ -representations.

Lemma 6.5. Let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \in {}^* \text{-mod}_{\mathcal{A}, H}(\mathcal{B})$. Then on $\mathcal{E} \otimes H$ we have a $(\mathcal{B} \rtimes H, \mathcal{A} \rtimes H)$ -bimodule structure defined by

$$(b \otimes g) \cdot (x \otimes h) = (b \cdot (g_{(1)} \triangleright x)) \otimes g_{(2)} h \quad (6.12)$$

and

$$(x \otimes g) \cdot (a \otimes h) = (x \cdot (g_{(1)} \triangleright a)) \otimes g_{(2)} h. \quad (6.13)$$

Moreover,

$$\langle x \otimes g, y \otimes h \rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \otimes H} = (g_{(1)}^* \triangleright \langle x, y \rangle_{\mathcal{A}}^{\mathcal{E}}) \otimes g_{(2)}^* h \quad (6.14)$$

defines a $(\mathcal{A} \rtimes H)$ -valued inner product on $\mathcal{E} \otimes H$, such that

$$\langle (b \otimes g) \cdot (x \otimes h), y \otimes k \rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \otimes H} = \langle x \otimes h, (b \otimes g)^* \cdot (y \otimes k) \rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \otimes H}. \quad (6.15)$$

Proof. It is a well-known straightforward computation to show that the definitions (6.12) and (6.13) indeed give the described bimodule structure. Thus, we have to prove that (6.14) is a $(\mathcal{A} \rtimes H)$ -valued inner product. Clearly, it extends \mathbb{C} -sesquilinearly to $\mathcal{E} \otimes H$.

We compute

$$\begin{aligned} (\langle x \otimes g, y \otimes h \rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \otimes H})^* &= (g_{(1)}^* \triangleright \langle x, y \rangle_{\mathcal{A}}^{\mathcal{E}} \otimes g_{(2)}^* h)^* \\ &= (h_{(1)}^* g_{(2)} S(g_{(1)}^*)^*) \triangleright (\langle x, y \rangle_{\mathcal{A}}^{\mathcal{E}})^* \otimes h_{(2)}^* g_{(3)} \\ &= h_{(1)}^* \triangleright \langle y, x \rangle_{\mathcal{A}}^{\mathcal{E}} \otimes h_{(2)}^* g \\ &= \langle y \otimes h, x \otimes g \rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \otimes H}. \end{aligned}$$

Moreover,

$$\begin{aligned} (x \otimes g, (y \otimes h) \cdot (a \otimes k))_{\mathcal{A} \rtimes H}^{\mathcal{E} \otimes H} &= g_{(1)}^* \triangleright \langle x, y \cdot (h_{(1)} \triangleright a) \rangle_{\mathcal{A}}^{\mathcal{E}} \otimes g_{(2)}^* h_{(2)} k \\ &= (g_{(1)}^* \triangleright \langle x, y \rangle_{\mathcal{A}}^{\mathcal{E}}) ((g_{(2)}^* h_{(1)}) \triangleright a) \otimes g_{(3)}^* h_{(2)} k \\ &= (g_{(1)}^* \triangleright \langle x, y \rangle_{\mathcal{A}}^{\mathcal{E}} \otimes g_{(2)}^* h) (a \otimes k) \\ &= \langle x \otimes g, y \otimes h \rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \otimes H} (a \otimes k), \end{aligned}$$

whence $\langle \cdot, \cdot \rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \otimes H}$ is indeed a $(\mathcal{A} \rtimes H)$ -valued inner product. Finally, we compute

$$\begin{aligned} ((b \otimes g) \cdot (x \otimes h), y \otimes k)_{\mathcal{A} \rtimes H}^{\mathcal{E} \otimes H} &= \langle (b \cdot g_{(1)} \triangleright x) \otimes g_{(2)} h, y \otimes k \rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \otimes H} \\ &= ((h_{(1)}^* g_{(2)}^*) \triangleright \langle b \cdot g_{(1)} \triangleright x, y \rangle_{\mathcal{A}}^{\mathcal{E}}) \otimes h_{(2)}^* g_{(3)}^* k \\ &= ((h_{(1)}^* g_{(2)}^*) \triangleright \langle g_{(1)} \triangleright x, b^* \cdot y \rangle_{\mathcal{A}}^{\mathcal{E}}) \otimes h_{(2)}^* g_{(3)}^* k \\ &= \left(h_{(1)}^* \triangleright \left((S(g_{(2)}^*)^* g_{(1)}) \triangleright x, g_{(3)}^* \triangleright (b^* \cdot y) \right)_{\mathcal{A}}^{\mathcal{E}} \right) \otimes h_{(2)}^* g_{(4)}^* k \\ &= \left(h_{(1)}^* \triangleright \langle x, g_{(1)}^* \triangleright (b^* \cdot y) \rangle_{\mathcal{A}}^{\mathcal{E}} \right) \otimes h_{(2)}^* g_{(2)}^* k \\ &= \langle x \otimes h, g_{(1)}^* \triangleright (b^* \cdot y) \otimes g_{(2)}^* k \rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \otimes H} \\ &= \langle x \otimes h, (b \otimes g)^* \cdot (y \otimes k) \rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \otimes H}, \end{aligned}$$

whence (6.15) follows. \square

It may happen that the inner product $\langle \cdot, \cdot \rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \otimes H}$ on $\mathcal{E} \otimes H$ is degenerate. Thus, we can pass in the usual way to the quotient by the degeneracy space which is compatible with the $(\mathcal{B} \rtimes H, \mathcal{A} \rtimes H)$ -bimodule structure as usual. We end up with an object in ${}^* \text{-mod}_{\mathcal{A} \rtimes H}(\mathcal{B} \rtimes H)$ which we shall denote by

$$\mathcal{E} \rtimes H = (\mathcal{E} \otimes H) / (\mathcal{E} \otimes H)^\perp, \tag{6.16}$$

always understood to be endowed with the $(\mathcal{B} \rtimes H, \mathcal{A} \rtimes H)$ -bimodule structure and the induced $(\mathcal{A} \rtimes H)$ -valued inner product which we shall denote by $\langle \cdot, \cdot \rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \rtimes H}$. The next lemma shows that complete positivity as well as strong non-degeneracy is always preserved:

Lemma 6.6. *Let ${}_{\mathcal{B}} \mathcal{E}_{\mathcal{A}} \in {}^* \text{-rep}_{\mathcal{A}, H}(\mathcal{B})$. Then the inner product $\langle \cdot, \cdot \rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \rtimes H}$ is completely positive, whence ${}_{\mathcal{B} \rtimes H} \mathcal{E} \rtimes H_{\mathcal{A} \rtimes H} \in {}^* \text{-rep}_{\mathcal{A} \rtimes H}(\mathcal{B} \rtimes H)$ is a * -representation on a pre-Hilbert*

module. Moreover, if $\mathcal{B}^{\mathcal{E}}_{\mathcal{A}} \in^* \text{-Mod}_{\mathcal{A}, H}(\mathcal{B})$ then $\mathcal{B}^{\mathcal{E}}_{\mathcal{A} \rtimes H} \in^* \text{-Mod}_{\mathcal{A} \rtimes H}(\mathcal{B} \rtimes H)$ and hence $\mathcal{B}^{\mathcal{E}}_{\mathcal{A}} \in^* \text{-Rep}_{\mathcal{A}, H}(\mathcal{B})$ implies $\mathcal{B}^{\mathcal{E}}_{\mathcal{A} \rtimes H} \in^* \text{-Rep}_{\mathcal{A} \rtimes H}(\mathcal{B} \rtimes H)$.

Proof. Let $\Phi^{(1)}, \dots, \Phi^{(n)} \in \mathcal{E} \otimes H$ be given and let $\Phi^{(\alpha)} = \sum_{i=1}^N x_i^{(\alpha)} \otimes g_i^{(\alpha)}$ with some $x_i^{(\alpha)} \in \mathcal{E}$ and $g_i^{(\alpha)} \in H$, where without restriction N is the same for all $\alpha = 1, \dots, n$. Then

$$\langle \Phi^{(\alpha)}, \Phi^{(\beta)} \rangle_{\mathcal{E} \otimes H, \mathcal{A} \rtimes H} = \sum_{i,j=1}^N \left((g_i^{(\alpha)})^*_{(1)} \triangleright \langle x_i^{(\alpha)}, x_j^{(\beta)} \rangle_{\mathcal{E}} \right) \otimes (g_i^{(\alpha)})^*_{(2)} g_j^{(\beta)}.$$

Now the map $f : M_{nN}(\mathcal{A}) \rightarrow M_{nN}(\mathcal{A} \rtimes H)$ defined by

$$f : A = (A_{ij}^{\alpha\beta}) \mapsto \left((g_i^{(\alpha)})^*_{(1)} \triangleright A_{ij}^{\alpha\beta} \otimes (g_i^{(\alpha)})^*_{(2)} g_j^{(\beta)} \right)$$

is a positive map. Indeed, we have

$$\begin{aligned} f(A^*A) &= \sum_{\gamma,k} \left((g_i^{(\alpha)})^*_{(1)} \triangleright \left((A_{ki}^{\gamma\alpha})^* A_{kj}^{\gamma\beta} \right) \otimes (g_i^{(\alpha)})^*_{(2)} g_j^{(\beta)} \right) \\ &= \sum_{\gamma,k} \left(\left((g_i^{(\alpha)})^*_{(1)} \triangleright (A_{ki}^{\gamma\alpha})^* \otimes (g_i^{(\alpha)})^*_{(2)} \right) (A_{kj}^{\gamma\beta} \otimes g_j^{(\beta)}) \right) \\ &= \sum_{\gamma,k} \left(A_{ki}^{\gamma\alpha} \otimes g_i^{(\alpha)} \right)^* (A_{kj}^{\gamma\beta} \otimes g_j^{(\beta)}) \\ &= (A \otimes g)^* (A \otimes g), \end{aligned}$$

where $A \otimes g \in M_{nN}(\mathcal{A} \rtimes H)$ is given by its matrix coefficients $(A \otimes g)_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta} \otimes g_j^{(\beta)}$. Thus $f(A^*A) \in M_{nN}(\mathcal{A} \rtimes H)^{++}$ whence f is positive. Since the matrix $\left(\langle x_i^{(\alpha)}, x_j^{(\beta)} \rangle_{\mathcal{E}} \right)$ is a positive matrix in $M_{nN}(\mathcal{A})$, by complete positivity of $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ we conclude that the matrix $f\left(\left(\langle x_i^{(\alpha)}, x_j^{(\beta)} \rangle_{\mathcal{E}}\right)\right)$ is positive as well. Then the summation over i, j is the positive map τ from [11, Example 2.1] whence the result is positive again. This is precisely the matrix $\langle \langle \Phi^{(\alpha)}, \Phi^{(\beta)} \rangle_{\mathcal{E} \otimes H} \rangle_{\mathcal{A} \rtimes H}$. Thus, the complete positivity of the inner product $\langle \cdot, \cdot \rangle_{\mathcal{E} \otimes H, \mathcal{A} \rtimes H}$ is shown and the complete positivity of $\langle \cdot, \cdot \rangle_{\mathcal{E} \otimes H, \mathcal{A} \rtimes H}$ follows. The statement on the strong non-degeneracy is trivial. \square

In the unital case one can simplify the above argument by observing that

$$\langle x \otimes g, y \otimes h \rangle_{\mathcal{E} \otimes H, \mathcal{A} \rtimes H} = (\mathbb{1}_{\mathcal{A}} \otimes g)^* \left(\langle x, y \rangle_{\mathcal{E}} \otimes \mathbb{1}_H \right) (\mathbb{1}_{\mathcal{A}} \otimes h). \tag{6.17}$$

From this the complete positivity of $\langle \cdot, \cdot \rangle_{\mathcal{E} \otimes H, \mathcal{A} \rtimes H}$ can be deduced more easily.

Remark 6.7. For a left \mathcal{B} -linear H -covariant \mathcal{B} -valued inner product the corresponding definition of the left $(\mathcal{B} \rtimes H)$ -linear $(\mathcal{B} \rtimes H)$ -valued inner product on $\mathcal{E} \rtimes H$ is

$$\begin{aligned} \mathcal{B} \rtimes H \langle x \otimes g, y \otimes h \rangle^{\mathcal{E} \rtimes H} &= \left(g_{(2)} \triangleright_{\mathcal{B}} \langle S^{-1}(g_{(1)}) \triangleright x, S^{-1}(h_{(1)}) \triangleright y \rangle^{\mathcal{E}} \right) \otimes g_{(3)} h_{(2)}^* \\ &= \mathcal{B} \left\langle x, S(g_{(1)})^* S^{-1}(h_{(1)}) \triangleright y \right\rangle^{\mathcal{E}} \otimes g_{(2)} h_{(2)}^*. \end{aligned} \tag{6.18}$$

The motivation for this formula comes from the isomorphism I_2 in Proposition 6.10 below which identifies the complex conjugated bimodule $\overline{\mathcal{E} \rtimes H}$ canonically with $\overline{\mathcal{E}} \rtimes H$. One can prove by an analogous computation that $\mathcal{B} \rtimes H \langle \cdot, \cdot \rangle^{\mathcal{E} \rtimes H}$ is indeed $(\mathcal{B} \rtimes H)$ -left linear and enjoys the correct symmetry properties. Moreover, it is compatible with the right $(\mathcal{A} \rtimes H)$ -module structure, whence it gives a non-degenerate inner product $\mathcal{B} \rtimes H \langle \cdot, \cdot \rangle^{\mathcal{E} \rtimes H}$ on the corresponding quotient $\mathcal{E} \rtimes H$. Finally, $\mathcal{B} \rtimes H \langle \cdot, \cdot \rangle^{\mathcal{E} \rtimes H}$ is completely positive if $\mathcal{B} \langle \cdot, \cdot \rangle^{\mathcal{E}}$ is completely positive, as one can check directly in the same spirit as for $\langle \cdot, \cdot \rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \rtimes H}$. Alternatively, we shall see an argument in Remark 6.11.

The next lemma ensures the functoriality of the construction of $\mathcal{E} \rtimes H$:

Lemma 6.8. Let $T : \mathcal{B}^{\mathcal{E}, \mathcal{A}} \rightarrow \mathcal{B}^{\mathcal{E}', \mathcal{A}}$ be an intertwiner between $\mathcal{B}^{\mathcal{E}, \mathcal{A}}, \mathcal{B}^{\mathcal{E}', \mathcal{A}} \in {}^* \text{-mod}_{\mathcal{A}, H}(\mathcal{B})$. Then the map $T \otimes \text{id}_H : \mathcal{E} \rtimes H \rightarrow \mathcal{E}' \rtimes H$ descends to an intertwiner between $\mathcal{E} \rtimes H$ and $\mathcal{E}' \rtimes H \in {}^* \text{-mod}_{\mathcal{A} \rtimes H}(\mathcal{B} \rtimes H)$. The adjoint of $T \otimes \text{id}$ is given by $T^* \otimes \text{id}$.

Proof. This is an easy verification using the H -equivariance of T as well as the existence of T^* . In fact, everything is already true on the level of $\mathcal{E} \rtimes H$ and $\mathcal{E}' \rtimes H$. \square

Collecting the results of the preceding lemmas we finally arrive at the following statement:

Proposition 6.9. The assignment $\mathcal{E} \mapsto \mathcal{E} \rtimes H$ on objects and $T \mapsto T \otimes \text{id}$ on morphisms gives a functor

$$\cdot \rtimes H : {}^* \text{-mod}_{\mathcal{A}, H}(\mathcal{B}) \rightarrow {}^* \text{-mod}_{\mathcal{A} \rtimes H}(\mathcal{B} \rtimes H) \tag{6.19}$$

which restricts to functors

$$\cdot \rtimes H : {}^* \text{-Mod}_{\mathcal{A}, H}(\mathcal{B}) \rightarrow {}^* \text{-Mod}_{\mathcal{A} \rtimes H}(\mathcal{B} \rtimes H) \tag{6.20}$$

$$\cdot \rtimes H : {}^* \text{-rep}_{\mathcal{A}, H}(\mathcal{B}) \rightarrow {}^* \text{-rep}_{\mathcal{A} \rtimes H}(\mathcal{B} \rtimes H) \tag{6.21}$$

$$\cdot \rtimes H : {}^* \text{-Rep}_{\mathcal{A}, H}(\mathcal{B}) \rightarrow {}^* \text{-Rep}_{\mathcal{A} \rtimes H}(\mathcal{B} \rtimes H). \tag{6.22}$$

In a next step we want to discuss the compatibility of the *crossed product functors* (6.19), (6.20), (6.21) and (6.22), respectively, with the tensor product functors from (3.2) and (3.4), respectively. Again, we only have to investigate the case of ${}^* \text{-mod}$, the other cases will follow analogously.

Proposition 6.10. *Let ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \in {}^* \text{-mod}_{\mathcal{B}, H}(\mathcal{C})$ and ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \in {}^* \text{-mod}_{\mathcal{A}, H}(\mathcal{B})$. Then we have*

(i) *The map*

$$\begin{aligned} I_1 : (\mathcal{F} \rtimes H) \widehat{\otimes}_{\mathcal{B} \rtimes H} (\mathcal{E} \rtimes H) &\ni (x \otimes g) \widehat{\otimes}_{\mathcal{B} \rtimes H} (y \otimes h) \\ &\mapsto (x \widehat{\otimes}_{\mathcal{B}} (g(1) \triangleright y)) \otimes g(2)h \in (\mathcal{F} \widehat{\otimes}_{\mathcal{B}} \mathcal{E}) \otimes H \end{aligned} \tag{6.23}$$

is a canonical isomorphism of * -representations of $\mathcal{C} \rtimes H$ on $(\mathcal{A} \rtimes H)$ -inner product modules.

(ii) *The map*

$$I_2 : \overline{\mathcal{E} \rtimes H} \ni \overline{x \otimes g} \mapsto g_{(1)}^* \overline{\triangleright x} \otimes g_{(2)}^* = \overline{S^{-1}(g(1)) \triangleright x} \otimes g_{(2)}^* \in \overline{\mathcal{E}} \rtimes H \tag{6.24}$$

is a canonical isomorphism of (right) $\mathcal{B} \rtimes H$ -representations on (left) $(\mathcal{A} \rtimes H)$ -inner product modules with inverse given explicitly by

$$I_2^{-1}(\overline{x} \otimes g) = \overline{g_{(1)}^* \triangleright x} \otimes g_{(2)}^*. \tag{6.25}$$

Proof. For the first part one checks easily that I_1 is well-defined over $\otimes_{\mathcal{B} \rtimes H}$. Moreover, it is a straightforward computation that I_1 is a bimodule map as specified. For the isometry we compute

$$\begin{aligned} &\langle (x \otimes_{\mathcal{B}} (g(1) \triangleright y)) \otimes g(2)h, (x' \otimes_{\mathcal{B}} (g'(1) \triangleright y')) \otimes g'(2)h' \rangle_{\mathcal{A} \rtimes H}^{(\mathcal{F} \widehat{\otimes}_{\mathcal{B}} \mathcal{E}) \rtimes H} \\ &= \left\langle \left(h_{(1)}^* g_{(2)}^* \right) \triangleright \langle x \otimes_{\mathcal{B}} (g(1) \triangleright y), x' \otimes_{\mathcal{B}} (g'(1) \triangleright y') \rangle_{\mathcal{A}}^{\mathcal{F} \otimes \mathcal{E}} \right\rangle \otimes h_{(2)}^* g_{(3)}^* g'_{(2)} h' \\ &= \left\langle \left(h_{(1)}^* g_{(2)}^* \right) \triangleright \left\langle g(1) \triangleright y, \langle x, x' \rangle_{\mathcal{B}}^{\mathcal{F}} \cdot (g'(1) \triangleright y') \right\rangle_{\mathcal{A}}^{\mathcal{E}} \right\rangle \otimes h_{(2)}^* g_{(3)}^* g'_{(2)} h' \\ &= \left\langle h_{(1)}^* \triangleright \left\langle (S(g_{(2)}^*)^* g_{(1)}) \triangleright y, g_{(3)}^* \triangleright \left(\langle x, x' \rangle_{\mathcal{B}}^{\mathcal{F}} \cdot (g'(1) \triangleright y') \right) \right\rangle_{\mathcal{A}}^{\mathcal{E}} \right\rangle \otimes h_{(2)}^* g_{(4)}^* g'_{(2)} h' \\ &= \left\langle h_{(1)}^* \triangleright \left\langle y, (g_{(1)}^* \triangleright \langle x, x' \rangle_{\mathcal{B}}^{\mathcal{F}}) \cdot (g_{(2)}^* g'_{(1)}) \triangleright y' \right\rangle_{\mathcal{A}}^{\mathcal{E}} \right\rangle \otimes h_{(2)}^* g_{(3)}^* g'_{(2)} h' \\ &= \left\langle y \otimes h, \left((g_{(1)}^* \triangleright \langle x, x' \rangle_{\mathcal{B}}^{\mathcal{F}}) \cdot (g_{(2)}^* g'_{(1)}) \triangleright y' \right) \otimes (g_{(2)}^* g'_{(2)}) h' \right\rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \rtimes H} \\ &= \left\langle y \otimes h, \left((g_{(1)}^* \triangleright \langle x, x' \rangle_{\mathcal{B}}^{\mathcal{F}}) \otimes g_{(2)}^* g'_{(1)} \right) \cdot (y' \otimes h') \right\rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \rtimes H} \\ &= \left\langle y \otimes h, \langle x \otimes g, x' \otimes g' \rangle_{\mathcal{B} \rtimes H}^{\mathcal{F} \rtimes H} \cdot (y' \otimes h') \right\rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \rtimes H} \\ &= \langle (x \otimes g) \otimes_{\mathcal{B}} (y \otimes h), (x' \otimes g') \otimes_{\mathcal{B}} (y' \otimes h') \rangle_{\mathcal{A} \rtimes H}^{(\mathcal{F} \rtimes H) \widehat{\otimes} (\mathcal{E} \rtimes H)}, \end{aligned}$$

whence I_1 is isometric already on the level of $\otimes_{\mathcal{B}}$ instead of $\widehat{\otimes}_{\mathcal{B}}$. Finally, surjectivity is clear since $(x \otimes \mathbb{1}_H) \otimes_{\mathcal{B}} (y \otimes g)$ is mapped to $(x \otimes_{\mathcal{B}} y) \otimes g$. The injectivity follows as on the quotients both inner products are, by definition, non-degenerate whence an isometric map is injective. This shows that I_1 is an isomorphism indeed. Moreover, it is canonical in the following sense: Let $S : \mathcal{F} \rightarrow \mathcal{F}'$ and $T : \mathcal{E} \rightarrow \mathcal{E}'$ be morphisms in ${}^* \text{-mod}_{\mathcal{B}, H}(\mathcal{C})$ and ${}^* \text{-mod}_{\mathcal{A}, H}(\mathcal{B})$, respectively. Then $S \widehat{\otimes} T$ is a morphism in ${}^* \text{-mod}_{\mathcal{A}, H}(\mathcal{C})$ and $S \otimes \text{id}$ and $T \otimes \text{id}$ are the corresponding morphisms in ${}^* \text{-mod}_{\mathcal{B} \rtimes H}(\mathcal{C} \rtimes H)$ and ${}^* \text{-mod}_{\mathcal{A} \rtimes H}(\mathcal{B} \rtimes H)$,

respectively, according to Lemma 6.8. Then I_1 is compatible with morphisms as it is easy to check that

$$I_1 \circ ((S \otimes \text{id}) \widehat{\otimes} (T \otimes \text{id})) = ((S \widehat{\otimes} T) \otimes \text{id}) \circ I_1.$$

This proves the first part. For the second we first observe that I_2 certainly has the correct \mathbb{C} -linearity properties. A lengthy but straightforward computation shows by successively unwinding the definitions that I_2 is a bimodule map. Thus, we compute using (2.20) and (6.18)

$$\begin{aligned} & \mathcal{A} \rtimes H \left(I_2(\overline{x \otimes g}), I_2(\overline{y \otimes h}) \right)_{\overline{\mathcal{E} \rtimes H}} \\ &= \mathcal{A} \rtimes H \left((g_{(1)}^* \triangleright \overline{x}) \otimes g_{(2)}^*, (h_{(1)}^* \triangleright \overline{y}) \otimes h_{(2)}^* \right)_{\overline{\mathcal{E} \rtimes H}} \\ &= \mathcal{A} \rtimes H \left(\overline{S(g_{(1)}^*)^* \triangleright x} \otimes g_{(2)}^*, \overline{S(h_{(1)}^*)^* \triangleright y} \otimes h_{(2)}^* \right)_{\overline{\mathcal{E} \rtimes H}} \\ &= \left(g_{(3)}^* \triangleright \mathcal{A} \left(S^{-1}(g_{(2)}^*) \triangleright \overline{S^{-1}(g_{(1)} \triangleright x)}, S^{-1}(h_{(2)}^*) \triangleright \overline{S^{-1}(h_{(1)} \triangleright y)} \right)_{\overline{\mathcal{E}}} \right) \otimes g_{(4)}^* h_{(3)} \\ &= \left(g_{(3)}^* \triangleright \mathcal{A} \left(\overline{(g_{(2)} S^{-1}(g_{(1)})) \triangleright x}, \overline{(h_{(2)} S^{-1}(h_{(1)})) \triangleright y} \right)_{\overline{\mathcal{E}}} \right) \otimes g_{(4)}^* h_{(3)} \\ &= \left(g_{(1)}^* \triangleright \langle x, y \rangle_{\mathcal{A}} \right) \otimes g_{(2)}^* h \\ &= \mathcal{A} \rtimes H \left(\overline{x \otimes g}, \overline{y \otimes h} \right)_{\overline{\mathcal{E} \rtimes H}}, \end{aligned}$$

whence I_2 is isometric. It is a simple computation that (6.25) provides an inverse for I_2 . The compatibility with intertwiners is shown analogously as for I_1 . \square

Remark 6.11. Using the second part of the proposition we also obtain an easy proof for the complete positivity of the inner product on $\overline{\mathcal{E} \rtimes H}$ if we had a left \mathcal{B} -linear \mathcal{B} -valued inner product on \mathcal{E} . In this case we can pass to $\overline{\mathcal{E}}$ instead, making the inner product right \mathcal{B} -linear and use $\overline{\mathcal{E} \rtimes H}$, which has, by Lemma 6.6, a completely positive right $(\mathcal{B} \rtimes H)$ -linear $(\mathcal{B} \rtimes H)$ -valued inner product. This is isometric to the corresponding inner product on $\overline{\mathcal{E} \rtimes H}$ by Proposition 6.10 and by Remark 2.2 the complete positivity of the inner product on $\overline{\mathcal{E} \rtimes H}$ follows.

Rephrasing the statement of the proposition in terms of the functors $\widehat{\otimes}$ and $\cdot \rtimes H$ we have the following result:

Corollary 6.12. *The diagram*

$$\begin{array}{ccc} * \text{-mod}_{\mathcal{B}, H}(\mathcal{C}) \times * \text{-mod}_{\mathcal{A}, H}(\mathcal{B}) & \xrightarrow{\widehat{\otimes}} & * \text{-mod}_{\mathcal{A}, H}(\mathcal{C}) \\ \downarrow (\cdot \rtimes H) \times (\cdot \rtimes H) & & \downarrow \cdot \rtimes H \\ * \text{-mod}_{\mathcal{B} \rtimes H}(\mathcal{C} \rtimes H) \times * \text{-mod}_{\mathcal{A} \rtimes H}(\mathcal{B} \rtimes H) & \xrightarrow{\widehat{\otimes}} & * \text{-mod}_{\mathcal{A} \rtimes H}(\mathcal{C} \rtimes H) \end{array} \tag{6.26}$$

commutes in the sense of functors, i.e. up to the natural transformation I_1 .

Analogous statements hold for the complex conjugation exchanging left and right linear inner products. Let us also remark that Proposition 6.10 still holds if we restrict ourselves to $*$ -representations in $*$ -Mod, $*$ -rep or $*$ -Rep, respectively.

6.3. The Picard groupoid of crossed products

After our discussion of crossed product constructions for general $*$ -representations we turn now to the equivalence bimodules. The first lemma ensures that the functor $\cdot \rtimes H$ applied to equivalence bimodules gives again equivalence bimodules:

Lemma 6.13. *Let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be an H -covariant $*$ -Morita equivalence bimodule. Then ${}_{\mathcal{B} \rtimes H}\mathcal{E} \rtimes H_{\mathcal{A} \rtimes H}$, endowed with the induced $(\mathcal{B} \rtimes H)$ -left linear $(\mathcal{B} \rtimes H)$ -valued inner product ${}_{\mathcal{B} \rtimes H}\langle \cdot, \cdot \rangle^{\mathcal{E} \rtimes H}$ and the induced right $(\mathcal{A} \rtimes H)$ -linear $(\mathcal{A} \rtimes H)$ -valued inner product $\langle \cdot, \cdot \rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \rtimes H}$, is a $*$ -Morita equivalence bimodule for $\mathcal{B} \rtimes H$ and $\mathcal{A} \rtimes H$. Moreover, if ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is even a strong equivalence bimodule then ${}_{\mathcal{B} \rtimes H}\mathcal{E} \rtimes H_{\mathcal{A} \rtimes H}$ is a strong equivalence bimodule as well.*

Proof. We already know that on $\mathcal{E} \otimes H$ we have two inner products ${}_{\mathcal{B} \rtimes H}\langle \cdot, \cdot \rangle^{\mathcal{E} \otimes H}$ and $\langle \cdot, \cdot \rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \otimes H}$ which have the correct linearity and compatibility with respect to the $(\mathcal{B} \rtimes H, \mathcal{A} \rtimes H)$ -bimodule structure. To show the compatibility of the inner products we compute

$$\begin{aligned} & {}_{\mathcal{B} \rtimes H}\langle x \otimes g, y \otimes h \rangle^{\mathcal{E} \otimes H} \cdot (z \otimes k) \\ &= \left({}_{\mathcal{B}}\langle x, (S(g(1))^* S^{-1}(h(1))) \triangleright y \rangle^{\mathcal{E}} \right) \cdot (z \otimes k) \\ &= {}_{\mathcal{B}}\langle x, (S(g(1))^* S^{-1}(h(1))) \triangleright y \rangle^{\mathcal{E}} \cdot \left((g(2)h(2)^*) \triangleright z \right) \otimes g(3)h(3)^* k \\ &= x \cdot \left\langle (S(g(1))^* S^{-1}(h(1))) \triangleright y, (g(2)h(2)^*) \triangleright z \right\rangle_{\mathcal{A}}^{\mathcal{E}} \otimes g(3)h(3)^* k \\ &= (x \otimes g) \cdot \left(\langle S^{-1}(h(1)) \triangleright y, h(2)^* \triangleright z \rangle_{\mathcal{A}}^{\mathcal{E}} \otimes h(3)^* k \right) \\ &= (x \otimes g) \cdot \left(h(1)^* \triangleright \langle y, z \rangle_{\mathcal{A}}^{\mathcal{E}} \otimes h(2)^* k \right) \\ &= (x \otimes g) \cdot \langle y \otimes h, z \otimes k \rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \otimes H}, \end{aligned}$$

whence (3.9) follows. Thus, it follows that their degeneracy spaces coincide whence we can non-ambiguously define $\mathcal{E} \rtimes H$ and obtain a $(\mathcal{B} \rtimes H, \mathcal{A} \rtimes H)$ -bimodule with compatible non-degenerate inner products ${}_{\mathcal{B} \rtimes H}\langle \cdot, \cdot \rangle^{\mathcal{E} \rtimes H}$ and $\langle \cdot, \cdot \rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \rtimes H}$. Moreover, since $\mathcal{B} \cdot \mathcal{E} = \mathcal{E} \cdot \mathcal{A}$ it is easy to check that $\mathcal{E} \rtimes H$ is also strongly non-degenerate for both module structures (in the unital case this is trivial). It remains to check whether the inner products are full. If $a = \sum_i \langle x_i, y_i \rangle_{\mathcal{A}}^{\mathcal{E}}$ by fullness of $\langle \cdot, \cdot \rangle_{\mathcal{A}}^{\mathcal{E}}$ then

$$a \otimes g = \sum_i \langle x_i \otimes \mathbb{1}_H, y_i \otimes g \rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \rtimes H}$$

implies fullness of $\langle \cdot, \cdot \rangle_{\mathcal{A} \rtimes H}^{\mathcal{E} \rtimes H}$ and analogously for ${}_{\mathcal{B} \rtimes H}\langle \cdot, \cdot \rangle^{\mathcal{E} \rtimes H}$. Thus ${}_{\mathcal{B} \rtimes H}\mathcal{E} \rtimes H_{\mathcal{A} \rtimes H}$ is indeed a $*$ -Morita equivalence bimodule. Since complete positivity of inner products is preserved

under the crossed product construction by Lemma 6.6, the remaining statement follows as well. \square

Remark 6.14. This lemma has a remarkable and well-known counterpart in C^* -algebra theory: here it is known that for a locally compact group G acting (strongly) continuously on C^* -algebras, G -covariant strong Morita equivalence implies strong Morita equivalence of the corresponding C^* -algebraic crossed products, see [16,18]. The above lemma reproduces this statements e.g. for the case of a discrete group by using the group algebra $H = \mathbb{C}[G]$. It will be left to a future project to investigate topological versions of the above lemma in order to recover the statements of [16,18] fully. We also refer to [20] for more general crossed product constructions in the C^* -algebraic framework related to Morita theory and for further references.

Adapting Lemma 6.8 to equivalence bimodules we immediately have to following statement:

Lemma 6.15. *Let $\mathcal{B}^{\mathcal{E}}_{\mathcal{A}}$, $\mathcal{B}^{\mathcal{E}'}$ be isomorphic H -covariant $*$ -Morita equivalence bimodules such that $T : \mathcal{B}^{\mathcal{E}}_{\mathcal{A}} \rightarrow \mathcal{B}^{\mathcal{E}'}$ is an isomorphism. Then $T \otimes \text{id} : \mathcal{B}^{\mathcal{E}}_{\mathcal{A}} \rtimes H_{\mathcal{A} \rtimes H} \rightarrow \mathcal{B}^{\mathcal{E}'}$ is an isomorphism of $*$ -Morita equivalence bimodules. The analogous statement holds for strong Morita equivalence bimodules.*

To obtain a good Morita theory for crossed products we have to guarantee that $\mathcal{A} \rtimes H$ is idempotent and non-degenerate. While it is easy to see that $\mathcal{A} \rtimes H$ is idempotent if \mathcal{A} is idempotent, there could be some torsion-effects due to $\otimes_{\mathbb{C}}$ which spoil the non-degeneracy of $\mathcal{A} \rtimes H$ even if \mathcal{A} was non-degenerate. Nevertheless it is safe to assume that $\mathcal{A} \rtimes H$ is non-degenerate as e.g. in the unital case $\mathcal{A} \rtimes H$ is unital and thus non-degenerate. Also if \mathbb{C} is a field we will have no problems. Thus we shall ignore these subtleties in the following and always assume that all occurring crossed products are non-degenerate.

Lemma 6.16. *The map*

$$I_3 : \mathcal{A} \rtimes \mathcal{A} \rtimes H \ni x \otimes g \mapsto x \otimes g \in \mathcal{A} \rtimes H \rtimes \mathcal{A} \rtimes H_{\mathcal{A} \rtimes H} \tag{6.27}$$

is an isomorphism of strong Morita equivalence bimodules.

Proof. Thanks to the assumption that $\mathcal{A} \rtimes H$ is non-degenerate, the canonical inner products on $\mathcal{A} \rtimes H$ are non-degenerate whence there is no quotient procedure necessary. Then, an easy check shows that the bimodule structures and the inner products on $\mathcal{A} \otimes H$ coming from both interpretations simply coincide. \square

Now we can formulate the main result of this section which relates the H -covariant Picard groupoid to the Picard groupoid of the corresponding crossed products:

Theorem 6.17. *The crossed product with H induces groupoid morphisms*

$$\cdot \rtimes H : \text{Pic}^*_H \rightarrow \text{Pic}^* \tag{6.28}$$

and

$$\cdot \rtimes H : \text{Pic}_H^{\text{str}} \longrightarrow \text{Pic}^{\text{str}}, \tag{6.29}$$

where units $\mathcal{A} \in \text{Pic}_H^*$ (or $\text{Pic}_H^{\text{str}}$, respectively) are mapped to the crossed product algebras $\mathcal{A} \rtimes H$ and arrows $[\mathcal{B} \xrightarrow{\mathcal{E}} \mathcal{A}] \in \text{Pic}_H^*(\mathcal{B}, \mathcal{A})$ (or $\text{Pic}_H^{\text{str}}(\mathcal{B}, \mathcal{A})$, respectively) are mapped to the arrows $[\mathcal{B} \rtimes H \xrightarrow{\mathcal{E} \rtimes H} \mathcal{A} \rtimes H] \in \text{Pic}^*(\mathcal{B} \rtimes H, \mathcal{A} \rtimes H)$ (or $\text{Pic}^{\text{str}}(\mathcal{B} \rtimes H, \mathcal{A} \rtimes H)$, respectively).

Proof. From Lemma 6.15 we see that $\cdot \rtimes H$ is well-defined on isomorphism classes. Moreover, Lemma 6.16 ensures that units are mapped to units indeed, in the stated way. Finally, Proposition 6.10 can easily be adapted to the case of two compatible inner products whence it shows that tensor products are mapped to tensor products: for equivalence bimodules one inner product determines the other by compatibility (3.9). Again, Proposition 6.10 shows that complex conjugated bimodules are mapped to complex conjugated bimodules whence products and inverses in Pic_H^* are mapped to products and inverses in Pic^* . The same holds for $\text{Pic}_H^{\text{str}}$ and Pic^{str} , whence (6.28) and (6.29) are groupoid morphisms indeed. \square

Let us now collect a few conclusions from this theorem. The first transfers well-known results from C^* -algebra theory [16,18], where strongly continuous actions of a locally compact group is considered, to our algebraic framework.

Corollary 6.18. *If \mathcal{A} and \mathcal{B} are H -covariantly strongly Morita equivalent (or $*$ -Morita equivalent, respectively) then $\mathcal{A} \rtimes H$ and $\mathcal{B} \rtimes H$ are strongly Morita equivalent ($*$ -Morita equivalent, respectively).*

Corollary 6.19. *There are group homomorphisms*

$$\text{Pic}_H^*(\mathcal{A}) \longrightarrow \text{Pic}^*(\mathcal{A} \rtimes H) \tag{6.30}$$

and

$$\text{Pic}_H^{\text{str}}(\mathcal{A}) \longrightarrow \text{Pic}^{\text{str}}(\mathcal{A} \rtimes H). \tag{6.31}$$

The following easy proposition shows that the groupoid morphism in Theorem 6.17 is compatible with the (much easier) groupoid morphism from (6.4) via the canonical groupoid morphism ℓ from Proposition 4.6.

Proposition 6.20. *The groupoid morphism $\cdot \rtimes H$ is compatible with the groupoid morphism ℓ , i.e. the diagram*

$$\begin{array}{ccc} \text{Iso}_H^* & \xrightarrow{\cdot \rtimes H} & \text{Iso}^* \\ \ell \downarrow & & \downarrow \ell \\ \text{Pic}_H^{\text{str}} & \xrightarrow{\cdot \rtimes H} & \text{Pic}^{\text{str}} \end{array} \tag{6.32}$$

commutes.

Proof. Let $\Phi \in \text{Iso}_H^*(\mathcal{B}, \mathcal{A})$. Then $\ell(\Phi \otimes \text{id})$ is represented by the bimodule $\mathcal{B} \otimes H$ endowed with the usual left $(\mathcal{B} \rtimes H)$ -module structure and the right $(\mathcal{A} \rtimes H)$ -module structure

$$\begin{aligned} (b \otimes g) \cdot_{\Phi \otimes \text{id}} (a \otimes h) &= (b \otimes g) \cdot (\Phi(a) \otimes h) = (b\Phi(g_{(1)} \triangleright a)) \otimes g_{(2)}h \\ &= (b \cdot_{\Phi} (g_{(1)} \triangleright a)) \otimes g_{(2)}h, \end{aligned}$$

which coincides with the right $(\mathcal{A} \rtimes H)$ -module structure on ${}_{\mathcal{B}}\mathcal{B}_{\mathcal{A}}^{\Phi} \rtimes H$ thanks to the H -equivariance of Φ . Analogously, for the $(\mathcal{A} \rtimes H)$ -valued inner products we have

$$\begin{aligned} \langle b \otimes g, b' \otimes g' \rangle_{\mathcal{A} \rtimes H}^{\mathcal{B} \rtimes H, \Phi \otimes \text{id}} &= (\Phi \otimes \text{id})^{-1} \left(\langle b \otimes g, b' \otimes g' \rangle_{\mathcal{B} \rtimes H}^{\mathcal{B} \rtimes H} \right) \\ &= (\Phi^{-1} \otimes \text{id}) \left((g_{(1)}^* \triangleright b^*) (g_{(2)}^* \triangleright b') \otimes g_{(3)}^* g' \right) \\ &= (g_{(1)}^* \triangleright \Phi^{-1}(b^* b')) \otimes g_{(2)}^* g' \\ &= (g_{(1)}^* \triangleright \langle b, b' \rangle_{\mathcal{A}}^{\mathcal{B}^{\Phi}}) \otimes g_{(2)}^* g' \\ &= \langle b \otimes g, b' \otimes g' \rangle_{\mathcal{A} \rtimes H}^{\mathcal{B}^{\Phi} \rtimes H}, \end{aligned}$$

whence they coincide, too. The left $(\mathcal{B} \rtimes H)$ -module structure and the $(\mathcal{B} \rtimes H)$ -valued inner products are unchanged whence the statement follows. \square

6.4. An example: $\text{Pic}_H^{\text{str}}(\mathbb{C}) \longrightarrow \text{Pic}^{\text{str}}(H)$

We illustrate our general techniques by a simple but yet interesting example where $\mathcal{A} = \mathbb{C}$, endowed with the trivial action of H . First, we need the following lemma:

Lemma 6.21. Let $\chi \in \text{GL}(H, \mathbb{C})$.

- (i) $\chi^{-1}(g) = \chi(S^{-1}(g)) = \chi(S(g))$.
- (ii) $\chi \in U(H, \mathbb{C})$ if and only if $\chi(g^*) = \overline{\chi(g)}$.
- (iii) $\Phi^{\chi}(g) = \chi(S(g_{(1)}))g_{(2)}$ defines an automorphism $\Phi^{\chi} \in \text{Aut}(H)$ and $\Phi^{\chi} \in \text{Aut}^*(H)$ if $\chi \in U(H, \mathbb{C})$.
- (iv) The map

$$\text{GL}(H, \mathbb{C}) \ni \chi \mapsto \Phi^{\chi} \in \text{Aut}(H) \tag{6.33}$$

is an injective group homomorphism.

- (v) Φ^{χ} is an inner automorphism if and only if $\chi = e$.

Proof. Clearly, $\chi^{-1}(g) = \chi(S^{-1}(g))$ by (A.3) since the action is trivial. Moreover, one easily checks that $\tilde{\chi}(g) = \chi(S(g))$ defines an inverse with respect to the convolution product whence by uniqueness $\tilde{\chi} = \chi^{-1}$. For the second part we define $\bar{\chi}(g) = \overline{\chi(g)}$. Then a straightforward computation using the first part and the unitarity condition for χ shows that $\bar{\chi}$ is a convolution inverse of χ^{-1} and thus equal to χ . The converse direction is trivial. For the third and fourth part we have $\Phi^e = \text{id}$ and $\Phi^{\chi}(gh) = \Phi^{\chi}(g)\Phi^{\chi}(h)$ by a little computation. Hence, Φ^{χ} is a homomorphism and if $\chi \in U(H, \mathbb{C})$ one immediately has $\Phi^{\chi}(g^*) = \Phi^{\chi}(g)^*$. Next, we prove $\Phi^{\chi}(\Phi^{\bar{\chi}}(g)) = \Phi^{\chi * \bar{\chi}}(g)$ again by a simple computation. Then it follows that Φ^{χ} is bijective

and (6.33) is a group homomorphism. For the injectivity assume $\Phi^\chi(g) = g$. Applying ε to this equation gives immediately $\chi(S(g)) = \varepsilon(g)$ whence $\chi = e$. \square

This lemma is the Hopf-algebraic version of the well-known construction of automorphisms of the group algebra $\mathbb{C}[G]$ out of group characters of a group G . It shows that $U(H, \mathbb{C})$ always gives a non-trivial contribution to $\text{Pic}^{\text{str}}(H)$: In fact, from [12, Eq. (2.4)], i.e. the non-covariant version of (4.5), we have an injective group homomorphism

$$U(H, \mathbb{C}) \ni \chi \mapsto \ell(\Phi^\chi) \in \text{Pic}^{\text{str}}(H), \tag{6.34}$$

where injectivity follows from the last part of the lemma. On the other hand, we know that the crossed product algebra $\mathbb{C} \rtimes H$ is just H itself where the canonical identification is simply $z \otimes g \mapsto zg$. Thus the general groupoid morphism $\cdot \rtimes H$ from (6.29) gives a group homomorphism

$$\text{Pic}_H^{\text{str}}(\mathbb{C}) \longrightarrow \text{Pic}^{\text{str}}(H), \tag{6.35}$$

which we shall relate to (6.34). From Remark A.4 we know that $U(H, \mathbb{C}) = U_0(H, \mathbb{C})$ and from Proposition 5.16 we know that we can view $U_0(H, \mathbb{C})$ as a subgroup of $\text{Pic}_H^{\text{str}}(\mathbb{C})$. Putting these group homomorphisms together we obtain the following statement:

Proposition 6.22. *The diagram of group homomorphisms*

$$\begin{array}{ccc} U(H, \mathbb{C}) & \hookrightarrow & \text{Pic}_H^{\text{str}}(\mathbb{C}) \\ \downarrow & & \downarrow \cdot \rtimes H \\ \text{Aut}^*(H) & \xrightarrow{\ell} & \text{Pic}^{\text{str}}(H) \end{array} \tag{6.36}$$

commutes whence $U(H, \mathbb{C})$ can be viewed as a subgroup of $\text{Pic}^{\text{str}}(H)$.

Proof. Let $\chi \in U(H, \mathbb{C}) = U_0(H, \mathbb{C})$. Then the image of χ in $\text{Pic}_H^{\text{str}}(\mathbb{C})$ is given by the isomorphism class of the trivial bimodule \mathbb{C} with canonical inner products and H -action $g \triangleright z = \chi(g_{(1)})g_{(2)} \triangleright z = \chi(g)z$. We denote this bimodule by \mathbb{C}^χ . Then $[\mathbb{C}^\chi]$ is mapped to $[\mathbb{C}^\chi \rtimes H]$ where $\mathbb{C}^\chi \rtimes H \cong H$ as \mathbb{C} -modules and the left H -module structure is given by

$$g \cdot h = \chi(g_{(1)})g_{(2)}h = \Phi^{\chi^{-1}}(g)h = g \cdot \Phi^\chi h,$$

while the right H -module structure is the canonical one. The left-linear inner product is easily shown to be $\Phi^\chi(g h^*)$ and the right-linear inner product is the canonical one. Thus $\mathbb{C}^\chi \rtimes H$ is isomorphic to $\Phi^\chi H_H$ whose class in $\text{Pic}^{\text{str}}(H)$ is just $\ell(\Phi^\chi)$. This proves the commutativity of (6.36). The injectivity of the inclusion of $U(H, \mathbb{C})$ into $\text{Pic}^{\text{str}}(H)$ was shown in (6.34). \square

If \mathbb{C} is even an algebraically closed field (which is the case if \mathbb{R} is a real closed field, see e.g. [24, Section 5.1]) then we can make (6.36) more precise:

Corollary 6.23. *Let R be real closed field whence C is algebraically closed.*

- (i) $\text{Pic}^{\text{str}}(C) = \{\text{id}\}$.
- (ii) $\text{Pic}_H^{\text{str}}(C) = U(H, C) = U_0(H, C)$.
- (iii) $\text{Pic}_H^{\text{str}}(C) \longrightarrow \text{Pic}^{\text{str}}(H)$ is injective and its image is determined by (6.36).

Proof. The first part is clear since the only equivalence bimodule (up to isomorphism) is the one-dimensional vector space C with the (uniquely determined up to isometries) canonical positive definite inner product $\langle z, w \rangle = \bar{z}w$. Note that $\text{Pic}^*(C) = \mathbb{Z}_2$ in this case. Then the second part follows from Proposition 5.16 whence the third part follows from Proposition 6.22. \square

Remark 6.24. Though $\cdot \rtimes H : \text{Pic}_H^{\text{str}}(C) \longrightarrow \text{Pic}^{\text{str}}(H)$ is injective in this example it needs not to be surjective: Neither the map $U(H, C) \longrightarrow \text{Aut}^*(H)$ nor the map ℓ need to be surjective. An example can be obtained for a commutative Hopf algebra H . In this case the antipode S is a $*$ -automorphism, $S \in \text{Aut}^*(H)$, and if $\Phi^\chi = S$ for some χ then applying ε gives $\chi(g) = \varepsilon(g)$ whence $S = \text{id}$ follows. Beside this case, which is rarely of interest, we conclude that $\ell(S) \in \text{Pic}^{\text{str}}(H)$ gives a non-trivial element not in the image of $\text{Pic}_H^{\text{str}}(C)$. In general, the interesting elements of $\text{Pic}^{\text{str}}(H)$ are those which are *not* in the image of ℓ anyway, see e.g. the discussion in [12, Section 2]. According to Proposition 6.22 they can never be obtained from $\text{Pic}_H^{\text{str}}(C)$.

Example 6.25. To have a more concrete example we consider again $H = U_C(\mathfrak{g})$. Then $a \in U(H, C)$ satisfies $a(\xi X) = a(\xi)a(X)$ by use of the action condition, where $\xi \in \mathfrak{g}$ and $X \in U_C(\mathfrak{g})$ since $\varepsilon(\xi) = 0$. Since \mathfrak{g} together with $\mathbb{1}$ generates $U_C(\mathfrak{g})$, we obtain $a(XY) = a(X)a(Y)$ for all $X, Y \in U_C(\mathfrak{g})$. From the normalization $a(\mathbb{1}) = 1$ and the unitarity $a(X^*) = \overline{a(X)}$ we finally see that any $a \in U(H, C)$ is given by a $*$ -homomorphism $a : U_C(\mathfrak{g}) \longrightarrow C$. Thus the group $U(H, C)$ coincides with the unitary C -valued characters of \mathfrak{g} .

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Appendix A. The groups $GL(H, \mathcal{A})$, $GL_0(H, \mathcal{A})$, $U(H, \mathcal{A})$ and $U_0(H, \mathcal{A})$

In this appendix we shall describe several groups naturally associated to any unital $*$ -algebra with a $*$ -action of a Hopf $*$ -algebra on it.

A.1. Definitions and fundamental properties

Recall that on $\text{Hom}_C(H, \mathcal{A})$ one has the associative convolution product

$$(a * b)(g) = a(g_{(1)})b(g_{(2)}) \tag{A.1}$$

where $a, b \in \text{Hom}_{\mathbb{C}}(H, \mathcal{A})$. Since \mathcal{A} is unital and H counital, $\text{Hom}_{\mathbb{C}}(H, \mathcal{A})$ is known to be unital with unit e given by

$$e(g) = \varepsilon(g)\mathbb{1}_{\mathcal{A}}, \quad (\text{A.2})$$

see e.g. [27,35]. We are now interested in particular subgroups of the group of all invertible elements in the convolution algebra $\text{Hom}_{\mathbb{C}}(H, \mathcal{A})$.

Definition A.1. An element $a \in \text{Hom}_{\mathbb{C}}(H, \mathcal{A})$ belongs to $\text{GL}(H, \mathcal{A})$ if for all $g, h \in H$ and $b \in \mathcal{A}$, we have

- (i) $a(\mathbb{1}_H) = \mathbb{1}_{\mathcal{A}}$ (normalization),
 - (ii) $a(gh) = a(g_{(1)})(g_{(2)} \triangleright a(h))$ (action condition),
 - (iii) $(g_{(1)} \triangleright b)a(g_{(2)}) = a(g_{(1)})(g_{(2)} \triangleright b)$ (module condition),
- and it belongs to $U(H, \mathcal{A})$ if in addition
- (iv) $a(g_{(1)})(a(S(g_{(2)}))^*)^* = \varepsilon(g)\mathbb{1}_{\mathcal{A}}$ (unitarity condition).

The ‘action condition’ can also be interpreted as a cocycle condition while the ‘module condition’ expresses a certain centrality property of the values $a(g) \in \mathcal{A}$. The subsets $U(H, \mathcal{A}) \subseteq \text{GL}(H, \mathcal{A})$ turn out to be subgroups of the group of invertible elements $\text{GL}(\text{Hom}_{\mathbb{C}}(H, \mathcal{A}), *)$:

Proposition A.2. The set $\text{GL}(H, \mathcal{A})$ becomes a group with respect to the convolution product $*$ and $U(H, \mathcal{A})$ is a subgroup. The inverse of $a \in \text{GL}(H, \mathcal{A})$ is explicitly given by

$$a^{-1}(g) = g_{(2)} \triangleright a(S^{-1}(g_{(1)})). \quad (\text{A.3})$$

Proof. Clearly $e \in U(H, \mathcal{A}) \subseteq \text{GL}(H, \mathcal{A})$ and $*$ is associative. Now let $a, b \in \text{GL}(H, \mathcal{A})$. Then $a * b$ fulfills the normalization condition. Moreover, by (ii) and (iii)

$$\begin{aligned} (a * b)(gh) &= a(g_{(1)}h_{(1)})b(g_{(2)}h_{(2)}) \\ &= a(g_{(1)})(g_{(2)} \triangleright a(h_{(1)}))b(g_{(3)})(g_{(4)} \triangleright b(h_{(2)})) \\ &= a(g_{(1)})b(g_{(2)})(g_{(3)} \triangleright a(h_{(1)}))(g_{(4)} \triangleright b(h_{(2)})) \\ &= a(g_{(1)})b(g_{(2)})(g_{(3)} \triangleright (a(h_{(1)})b(h_{(2)}))) \\ &= (a * b)(g_{(1)})(g_{(2)} \triangleright (a * b)(h)), \end{aligned}$$

whence $a * b$ fulfills the action condition. For the module condition we compute

$$\begin{aligned} (g_{(1)} \triangleright c)((a * b)(g_{(2)})) &= (g_{(1)} \triangleright c)a(g_{(2)})b(g_{(3)}) \\ &= a(g_{(1)})(g_{(2)} \triangleright c)b(g_{(3)}) \\ &= a(g_{(1)})b(g_{(2)})(g_{(3)} \triangleright c) \\ &= ((a * b)(g_{(1)}))(g_{(2)} \triangleright c), \end{aligned}$$

whence $a * b \in \text{GL}(H, \mathcal{A})$, indeed. Now let $a^{-1} \in \text{Hom}_{\mathbb{C}}(H, \mathcal{A})$ be defined as in (A.3) then a^{-1} satisfies the normalization condition. For the action condition we compute

using $S \otimes S \circ \Delta^{\text{op}} = \Delta \circ S$

$$\begin{aligned}
 & a^{-1}(g_{(1)})(g_{(2)} \triangleright a^{-1}(h)) \\
 &= (g_{(2)} \triangleright a(S^{-1}(g_{(1)}))) \left((g_{(3)}h_{(2)}) \triangleright a(S^{-1}(h_{(1)})) \right) \\
 &= (g_{(4)}h_{(3)}) \triangleright \left(\left(S^{-1}(g_{(3)}h_{(2)}) \triangleright (g_{(2)} \triangleright a(S^{-1}(g_{(1)}))) \right) a \left(S^{-1}(h_{(1)}) \right) \right) \\
 &= (g_{(2)}h_{(3)}) \triangleright \left(\left(S^{-1}(h_{(2)}) \triangleright a(S^{-1}(g_{(1)})) \right) a \left(S^{-1}(h_{(1)}) \right) \right) \\
 &= (g_{(2)}h_{(2)}) \triangleright \left(\left(S^{-1}(h_{(1)})_{(1)} \triangleright a(S^{-1}(g_{(1)})) \right) a(S^{-1}(h_{(1)})_{(2)}) \right) \\
 &\stackrel{\text{(iii)}}{=} (g_{(2)}h_{(2)}) \triangleright \left(a(S^{-1}(h_{(1)})_{(1)}) \left(S^{-1}(h_{(1)})_{(2)} \triangleright a \left(S^{-1}(g_{(1)}) \right) \right) \right) \\
 &\stackrel{\text{(ii)}}{=} (g_{(2)}h_{(2)}) \triangleright \left(a \left(S^{-1}(h_{(1)})S^{-1}(g_{(1)}) \right) \right) \\
 &= (gh)_{(2)} \triangleright a(S^{-1}((gh)_{(1)})) \\
 &= a^{-1}(gh).
 \end{aligned}$$

For the module condition we compute

$$\begin{aligned}
 (g_{(1)} \triangleright b)a^{-1}(g_{(2)}) &= (g_{(1)} \triangleright b) \left(g_{(3)} \triangleright a(S^{-1}(g_{(2)})) \right) \\
 &= g_{(4)} \triangleright \left((S^{-1}(g_{(3)}) \triangleright (g_{(1)} \triangleright b)) a(S^{-1}(g_{(2)})) \right) \\
 &= g_{(3)} \triangleright \left((S^{-1}(g_{(2)})_{(1)} \triangleright (g_{(1)} \triangleright b)) a(S^{-1}(g_{(2)})_{(2)}) \right) \\
 &\stackrel{\text{(iii)}}{=} g_{(3)} \triangleright \left(a(S^{-1}(g_{(2)})_{(1)}) (S^{-1}(g_{(2)})_{(2)} \triangleright (g_{(1)} \triangleright b)) \right) \\
 &= g_{(2)} \triangleright \left(a(S^{-1}(g_{(1)})) b \right) \\
 &= (g_{(2)} \triangleright a(S^{-1}(g_{(1)}))) (g_{(3)} \triangleright b) \\
 &= a^{-1}(g_{(1)})(g_{(2)} \triangleright b),
 \end{aligned}$$

whence $a^{-1} \in \text{GL}(H, \mathcal{A})$ is shown. It remains to show that a^{-1} is the convolution inverse of a . Indeed,

$$\begin{aligned}
 (a^{-1} * a)(g) &= (g_{(2)} \triangleright a(S^{-1}(g_{(1)}))) a(g_{(3)}) \\
 &\stackrel{\text{(iii)}}{=} a(g_{(2)})(g_{(3)} \triangleright a(S^{-1}(g_{(1)}))) \\
 &\stackrel{\text{(ii)}}{=} a(g_{(2)}S^{-1}(g_{(1)})) \\
 &\stackrel{\text{(i)}}{=} \varepsilon(g) \mathbb{1}_{\mathcal{A}},
 \end{aligned}$$

and similarly for $a * a^{-1} = e$. Thus $\text{GL}(H, \mathcal{A})$ is a group and the inverses are given by formula (A.3). Thus let $a, b \in U(H, \mathcal{A})$ be given. Then

$$\begin{aligned}
 (a * b)(g_{(1)})((a * b)(S(g_{(2)}))^*)^* &= a(g_{(1)})b(g_{(2)})(b(S(g_{(3)}))^*)^*(a(S(g_{(4)}))^*)^* \\
 &\stackrel{\text{(iv)}}{=} a(g_{(1)})\varepsilon(g_{(2)})(a(S(g_{(3)}))^*)^* \\
 &\stackrel{\text{(iv)}}{=} \varepsilon(g) \mathbb{1}_{\mathcal{A}}
 \end{aligned}$$

whence $a * b \in U(H, \mathcal{A})$. Finally,

$$\begin{aligned} \varepsilon(g)\mathbb{1}_{\mathcal{A}} &= \varepsilon(g_{(1)})\overline{\varepsilon(S(g_{(2)})^*)}\mathbb{1}_{\mathcal{A}} \\ &= (a^{-1} * a)_{(g_{(1)})} \left((a^{-1} * a)_{(S(g_{(2)})^*)} \right)^* \\ &= a^{-1}(g_{(1)})a(g_{(2)})(a(S(g_{(3)})^*))^* \left(a^{-1}(S(g_{(4)})^*) \right)^* \\ &\stackrel{(iv)}{=} a^{-1}(g_{(1)})\varepsilon(g_{(2)}) \left(a^{-1}(S(g_{(3)})^*) \right)^* \\ &= a^{-1}(g_{(1)}) \left(a^{-1}(S(g_{(2)})^*) \right)^* \end{aligned}$$

shows $a^{-1} \in U(H, \mathcal{A})$ as well. This completes the proof. \square

Note that the group $GL(H, \mathcal{A})$ is defined for any action of a Hopf algebra H on an unital associative algebra as long as the antipode of H is invertible. For $U(H, \mathcal{A})$ we need the $*$ -involutions.

The next proposition describes how certain central elements of \mathcal{A} contribute to $GL(H, \mathcal{A})$ and $U(H, \mathcal{A})$, respectively. We denote by $GL(\mathcal{Z}(\mathcal{A}))$ the abelian group of invertible central elements in \mathcal{A} and $U(\mathcal{Z}(\mathcal{A}))$ denotes the subgroup of unitary central elements. Moreover, $GL(\mathcal{Z}(\mathcal{A}))^H$ and $U(\mathcal{Z}(\mathcal{A}))^H$ denote the H -invariant elements in $GL(\mathcal{Z}(\mathcal{A}))$ and $U(\mathcal{Z}(\mathcal{A}))$, respectively, which are subgroups.

Proposition A.3. *Let $c \in GL(\mathcal{Z}(\mathcal{A}))$, then*

$$\hat{c}(g) = c(g \triangleright c^{-1}) \tag{A.4}$$

defines an element $\hat{c} \in GL(H, \mathcal{A})$ and $c \mapsto \hat{c}$ is a group homomorphism such that

$$1 \longrightarrow GL(\mathcal{Z}(\mathcal{A}))^H \longrightarrow GL(\mathcal{Z}(\mathcal{A})) \xrightarrow{\hat{}} GL(H, \mathcal{A}) \tag{A.5}$$

is exact. Similarly, for $c \in U(\mathcal{Z}(\mathcal{A}))$ we have $\hat{c} \in U(H, \mathcal{A})$ and

$$1 \longrightarrow U(\mathcal{Z}(\mathcal{A}))^H \longrightarrow U(\mathcal{Z}(\mathcal{A})) \xrightarrow{\hat{}} U(H, \mathcal{A}) \tag{A.6}$$

is an exact sequence of group homomorphisms. Moreover, the image of $GL(\mathcal{Z}(\mathcal{A}))$ under $\hat{}$ is in the center of $GL(H, \mathcal{A})$.

Proof. First we check that $\hat{c} \in GL(H, \mathcal{A})$. The normalization is clear. For the action condition we compute

$$\begin{aligned} \hat{c}(g_{(1)})(g_{(2)} \triangleright \hat{c}(h)) &= c(g_{(1)} \triangleright c^{-1}) \left(g_{(2)} \triangleright \left(c(h \triangleright c^{-1}) \right) \right) \\ &= c \left(g_{(3)} \triangleright \left((S^{-1}(g_{(2)}g_{(1)}) \triangleright c^{-1}) c(h \triangleright c^{-1}) \right) \right) \\ &= c(g \triangleright (c^{-1} c(h \triangleright c^{-1}))) \\ &= c((gh) \triangleright c^{-1}) \\ &= \hat{c}(gh). \end{aligned}$$

The module condition is shown by

$$\begin{aligned} (g_{(1)} \triangleright b) \hat{c}(g_{(2)}) &= (g_{(1)} \triangleright b) c(g_{(2)} \triangleright c^{-1}) \\ &= c(g \triangleright (bc)^{-1}) \\ &= c(g_{(1)} \triangleright c^{-1})(g_{(2)} \triangleright b) \\ &= \hat{c}(g_{(1)})(g_{(2)} \triangleright b), \end{aligned}$$

whence $\hat{c} \in \text{GL}(H, \mathcal{A})$. Clearly $\widehat{1_{\mathcal{A}}} = e$ and for $c, d \in \text{GL}(\mathcal{Z}(\mathcal{A}))$ we have

$$\begin{aligned} \widehat{cd}(g) &= cd(g_{(1)} \triangleright c^{-1})(g_{(2)} \triangleright d^{-1}) = c(g_{(1)} \triangleright c^{-1})d(g_{(2)} \triangleright d^{-1}) = \hat{c}(g_{(1)})\hat{d}(g_{(2)}) \\ &= (\hat{c} * \hat{d})(g), \end{aligned}$$

whence \wedge is a group morphism. If c is H -invariant it is easy to see that $\hat{c} = e$. Conversely, if $\hat{c} = e$, then $c(g \triangleright c^{-1}) = \varepsilon(g)$ whence $g \triangleright c^{-1} = \varepsilon(g)c^{-1}$. Hence c^{-1} and thus c is H -invariant. This proves the exactness of (A.5). Now let $a \in \text{GL}(H, \mathcal{A})$ be arbitrary and $c \in \text{GL}(\mathcal{Z}(\mathcal{A}))$. Then using the centrality of c as well as (iii) for a we get

$$\begin{aligned} (a * \hat{c})(g) &= a(g_{(1)})c(g_{(2)} \triangleright c^{-1}) = ca(g_{(1)})(g_{(2)} \triangleright c^{-1}) = c(g_{(1)} \triangleright c^{-1})a(g_{(2)}) \\ &= (\hat{c} * a)(g), \end{aligned}$$

whence \hat{c} is central in $\text{GL}(H, \mathcal{A})$. For the last part let $c \in U(\mathcal{Z}(\mathcal{A}))$ be unitary. Then

$$\begin{aligned} \hat{c}(g_{(1)})(\hat{c}(S(g_{(2)}))^*)^* &= c(g_{(1)} \triangleright c^{-1})(S(g_{(2)})^* \triangleright c^{-1})^* c^* \\ &= (g_{(1)} \triangleright c^{-1})(S(S(g_{(2)}))^* \triangleright c) \\ &= g \triangleright (c^{-1}c) \\ &= \varepsilon(g)1_{\mathcal{A}} \end{aligned}$$

shows $\hat{c} \in U(H, \mathcal{A})$. The remaining statements follow easily. \square

Thus, we can divide by the image of $\text{GL}(\mathcal{Z}(\mathcal{A}))$ under \wedge and obtain the quotient groups

$$\text{GL}_0(H, \mathcal{A}) = \text{GL}(H, \mathcal{A}) / \widehat{\text{GL}(\mathcal{Z}(\mathcal{A}))} \tag{A.7}$$

and

$$U_0(H, \mathcal{A}) = U(H, \mathcal{A}) / \widehat{U(\mathcal{Z}(\mathcal{A}))}. \tag{A.8}$$

This way, we can complete the sequences (A.5) and (A.6) to the exact sequences

$$1 \longrightarrow \text{GL}(\mathcal{Z}(\mathcal{A}))^H \longrightarrow \text{GL}(\mathcal{Z}(\mathcal{A})) \xrightarrow{\wedge} \text{GL}(H, \mathcal{A}) \longrightarrow \text{GL}_0(H, \mathcal{A}) \longrightarrow 1 \tag{A.9}$$

and

$$1 \longrightarrow U(\mathcal{Z}(\mathcal{A}))^H \longrightarrow U(\mathcal{Z}(\mathcal{A})) \xrightarrow{\wedge} U(H, \mathcal{A}) \longrightarrow U_0(H, \mathcal{A}) \longrightarrow 1. \tag{A.10}$$

Remark A.4. If the center $\mathcal{Z}(\mathcal{A}) = \mathbb{C}1_{\mathcal{A}}$ is trivial then $\text{GL}(\mathcal{Z}(\mathcal{A}))^H = \text{GL}(\mathcal{Z}(\mathcal{A}))$ as well as $U(\mathcal{Z}(\mathcal{A}))^H = U(\mathcal{Z}(\mathcal{A}))$. Thus we have in this case

$$\text{GL}(H, \mathcal{A}) = \text{GL}_0(H, \mathcal{A}) \quad \text{and} \quad U(H, \mathcal{A}) = U_0(H, \mathcal{A}). \tag{A.11}$$

The groups $GL(H, \mathcal{A})$, $GL_0(H, \mathcal{A})$, $U(H, \mathcal{A})$ and $U_0(H, \mathcal{A})$ enjoy nice functorial properties which we shall discuss now. Under general homomorphisms or $*$ -homomorphisms, respectively, we cannot conclude any good behavior of elements in $GL(H, \mathcal{A})$ or $U(H, \mathcal{A})$, respectively, as the module condition requires information about commutation relations with arbitrary algebra elements. Nevertheless, for *surjective* homomorphisms we have the following statement expressing the functorial properties:

Proposition A.5. *Let $\phi : \mathcal{A} \rightarrow \mathcal{B}$ be a H -equivariant surjective homomorphism.*

- (i) *For any $a \in GL(H, \mathcal{A})$ we have $\phi_* a = \phi \circ a \in GL(H, \mathcal{B})$.*
- (ii) *The map $\phi_* : GL(H, \mathcal{A}) \rightarrow GL(H, \mathcal{B})$ is a group homomorphism.*
- (iii) *If $\psi : \mathcal{B} \rightarrow \mathcal{C}$ is another H -equivariant surjective homomorphism then*

$$(\psi \circ \phi)_* = \psi_* \circ \phi_* \quad \text{and} \quad (\text{id}_{\mathcal{A}})_* = \text{id}_{GL(H, \mathcal{A})}. \tag{A.12}$$

- (iv) *The group homomorphism $\phi_* : GL(H, \mathcal{A}) \rightarrow GL(H, \mathcal{B})$ induces a group homomorphism $GL_0(H, \mathcal{A}) \rightarrow GL_0(H, \mathcal{B})$, also denoted by ϕ_* , such that the diagram*

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & GL(\mathcal{Z}(\mathcal{A}))^H & \longrightarrow & GL(\mathcal{Z}(\mathcal{A})) & \longrightarrow & GL(H, \mathcal{A}) & \longrightarrow & GL_0(H, \mathcal{A}) & \longrightarrow & 1 \\
 & & \downarrow \phi & & \downarrow \phi & & \downarrow \phi_* & & \downarrow \phi_* & & \\
 1 & \longrightarrow & GL(\mathcal{Z}(\mathcal{B}))^H & \longrightarrow & GL(\mathcal{Z}(\mathcal{B})) & \longrightarrow & GL(H, \mathcal{B}) & \longrightarrow & GL_0(H, \mathcal{B}) & \longrightarrow & 1
 \end{array} \tag{A.13}$$

commutes.

- (v) *If ϕ is in addition a $*$ -homomorphism we can replace ‘GL’ by ‘U’ everywhere. In particular we have a commutative diagram*

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & U(\mathcal{Z}(\mathcal{A}))^H & \longrightarrow & U(\mathcal{Z}(\mathcal{A})) & \longrightarrow & U(H, \mathcal{A}) & \longrightarrow & U_0(H, \mathcal{A}) & \longrightarrow & 1 \\
 & & \downarrow \phi & & \downarrow \phi & & \downarrow \phi_* & & \downarrow \phi_* & & \\
 1 & \longrightarrow & U(\mathcal{Z}(\mathcal{B}))^H & \longrightarrow & U(\mathcal{Z}(\mathcal{B})) & \longrightarrow & U(H, \mathcal{B}) & \longrightarrow & U_0(H, \mathcal{B}) & \longrightarrow & 1.
 \end{array} \tag{A.14}$$

Proof. The first part is a simple verification of the axioms and the second part follows from well-known properties of the convolution product. The third part is obvious. For the fourth part we have to show that the first three vertical arrows give commuting diagrams since the last arrow is induced precisely in the way that (A.13) commutes in total. For the first box in (A.13) this is obvious. Let $c \in GL(\mathcal{Z}(\mathcal{A}))$ then

$$(\phi_* \hat{c})(g) = \phi(c(g \triangleright c^{-1})) = \phi(c)(g \triangleright \phi(c)^{-1}) = \widehat{\phi(c)}(g)$$

shows the commutativity of the second box in (A.13). Note that since $\phi(\mathbb{1}_{\mathcal{A}}) = \mathbb{1}_{\mathcal{B}}$ we indeed have $\phi(c^{-1}) = \phi(c)^{-1}$. Then the last part is again a simple consequence of the fact that $\phi(U(\mathcal{Z}(\mathcal{A}))) \subseteq U(\mathcal{Z}(\mathcal{B}))$. \square

In a slightly more fancy way we can rephrase the content of the proposition as follows:

Corollary A.6. *The groupoid of H -equivariant isomorphisms Iso_H acts on the exact sequence (A.9) by isomorphisms, whence in particular the whole exact sequence of groups*

together with its $\text{Aut}_H(\mathcal{A})$ -action on it is an invariant of \mathcal{A} as associative algebra with H -action. Analogously, the groupoid Iso_H^* acts on the exact sequence (A.10) by isomorphisms, whence the exact sequence (A.10) together with its $\text{Aut}_H^*(\mathcal{A})$ -action on it is an invariant of \mathcal{A} as $*$ -algebra with $*$ -action of H .

A.2. The cocommutative case

We specialize now for a cocommutative Hopf $*$ -algebra H . In this case the situation simplifies as follows:

Proposition A.7. *Let H be cocommutative.*

- (i) $H \triangleright \mathcal{Z}(\mathcal{A}) \subseteq \mathcal{Z}(\mathcal{A})$ whence $\mathcal{Z}(\mathcal{A})$ inherits the H -action.
- (ii) For $\mathfrak{a} \in \text{GL}(H, \mathcal{A})$ we have $\mathfrak{a}(g) \in \mathcal{Z}(\mathcal{A})$ for all $g \in H$ whence

$$\text{GL}(H, \mathcal{A}) = \text{GL}(H, \mathcal{Z}(\mathcal{A})) \quad \text{and} \quad \text{GL}_0(H, \mathcal{A}) = \text{GL}_0(H, \mathcal{Z}(\mathcal{A})) \quad (\text{A.15})$$

and analogously

$$U(H, \mathcal{A}) = U(H, \mathcal{Z}(\mathcal{A})) \quad \text{and} \quad U_0(H, \mathcal{A}) = U_0(H, \mathcal{Z}(\mathcal{A})). \quad (\text{A.16})$$

- (iii) The groups $\text{GL}(H, \mathcal{A})$, $\text{GL}_0(H, \mathcal{A})$, $U(H, \mathcal{A})$ and $U_0(H, \mathcal{A})$ are abelian.
- (iv) The space of unital algebra homomorphisms $H \rightarrow \mathcal{Z}(\mathcal{A})^H$ is a subgroup of $\text{GL}(H, \mathcal{A})$ and the space of unital $*$ -homomorphisms $H \rightarrow \mathcal{Z}(\mathcal{A})^H$ is a subgroup of $U(H, \mathcal{A})$. The inverse of such a homomorphism $\mathfrak{a} : H \rightarrow \mathcal{Z}(\mathcal{A})$ is explicitly given by

$$\mathfrak{a}^{-1}(g) = \mathfrak{a}(S(g)) = \mathfrak{a}(S^{-1}(g)). \quad (\text{A.17})$$

Proof. The first statement is well-known. For the second we compute

$$\begin{aligned} \mathfrak{a}(g)b &= \mathfrak{a}(g_1)\varepsilon(g_2)b = \mathfrak{a}(g_1)((g_2S(g_3))\triangleright b) = (g_1\triangleright(S(g_3)\triangleright b))\mathfrak{a}(g_2) \\ &= \varepsilon(g_1)b\mathfrak{a}(g_2) = b\mathfrak{a}(g), \end{aligned}$$

using the module condition for \mathfrak{a} as well as the cocommutativity. The third part is then a simple consequence. For the fourth part we consider a unital homomorphism $\mathfrak{a} : H \rightarrow \mathcal{Z}(\mathcal{A})^H$. Then $\mathfrak{a}(1_H) = 1_{\mathcal{A}}$ by definition and

$$\mathfrak{a}(gh) = \mathfrak{a}(g)\mathfrak{a}(h) = \mathfrak{a}(g_1)\varepsilon(g_2)\mathfrak{a}(h) = \mathfrak{a}(g_1)(g_2\triangleright\mathfrak{a}(h)),$$

since $\mathfrak{a}(h)$ is H -invariant. Moreover,

$$(g_1\triangleright b)\mathfrak{a}(g_2) = \mathfrak{a}(g_2)(g_1\triangleright b) = \mathfrak{a}(g_1)(g_2\triangleright b),$$

since $\mathfrak{a}(g_2)$ is central and H is cocommutative. Finally, if \mathfrak{a} is even a $*$ -homomorphism then

$$\mathfrak{a}(g_1)(\mathfrak{a}(S(g_2))^*)^* = \mathfrak{a}(g_1)\mathfrak{a}(S(g_2)) = \mathfrak{a}(g_1S(g_2)) = \varepsilon(g)1_{\mathcal{A}}.$$

Now let $a, b : H \rightarrow \mathcal{Z}(\mathcal{A})^H$ be unital homomorphisms then a simple computation shows that $a * b$ is again a unital homomorphism taking its values in $\mathcal{Z}(\mathcal{A})^H$. Moreover, the general formula for the inverse (A.3) leads to

$$a^{-1}(g) = \varepsilon(g_{(1)})a(S^{-1}(g_{(2)})) = a(S^{-1}(g)) = a(S(g))$$

since $a(g)$ is invariant and $S^2 = \text{id}$ in the cocommutative case. It is easy to see that a^{-1} is still a homomorphism since S is an antihomomorphism and the images all commute. If a is even a $*$ -homomorphism then a^{-1} is a $*$ -homomorphism, too, since S commutes with the $*$ -involution in the cocommutative case. This completes the proof. \square

In particular, the unital algebra homomorphisms

$$\chi : H \rightarrow \mathbb{C}, \tag{A.18}$$

i.e. the *characters* of H , always contribute to $\text{GL}(H, \mathcal{A})$ by setting $a^\chi(g) = \chi(g)\mathbb{1}_{\mathcal{A}}$. If χ is in addition a $*$ -homomorphism we call χ a *unitary character*. In fact, if the center of \mathcal{A} is trivial the characters of H give the whole group $\text{GL}(H, \mathcal{A})$:

Proposition A.8. *Let H be cocommutative.*

- (i) *The characters of H constitute a subgroup of $\text{GL}(H, \mathcal{A})$ via $\chi \mapsto a^\chi$ and the unitary characters of H constitute a subgroup of $U(H, \mathcal{A})$.*
- (ii) *If the center of \mathcal{A} is trivial, $\mathcal{Z}(\mathcal{A}) = \mathbb{C}\mathbb{1}_{\mathcal{A}}$, then any element of $\text{GL}(H, \mathcal{A}) = \text{GL}_0(H, \mathcal{A})$ is a character and any element of $U(H, \mathcal{A}) = U_0(H, \mathcal{A})$ is a unitary character.*

Proof. First it is clear that a^χ is a unital algebra homomorphism $H \rightarrow \mathcal{Z}(\mathcal{A}) = \mathcal{Z}(\mathcal{A})^H = \mathbb{C}\mathbb{1}_{\mathcal{A}}$ and thus an element of $G(H, \mathcal{A})$. If in addition $\chi(g) = \chi(g^*)$ then a^χ is a $*$ -homomorphism and hence $a^\chi \in U(H, \mathcal{A})$. Since $\chi * \chi'$ is clearly a character if χ, χ' are characters and since $a^\chi * a^{\chi'} = a^{\chi * \chi'}$ we see that the elements of the form a^χ are closed under multiplication in $\text{GL}(H, \mathcal{A})$. Moreover, denoting the ‘inverse character’ of χ by $\chi^{-1}(g) = \chi(S(g))$ we see that $(a^\chi)^{-1} = a^{\chi^{-1}}$ whence the characters are a subgroup indeed. If $a \in \text{GL}(H, \mathcal{A})$ is of the form $a(g) = \chi(g)\mathbb{1}_{\mathcal{A}}$ for any $g \in H$ with some $\chi(g) \in \mathbb{C}$ then $g \mapsto \chi(g)$ is necessarily a character. The unitary case is treated analogously. Then the second part follows from $\text{GL}(H, \mathcal{A}) = \text{GL}_0(H, \mathcal{A})$ and $U(H, \mathcal{A}) = U_0(H, \mathcal{A})$ by Remark A.4. \square

This statement allows to construct easily $*$ -algebras \mathcal{A} with $*$ -actions of a cocommutative Hopf $*$ -algebra H such that the groups $\text{GL}_0(H, \mathcal{A})$ and $U_0(H, \mathcal{A})$ are non-trivial. Note however, that if the center $\mathcal{Z}(\mathcal{A})$ is non-trivial then it may well happen that characters of H , viewed as non-trivial elements of $\text{GL}(H, \mathcal{A})$, are killed when passing to the quotient group $\text{GL}_0(H, \mathcal{A})$. Hence in general $\text{GL}(H, \mathcal{A}) \neq \text{GL}_0(H, \mathcal{A})$ as well as $U(H, \mathcal{A}) \neq U_0(H, \mathcal{A})$.

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