On the Laplacian spectral ratio of connected graphs

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\section*{A B S T R A C T}

The Laplacian spectrum of a graph is the eigenvalues of the associated Laplacian matrix. The quotient between the largest and second smallest Laplacian eigenvalues of a connected graph, is called the Laplacian spectral ratio. Some bounds on the Laplacian spectral ratio are considered. We improve a relation on the Laplacian spectral ratio of regular graphs. Especially, the first two smallest Laplacian spectral ratios of graphs with given order are determined. And some operations on Laplacian spectral ratio are presented.

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\section*{1. Introduction}

Let $G$ be a simple graph with $n$ vertices and $m$ edges. Denote by $\delta(G)$ and $\Delta(G)$ the minimum and the maximum degree, respectively. The Laplacian matrix of $G$ is defined as $L = D - A$, where $D = \text{diag}(d_1, d_2, \ldots, d_n)$ is the diagonal matrix of vertex degrees and $A$ is the adjacency matrix of $G$. The Laplacian spectrum of $G$ is the spectrum of its Laplacian matrix, and consists of the values $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. Since $\mu_{n-1}(G) > 0$ if and only if $G$ is connected, Fiedler \cite{1} called $\mu_{n-1}(G)$ (or $\alpha(G)$) the algebraic connectivity of $G$.

The Laplacian spectral ratio of a connected graph $G$ with $n$ vertices is defined as

$$ r_L(G) = \frac{\mu_1}{\mu_{n-1}}. $$

In 2002, Barahona et al. \cite{2} showed that a graph exhibits better synchronizability if the ratio $r_L(G) = \frac{\mu_1}{\mu_{n-1}}$ is as small as possible.

\begin{theorem} \textsuperscript{[3]} \end{theorem}

If $G \neq K_n$ is a connected graph with $n$ vertices, then

$$ r_L(G) = \frac{\mu_1(G)}{\mu_{n-1}(G)} \geq \frac{\Delta(G) + 1}{\delta(G)}. $$

A vertex cut of $G$ is a subset $S$ of $V(G)$ such that $G - S$ is disconnected. $G$ is a $t$-tough graph ($t > 0$ and $t \in \mathbb{R}$) if, for every vertex cut $S$, the number of components of the graph $G - S$, denoted by $C(G - S)$, is at most $|S|/t$, i.e., $C(G - S) \leq |S|/t$.

\begin{theorem} \textsuperscript{[4]} \end{theorem}

Let $G$ be a simple graph with $n$ vertices, and Laplacian eigenvalues $0 = \mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_1$. If $\mu_{n-1} \geq \frac{\mu_1}{2}$, then $G$ is 2-tough.

From Theorem B, we can see that if $r_L(G) \leq \frac{3}{2}$, then $G$ is 2-tough.

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In this paper we obtain some bounds on the Laplacian spectral ratio. A relation on the Laplacian spectral ratio of regular graphs is improved. A problem on the extremal Laplacian spectral ratio among trees with \( n \) vertices is proposed. Moreover, some graph operations on the Laplacian spectral ratio are given.

**Lemma 1.1** (\([5]\)). Let \( X \) and \( Y \) be disjoint sets of vertices of \( G \), such that there is no edge between \( X \) and \( Y \). Then

\[
\frac{|X|}{|Y|} \leq \left( \frac{\mu_1 - \mu_{n-1}}{\mu_1 + \mu_{n-1}} \right)^2.
\]

**Proposition 1.1.** Let \( X \) and \( Y \) be disjoint sets of vertices of a connected graph \( G \), such that there is no edge between \( X \) and \( Y \). Then

\[
r_2(G) \geq \frac{2\sqrt{(n-|X|)(n-|Y|)}}{\sqrt{n-|X|} \sqrt{n-|Y|}} - 1.
\]

**Proof.** By Lemma 1.1, \( \frac{1}{r_2+1} = \frac{\mu_1 - \mu_{n-1}}{\mu_1 + \mu_{n-1}} \geq \sqrt{\frac{|X|}{|Y|}} \). Then

\[
1 - \frac{2}{r_2+1} \geq \sqrt{\frac{|X|}{|Y|}}.
\]

By the above inequality, the result follows. \( \square \)

**Lemma 1.2** (\([5]\)). If \( G \) is a connected graph with diameter \( d > 1 \), then

\[
d < 1 + \frac{\log(\mu_1) + \sqrt{\mu_{n-1}}} {- \log(\mu_1 - \sqrt{\mu_{n-1}})}.
\]

**Proposition 1.2.** Let \( G \) be a connected graph with \( n \) vertices and diameter \( d > 1 \). Then

\[
r_L > \left( 1 + \frac{2}{\sqrt{2(n-1)} - 1} \right)^2.
\]

**Proof.** By Lemma 1.2, \( n-1 > \frac{1}{2} \left( \sqrt{\mu_1 - \sqrt{\mu_{n-1}}} \right)^{d-1} \). Then

\[
d \sqrt{2(n-1)} > \frac{\sqrt{\mu_1} + \sqrt{\mu_{n-1}}}{\sqrt{\mu_1} - \sqrt{\mu_{n-1}}} = 1 + \frac{2}{\sqrt{\mu_1} - \sqrt{\mu_{n-1}}}.
\]

The above inequality transforms into the result by solving for \( r_L \). \( \square \)

2. **Bounds on the Laplacian spectral ratio**

Note that \( \sum_{i=1}^{n} \mu_i^2 = \sum_{u \in V} d_u^2 + 2m := M_1 + 2m \).

**Lemma 2.1** (\([6]\) Unweighted Cassels’ Inequality). Let \( \vec{a} = (a_1, \ldots, a_n) \) and \( \vec{b} = (b_1, \ldots, b_n) \) be two positive \( n \)-tuples. Then

\[
\left( \sum_{i=1}^{n} a_i^2 \right)^{1/2} \left( \sum_{i=1}^{n} b_i^2 \right)^{1/2} \leq \frac{(a + A)^2}{4aA},
\]

where \( a = \min_{1 \leq i \leq n} \left( \frac{a_i}{\sqrt{\mu_i}} \right) \) and \( A = \max_{1 \leq i \leq n} \left( \frac{a_i}{\sqrt{\mu_i}} \right) \).

**Theorem 2.1.** Let \( G \) be a connected graph with \( n \) vertices and \( m \) edges. Then

\[
\sqrt{r_L} + \frac{1}{\sqrt{r_L}} \geq \frac{\sqrt{(n-1)(M_1 + 2m)}}{m}.
\]

**Proof.** Let \( \vec{a} = (1, \ldots, 1) \) and \( \vec{\mu} = (\mu_{n-1}, \ldots, \mu_1) \) be two positive \( n-1 \)-tuples. Then \( a = \frac{1}{\mu_1} \) and \( A = \frac{1}{\mu_{n-1}} \).

By Lemma 2.1,

\[
\sqrt{(n-1)(M_1 + 2m)} \leq \frac{a + A}{2\sqrt{aA}} \quad \text{i.e.,}
\]

\[
\frac{\sqrt{(n-1)(M_1 + 2m)}}{(2m)^{1/2}} \leq \frac{a + A}{2\sqrt{aA}} = \frac{1}{2} \left( \sqrt{\frac{A}{a}} + \sqrt{\frac{a}{A}} \right) = \frac{1}{2} \left( \sqrt{r_L} + \frac{1}{\sqrt{r_L}} \right).
\]

The result follows. \( \square \)
Corollary 2.1 ([3]). Let $G$ be a connected $k$-regular graph with $n$ vertices. Then

$$\sqrt{r_L} + \frac{1}{r_L} \geq 2\sqrt{n - \frac{1}{k}}.$$

Remark 2.1. By Theorem A, Goldberg [3] obtained that for a $k$-regular graph $G$,

$$\sqrt{\frac{\mu_1(G)}{\mu_{n-1}(G)}} + \left(\sqrt{\frac{\mu_1(G)}{\mu_{n-1}(G)}}\right)^{-1} \geq \sqrt{\frac{k+1}{k}} + \sqrt{\frac{k}{k+1}}. \tag{1}$$

For $n \geq 3$, it is easy to check that $2\sqrt{n - \frac{1}{k}} > \sqrt{\frac{k+1}{k}} + \sqrt{\frac{k}{k+1}}$. Then the bound of Corollary 2.1 is better than (1).

The following additive version of unweighted Cassels’ inequality also holds.

Lemma 2.2 ([6]). Let $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_n)$ be two positive $n$-tuples. Then

$$\left(\sum_{i=1}^{n} a_i^2\right) \left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \frac{(A - a)^2}{4A} \left(\sum_{i=1}^{n} a_i b_i\right)^2,$$

where $a = \min\{\frac{\alpha}{\beta}\}$ and $A = \max\{\frac{\alpha}{\beta}\}$.

Theorem 2.2. Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then

$$\sqrt{r_L} + \frac{1}{r_L} \geq \sqrt{(n - 1)(M_1 + 2m) - 4m^2} \frac{2}{m}.$$  

Proof. Let $\alpha = (1, \ldots, 1)$ and $\beta = (\mu_{n-1}, \ldots, \mu_1)$ be two positive $n - 1$-tuples, $a = \frac{1}{\mu_1}$ and $A = \frac{1}{\mu_{n-1}}$.

By Lemma 2.2,

$$\sqrt{(n - 1)(M_1 + 2m) - 4m^2} \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \frac{A - a}{2\sqrt{aA}},$$

i.e.,

$$A - a \leq \frac{2}{2\sqrt{aA}} = \frac{1}{2} \left(\sqrt{\frac{a}{A}} - \sqrt{\frac{A}{a}}\right) = \frac{1}{2} \left(\sqrt{r_L} - \frac{1}{r_L}\right).$$

The result follows. $\square$

By Theorems 2.1 and 2.2, we have

Theorem 2.3. Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then

$$r_L(G) \geq \frac{1}{4} \left(\sqrt{\frac{(n - 1)(M_1 + 2m)}{m}} + \sqrt{\frac{(n - 1)(M_1 + 2m) - 4m^2}{m}}\right)^2.$$

By Theorem 2.3, it is easy to obtain the following.

Corollary 2.2. Let $G$ be a connected $k$-regular graph with $n$ vertices. Then

$$r_L(G) \geq \left(\sqrt{\frac{(n - 1)(k + 1)}{nk}} + \sqrt{\frac{(n - 1)(k + 1) - 1}{nk}}\right)^2.$$  

For a connected $k$-regular ($k \leq n - 2$) graph with $n$ vertices, by Theorem A, then

$$r_L(G) \geq \frac{k + 1}{k}. \tag{2}$$
Remark 2.2. Let
\[
 f(k) := \left( \frac{(n-1)(k+1)}{nk} + \frac{(n-1)(k+1)}{nk} - 1 \right)^2 - k + 1
\]
\[
 = \frac{n - 2k - 2}{nk} + 2\sqrt{\frac{(n-1)(k+1)}{nk}} - \frac{(n-1)(k+1)}{nk} - 1.
\]

We only need to show \(2\sqrt{\frac{(n-1)(k+1)}{nk}} - \frac{(n-1)(k+1)}{nk} - 1 \geq (\frac{2}{n} - \frac{k}{n})^2\). The inequality can be transformed into \(n(4k(n - 2 - k) + 3n - 4) \geq 0\). This inequality is evidently satisfied for \(k \leq n - 2\). Then \(f(k) > 0\), i.e., the bound of Corollary 2.2 is better than (2).

Remark 2.3. By Appendix A of [7], for a tree \(T\) with \(n (3 \leq n \leq 10)\) vertices, then \(r_1(S_n) = n \leq r_1(T) \leq r_1(P_n)\).

By Remark 2.3, it is naturally to conjecture that:

Conjecture 2.1. Let \(T\) be a tree with \(n \geq 3\) vertices, then \(r_1(S_n) = n \leq r_1(T) \leq r_1(P_n)\) and the left (right) equality holds if and only if \(G \cong S_n(G \cong P_n)\).

Lemma 2.3 ([8]). If \(T\) is a tree with diameter \(D(T)\), then \(\alpha(T) \leq 2(1 - \cos(\frac{\pi}{D(T)+1}))\).

Lemma 2.4 ([8]). Let \(T \neq K_{n-1}\) be a tree on \(n \geq 6\) vertices, then \(\alpha(G) = \mu_{n-1}(T) < 0.49\).

Lemma 2.5 ([9]). Let \(G\) be a graph with at least one edge and maximum vertex degree \(\Delta\). Then \(\mu_1 \geq \Delta + 1\) with equality for connected graph if and only if \(\Delta = n - 1\).

Proposition 2.1. Let \(T\) be a tree with \(n (n \geq 10)\) vertices. If \(\Delta(T) \geq \lceil \frac{n}{2} \rceil - 1\) or \(D(T) \geq \lceil \frac{n}{2} \rceil - 1\), where \(D(T)\) is the diameter of \(T\), then \(r_1(T) > n = r_1(S_n)\).

Proof. If \(\Delta(T) \geq \lceil \frac{n}{2} \rceil - 1\), by Lemmas 2.4 and 2.5, then \(r_1(T) \geq \frac{\Delta + 1}{\mu_{n-1}} > \frac{\lceil \frac{n}{2} \rceil - 1 + 1}{0.49} > n\).

If \(D(T) \geq \lceil \frac{n}{2} \rceil - 1\), note that \(\mu_1(T) \geq \mu_1(P_n) = 2(1 + \cos \frac{\pi}{n})\) and by Lemma 2.3, then
\[
r_1(T) \geq \frac{2 (1 + \cos \frac{\pi}{n})}{2 (1 - \cos (\frac{\pi}{D(T)+1}))} \geq \frac{1 + \cos \frac{\pi}{n}}{1 - \cos \frac{2\pi}{n}} = \frac{1}{2 \left( 1 - \cos \frac{\pi}{n} \right)}.
\]

Let \(f(x) := x(1 - \cos \frac{\pi}{x}) - \frac{1}{x}\). Consider the first derivative \(f'(x) = 1 - \cos \frac{\pi}{x} - \frac{\pi}{x} \sin \frac{\pi}{x} = 2 \sin \frac{\pi}{2x} \cos \frac{\pi}{2x} (\tan \frac{\pi}{2x} - \frac{\pi}{x})\).

Let \(g(y) := \tan y - 2y\). And \(g'(y) = \frac{\sin y \cos y + \sin y \cos y}{\cos^2 y} < 0\) for \(y < \frac{\pi}{4}\). Then \(g(y)\) is decreasing function on \(y\). Thus
\[
 g(\frac{\pi}{2x}) = \tan \frac{\pi}{2x} - \frac{\pi}{x} < g(0) = 0 \text{ for } x > 0.
\]

Then \(f(x)\) is a decreasing function on \(x\) and \(f(x) \leq f(10) = -0.0105 < 0\), i.e., \(\frac{1}{2(1-\cos \frac{\pi}{n})} > x\) for \(x \geq 10\). Hence \(r_1(T) \geq \frac{1}{2(1-\cos \frac{\pi}{n})} > n\).

The result follows. \(\square\)

Lemma 2.6 ([10]). Let \(G\) be a graph with \(n\) vertices. Then \(\mu_1(G) = \mu_2(G) = \cdots = \mu_{n-1}(G)\) if and only if \(G \cong K_n\) or \(G \cong \overline{K_n}\).

Theorem 2.4. Let \(G\) be a connected graph with \(n (n \geq 3)\) vertices. Then \(r_1(G) \geq 1\) with the equality holds if and only if \(G \cong K_n\).

Proof. Since \(\mu_1(G) \geq \alpha(G), r_1(G) = \frac{\mu_1(G)}{\alpha(G)} \geq 1\). The equality holds if and only if \(\mu_1(G) = \alpha(G)\), i.e., \(\mu_1(G) = \mu_2(G) = \cdots = \mu_{n-1}(G)\).

By Lemma 2.6, the result holds. \(\square\)

The following property [11] of the Laplacian eigenvalues is needed.

Let \(\overline{G}\) (or \(G^c\)) be the complement of the graph \(G\) with \(n\) vertices. The Laplacian eigenvalues of \(\overline{G}\) are \(n - \mu_{n-1}, n - \mu_{n-2}, \ldots, n - \mu_1, 0\).

Theorem 2.5. Let \(G \neq K_n\) be a connected graph with \(n\) vertices. Then \(r_1(G) \geq \frac{n}{n-2}\). The equality holds if and only if \(G \in \mathcal{G}\), where \(\mathcal{G}\) is the set of graphs such that \(\overline{G} = iK_2 \cup (n - 2i)K_1\) (\(i = 1, \ldots, \lfloor \frac{n}{2} \rfloor\)).
Lemma 3.4
By Lemmas 3.1 and 3.2, \( \mu \) and \( t \) (Lemma 3.2)
spectrum: \( r \)
Theorem 3.1.
Proof. Let \( G \) be a connected graph with \( n \) vertices and
Theorem 3.2. \( \mu \)
Lemma 3.1
Proposition 2.2.
Corollary 2.3.
Case
Proof. Let \( G \) be a connected graph with \( n \) vertices.
Theorem 3.1.
Corollary 2.3.
Proposition 2.2.
Corollary 2.3.
3. The Laplacian spectral ratio and graph operations
Lemma 3.1 ([12]). For \( e \notin E(G) \), the Laplacian eigenvalues of \( G \) and \( G' = G + e \) interlace, i.e., \( \mu_1(G') \geq \mu_1(G) \geq \mu_2(G') \geq \cdots \geq \mu_n(G') = \mu_n(G) = 0 \).
Lemma 3.2 ([13]). Let \( G \) be a connected graph on \( n \) vertices. If \( v \) is a pendant vertex of \( G \), then \( \mu_i(G) \leq \mu_{i-1}(G - v), \ 2 \leq i \leq n \).
Particularly, \( \alpha(G) \leq \alpha(G - v) \).
Theorem 3.1. Let \( G \) be a connected graph with \( n \) vertices. If \( v \) is a pendant vertex of \( G \), then \( r_L(G) \geq r_L(G - v) \).
Proof. By Lemmas 3.1 and 3.2, \( \mu_1(G) \geq \mu_1(G - v) \) and \( \alpha(G) \leq \alpha(G - v) \).
Remark 3.1. Theorem 3.1 shows that \( r_L(G) \) decreases when pendant vertices are deleted. But if we delete a non- pendant vertex, then \( r_L \) maybe increase or decrease. For instance, let \( U_1 \) be the unicyclic graph obtained from \( C_4 \) by attaching a pendant vertex to \( C_4 \). \( S_4 \) and \( P_4 \) are two subgraphs by deleting different vertices. Then \( r_L(U_1) \approx 5.3996, r_L(S_4) = 4 \) and \( r_L(P_4) \approx 5.82838 \).
Lemma 3.3 ([14]). For a connected graph \( G \), \( \mu_1(G) = 2\Delta(G) \) if and only if \( G \) is a bipartite \( \Delta(G) \)-regular graph.
Lemma 3.4 ([15]). Let \( G \) be a d-regular simple graph with \( m \) edges and \( n \) vertices. Then the line graph \( L(G) \) has the Laplacian spectrum: \( 2d - \left( m - n \right) \) times, \( \mu_1, \ldots, \mu_{n-1}, \mu_n = 0 \).
Theorem 3.2. Let \( G \) be a connected d-regular graph with \( n \) vertices and \( L(G) \) the line graph of \( G \). Then \( r_L(L(G)) \geq r_L(G) \), the equality holds if and only if \( G \) is bipartite d-regular graph.
Proof. By Lemma 3.4, \( r_L(L(G)) \geq \frac{2d}{\alpha(G)} \geq \frac{\mu_1(G)}{\alpha(G)} = r_L(G) \). The equality holds if and only if \( \mu_1 = 2d = 2\Delta, \) by Lemma 3.3, i.e., \( G \) is a bipartite d-regular graph.
Theorem 3.3. Let \( G \) be a connected graph with \( n \) vertices and \( \mu_1 \neq n \). Then \( r_L(G) \geq r_L(G) \) if and only if \( \mu_1 + \mu_{n-1} \leq n, \) i.e., \( \mu_1(G) \leq \mu_1(G) \).
Proof. Since \( \mu_1 \neq n, \mu_{n-1}(G) = n - \mu_1 > 0 \), then \( r_L(G) - r_L(G) = \frac{\mu_1}{\mu_{n-1}} - \frac{n - \mu_{n-1}}{n - \mu_1} = \frac{(\mu_1 - \mu_{n-1})(\mu_1 + \mu_{n-1})}{\mu_{n-1}(n - \mu_1)} \).
The result follows. 

Corollary 3.1. Let $T \neq K_{1,n-1}$ be a tree with $n$ ($n \geq 6$) vertices. Then $r_1(T) > r_1(T)$.  

Proof. By [16], $\mu_1(T) - \mu_{n-1}(T) \leq n - 1$. By Lemma 2.4,

$$\mu_1(T) + \mu_{n-1}(T) = \mu_1(T) - \mu_{n-1}(T) + 2\mu_{n-1}(T) \leq n - 1 + 2\mu_{n-1}(T) < n - 0.02.$$  

Hence the result follows by Theorem 3.3.  

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References