On a conjecture concerning the orientation number of a graph

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Abstract

For a connected graph $G$ containing no bridges, let $\mathcal{D}(G)$ be the family of strong orientations of $G$; and for any $D \in \mathcal{D}(G)$, we denote by $d(D)$ the diameter of $D$. The orientation number $\vec{d}(G)$ of $G$ is defined by $\vec{d}(G) = \min\{d(D) : D \in \mathcal{D}(G)\}$. Let $\mathcal{G}(p, q; m)$ denote the family of simple graphs obtained from the disjoint union of two complete graphs $K_p$ and $K_q$ by adding $m$ edges linking them in an arbitrary manner. The study of the orientation numbers of graphs in $\mathcal{G}(p, q; m)$ was introduced by Koh and Ng [K.M. Koh, K.L. Ng, The orientation number of two complete graphs with linkages, Discrete Math. 295 (2005) 91–106]. Define $\vec{d}(m) = \min\{\vec{d}(G) : G \in \mathcal{G}(p, q; m)\}$ and $\alpha = \min\{m : \vec{d}(m) = 2\}$. In this paper we prove a conjecture on $\alpha$ proposed by K.M. Koh and K.L. Ng in the above mentioned paper, for $q \geq p + 4$. © 2008 Elsevier B.V. All rights reserved.

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1. Introduction and Terminology

Let $D$ be a digraph with vertex set $V(D)$ and arc set $E(D)$. For $v \in V(D)$, the eccentricity $e(v)$ of $v$ is defined as $e(v) = \max\{d(v, x) : x \in V(D)\}$, where $d(v, x)$ denotes the distance from $v$ to $x$. The diameter of $D$, denoted by $d(D)$, is defined as $d(D) = \max\{e(v) : v \in V(D)\}$.

An orientation of a graph $G$ is a digraph obtained from $G$ by assigning to each edge in $G$ a direction. An orientation $D$ of $G$ is strong if every two vertices in $D$ are mutually reachable in $D$. An edge $e$ in a connected graph $G$ is a bridge if $G - e$ is disconnected. Robbins’ celebrated one-way street theorem [25] states that a connected graph $G$ has a strong orientation if and only if no edge of $G$ is a bridge. For a connected graph $G$ containing no bridges, let $\mathcal{D}(G)$ be the family of strong orientations of $G$. The orientation number of $G$, denoted by $\vec{d}(G)$, is defined by $\vec{d}(G) = \min\{d(D) : D \in \mathcal{D}(G)\}$.

An orientation $D$ of $G$ is an optimal orientation if $d(D) = \vec{d}(G)$.

The notion of orientation numbers has been studied for various classes of graphs including complete graphs [1,7,20,22,24], complete multipartite graphs [1–4,8,9,23,26], cartesian products of graphs [5,10–15,17–19,21] and $G$-vertex multiplications of graphs [16].

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Given a family of disjoint graphs, we shall study the orientation number and design a corresponding optimal orientation for a resulting graph obtained by linking the given graphs with a set of additional edges. This new direction in the study of orientation numbers was first considered by Koh and Ng in [6] where, in particular, additional edges were added to link two disjoint complete graphs. More precisely, given two fixed integers $p$ and $q$ with $q \geq p \geq 5$ and an integer $m$ with $2 \leq m \leq pq$, let $G(p, q; m)$, or $G_m$, if there is no danger of confusion, denote the family of graphs that are obtained from the disjoint union of two complete graphs $K_p$ and $K_q$ by adding $m$ edges linking them in an arbitrary manner.

Write $D(G_m) = \bigcup\{D(G) : G \in G_m\}$ and define the parameter $\vec{d}(m)$ as follows:

$$\vec{d}(m) = \min\{\vec{d}(G) : G \in G_m\} = \min\{d(D) : D \in D(G_m)\}.$$  

It is known [1, 20, 22, 24] that $\vec{d}(K_n) = 2$ if $n \geq 3$ and $n \neq 4$; and $\vec{d}(K_4) = 3$, therefore $\vec{d}(pq) = 2$. Also, as $\vec{d}(s) \leq \vec{d}(r)$ whenever $2 \leq r \leq s \leq pq$, it follows that $2 = \vec{d}(pq) \leq \vec{d}(m) \leq \vec{d}(2)$. The parameter $\alpha = \min\{m : \vec{d}(m) = 2\}$ tells us the minimum number of edges that we need to add between $K_p$ and $K_q$ so that the resulting graph has orientation number 2. The following theorem was established in [6].

**Theorem 1.** If $\alpha = \min\{m : \vec{d}(m) = 2\}$, then

$$\alpha = \begin{cases} 
2p & \text{if } q \in \{p, p + 1\} \\
2p + 1 & \text{if } q = p + 2 \\
2p + 2 & \text{if } q = p + 3 \\
(2p + 2, 2p + 3, 2p + 4) & \text{if } q \geq p + 4. 
\end{cases}$$

The exact value of $\alpha$ when $q \geq p + 4$ was conjectured in [6] as follows:

$$\alpha = \begin{cases} 
2p + 3 & \text{if } q = p + 4, \\
2p + 4 & \text{if } q \geq p + 5. 
\end{cases}$$

In this note we shall prove this conjecture, which together with Theorem 1, completely determines the value of $\alpha$. But before doing that, we introduce some terminologies.

Let $D$ be a digraph. For $x, y \in V(D)$, we write ‘$x \to y$’ or ‘$y \leftarrow x$’ if $x$ is adjacent to $y$ in $D$. More generally, for $A, B \subseteq V(D)$ with $A \cap B = \emptyset$, we write ‘$A \to B$’ if $x$ is adjacent to $y$ for all $x \in A$ and $y \in B$. For simplicity, we write $x \to B$ (resp., $A \to y$) for $\{x\} \to B$ (resp., $A \to \{y\}$). Also, let $O(x) = \{v \in V(D) : x \to v\}$ and $I(x) = \{v \in V(D) : v \to x\}$. The converse of $D$, denoted by $\tilde{D}$, is the digraph obtained from $D$ by reversing each arc in $D$. It is obvious that for any strong digraph $D$, $d(D) = d(\tilde{D})$.

Let $F \in D(G_m)$ then $V(F) = V(K_p) \cup V(K_q)$. Let $u \in V(K_p)$ and $v \in V(K_q)$. If $u \to v$ in $F$, then we say that the arc $uv$ is of type 1; if $v \to u$ in $F$, then we say that the arc $vu$ is of type 2. For each $x \in V(K_p)$, let $\delta^+(x) = |O(x) \cap V(K_q)|$ and $\delta^-(x) = |I(x) \cap V(K_q)|$. Likewise, for each $y \in V(K_q)$, let $\delta^+(y) = |O(y) \cap V(K_p)|$ and $\delta^-(y) = |I(y) \cap V(K_p)|$. For all $x \in V(F)$, we let $\delta(x) = \delta^+(x) + \delta^-(x)$.

2. An improved lower bound on $\alpha$ when $q = p + 4$ and $q \geq p + 5$

The following lemma proven in [6] will be found useful in this section.

**Lemma 1.** If $G \in G_n$ and $F$ is an orientation of $G$ such that $d(F) = 2$, then there are at least $p$ arcs of type i, for each $i = 1, 2$, and so $n \geq 2p$. Furthermore, if $n < p + q$, then $\delta^+(x) \geq 1$ and $\delta^-(x) \geq 1$ for each $x \in V(K_p)$.

The following theorem in [6] provides a lower bound on $\alpha$ when $q \geq p + 3$.

**Theorem 2.** If $q \geq p + 3$ and $\vec{d}(n) = 2$, then $n \geq 2p + 2$.

The main result in this note improves the lower bound on $\alpha$ when $q \geq p + 4$. This is presented in the following proposition.

**Proposition 1.** Let $G \in G_n$ and $F$ be an orientation of $G$ such that $d(F) = 2$. If

(i) $q \geq p + 4$, then $\alpha \geq 2p + 3$;
(ii) $q \geq p + 5$, then $\alpha \geq 2p + 4$. 

Proof. By Theorem 2, $\alpha \geq 2p + 2$. Suppose $q \geq p + m$, where $m = 4, 5$. Let $V(K_p) = \{x_1, \ldots, x_p\}$, $V(K_q) = \{y_1, \ldots, y_q\}$, $G \in \mathcal{G}_n$, $n = 2p + m - 2$ and $F$ be an orientation of $G$ such that $d(F) = 2$. Let $C$ be the set of all arcs of type 1 and $D$ be the set of all arcs of type 2 in $F$. By Lemma 1, $|C| \geq p$, $|D| \geq p$, $\delta^+(x_i) \geq 1$, $\delta^-(x_i) \geq 1$ for all $x_i \in V(K_p)$. Let $A = \{y_i \in V(K_q) | \delta^+(y_i) \geq 1\}$, $B = V(K_q) \setminus A$, $Y = \{y_i \in V(K_q) | \delta^-(y_i) \geq 1\}$ and $Z = V(K_q) \setminus Y$. This consideration may be split into two cases.

Case 1: Suppose there are exactly $p$ arcs of type 1, that is, $|C| = p$. Note that the case where $|D| = p$ is similar. By Lemma 1, we have $\delta^+(x_i) = 1$ for each $x_i \in V(K_p)$. This implies that $Y \to Z$ since $d(F) = 2$ (otherwise, say if $y \in Y$ and $z \in Z$ with $y \not\to z$, then $d(x_i, z) \geq 3$, where $x_i \to y$, which is a contradiction). Furthermore, it can be verified that, for any two different vertices $y, y' \in Y$, $I(y) \cap V(K_p)$ and $I(y') \cap V(K_p)$ are disjoint. Therefore each vertex $z \in Z$ is adjacent to at least one vertex in $I(y) \cap V(K_p)$ for each $y \in Y$ (otherwise we must have $d(z, y) \geq 3$, which is also a contradiction), i.e., $\delta^+(z) \geq |Y|$ for each $z \in Z$. Noticing that $|Z| = q - |Y| \geq p + m - |Y|$ and $1 \leq |Y| \leq p$, we have

$$n = |C| + |D| \geq p + \sum_{z \in Z} \delta^+(z) \geq p + |Y|(p + m - |Y|) > 2p + m - 2 = n,$$

again a contradiction.

Case 2: Suppose there are exactly $p + 1$ arcs of type 2, that is, $|D| = p + 1$ and $|C| = p + m - 3$. Note that the case where $|C| = p + 1$ is similar. In this case, by Lemma 1, we may assume that $\delta^-(x_i) = 2$ and $\delta^-(x_i) = 1$ for all $i = 2, \ldots, p$. Without loss of generality, let $\{y_1, y_2\} \to x_1$, where $y_1, y_2 \in A$. Since $|D| = p + 1$, we have $|A| \leq p + 1$, which implies $|B| \geq m - 1 > 0$. Define the following subset of $A$:

$$A' = \{y_i \in A | y_i \to x_j \text{ for some } j \neq 1\}.$$

Note that $y_1$ or $y_2$ or both may possibly belong to $A'$.

(2.1) Suppose $y_1, y_2 \in A'$. In this case, $A = A'$ and we may assume that $y_1 \to x_2$ and $y_2 \to x_3$ in $F$. Since $\delta^-(x_i) = 1$ for all $i = 2, \ldots, p$ and $d_F(b, x_i) \leq 2$ for all $b \in B$ and $i = 2, \ldots, p$, therefore $B \to A$. Now if there exists $b \in B$ such that $\delta^-(b) = 0$, then $d_F(y_1, b) \geq 3$, which is a contradiction. Thus we may assume that $\delta^-(b) \geq 1$ for all $b \in B$. As $|C| = p + m - 3$, we must have $|B| \leq p + m - 3$. Also, $\delta^+(y_i) \geq 2$ for $i = 1, 2$ and $|D| = p + 1$ imply that $|A| \leq p - 1$. Let $A = \{y_1, y_2, \ldots, y_k\}$, where $2 \leq k \leq p - 1$, and thus $|B| \geq p + m - k$.

Let $j \in \{1, \ldots, k\}$ be such that $\delta^+(y_j) \leq \delta^+(y_i)$ for all $i = 1, \ldots, k$. By our choice of $j$, $\delta^+(y_j) \leq \lceil \frac{p + 1}{k} \rceil$. Noticing that $|C| = p - |D| = p + m - 3$ and $2 \leq k \leq p - 1$, we have

$$\sum_{\{x_i | y_j \to x_i\}} \delta^+(x_i) = p + m - 3 - \sum_{\{x_i | y_j \not\to x_i\}} \delta^+(x_i) \leq p + m - 3 - (p - \delta^+(y_j)) \leq p + m - 3 - \left(p - \left\lfloor \frac{p + 1}{k} \right\rfloor \right) < p + m - k \leq |B|,$$

where the first inequality follows because $\delta^-(x_i) \geq 1$ for each $x_i \in V(K_p)$. Thus, there exists a vertex $b \in B$ such that $d(y_j, b) \geq 3$, a contradiction.

(2.2) Suppose exactly one of $y_1, y_2$ belongs to $A'$. Without loss of generality, we assume $y_2 \in A'$, $\{y_1, y_2\} \to x_1$, $y_2 \to x_2$ in $F$, $\delta^+(y_1) = 1$ and $\delta^+(y_2) \geq 2$. Similar to the discussion in (2.1), that $d_F(b, x_i) \leq 2$ for all $b \in B$ and $i = 2, \ldots, p$ implies $B \to A'$. Now if there exists $b \in B$ such that $\delta^-(b) = 0$, then $d_F(y_1, b) \leq 2$ for all $y_j \in A'$, which implies $y_j \to y_1 \to b$. Thus $A' \to y_1 \to b$. Now as $d_F(y_1, x_i)$ \leq 2 for all $i = 2, \ldots, p$, $x_1 \to \{x_2, \ldots, x_p\}$. As $d_F(x_1, x_i)$ \leq 2 for all $i = 2, \ldots, p$, $x_1 \to y_1$ or $y_2$. Thus $\delta^-(y_1) + \delta^-(y_2) \geq p - 1$.

If $|A'| \geq m$, where $m = 4, 5$, let $A' = \{y_2, y_3, \ldots, y_r\}$, where $r \geq m + 1$. If there exists $j \in \{3, \ldots, r\}$ such that $\delta^-(y_j) = 0$, then let $x_k \in \{x_2, \ldots, x_p\}$ be such that $\delta^+(x_k) = 1$. Note that such a $x_k$ always exists since $|C| = p + m - 3$ and $\delta^+(x_k) \geq 1$ for all $i = 1, \ldots, p$. As $d_F(x_k, x_1) \leq 2$, $x_k \to y_1$ or $y_2$. If $x_k \to y_2$, then $d_F(x_k, b) \geq 3$, which is a contradiction. Thus $x_k \to y_1$. But in this case, $d_F(x_k, y_j) \geq 3$, again a contradiction. Thus $\delta^-(y_j) \geq 1$ for all $j = 3, \ldots, r$, which implies $\sum_{j=3}^r \delta^-(y_j) \geq |A'| - 1 \geq m - 1$. Since $\delta^-(y_1) + \delta^-(y_2) \geq p - 1$, we have $\sum_{j=1}^2 \delta^-(y_1) \geq p + m - 2$, which contradicts the fact that $|C| = p + m - 3$. We shall consider the cases $|A'| = 1, 2, \ldots, m - 1$ separately.
When $|A'| = 1$, that is, $A' = \{y_2\}$, we have $y_1 \to x_1$ and $y_2 \to \{x_1, \ldots, x_p\}$. In this case, let $x_k \in \{x_2, \ldots, x_p\}$ be such that $\delta^+(x_k) = 1$. As $d_F(x_k, x_1) \leq 2$, $x_k \to y_1$. Now $d_F(x_k, y_2) \geq 3$, a contradiction.

When $|A'| = 2$, we shall consider $m = 4$ and $m = 5$ separately.

First consider $m = 4$. In this case, let $A' = \{y_2, y_3\}$. As $d_F(x_1, b) \leq 2$, $x_1 \to b' \to b$ for some $b' \in B$. Since $|C| = p + 1$, we must have $\delta^-(y_1) + \delta^-(y_2) = p - 1$, $\delta^-(y_3) = 1$, $\delta^-(b') = 1$ and $\delta^-(b'') = 0$ for all $b'' \in B \setminus \{b'\}$. As $d_F(x_1, b'') \leq 2$ for all $b'' \in B \setminus \{b', b''\}$, $d_F(b, b') \geq 3$, a contradiction.

If $m = 5$, let $A' = \{y_2, y_3, y_4\}$. As $d_F(x_1, b) \leq 2$, $x_1 \to b' \to b$ for some $b' \in B$. Let $x_k \in \{x_2, \ldots, x_p\}$ be such that $\delta^+(x_k) = 1$. As $d_F(x_k, x_1) \leq 2$, $x_k \to y_1$ or $y_2$. If $x_k \to y_2$, then $d_F(x_k, b) \geq 3$. Thus $x_k \to y_1$. As $d_F(x_k, y_2) \leq 2$, $x_k \to x_1$ or $x_k \to b^* \to b$ for some $b^* \in B \setminus \{b\}$. If $x_k \to y_1$, since $x_k \to \{y_1, y_2\}$ and $|C| = p + 2$, we must have $\delta^-(y_1) + \delta^-(y_2) = p$, $\delta^-(x_k) = 1$, $\delta^-(b') = 1$ and $\delta^-(b'') = 0$ for all $b'' \in B \setminus \{b'\}$. As $d_F(y_1, b'') \leq 2$ for all $b'' \in B \setminus \{b', b''\}$, $d_F(b, b') \geq 3$, a contradiction.

In the second case where $x_k \to b^* \to b$, either $b^* = b'$ or $b^* \neq b'$. If $b^* = b'$, we end up with a case similar to $x_k \to y_1$, leading to a contradiction. If $b^* \neq b'$, $d_F(x_k, b'') \leq 2$ for all $b'' \in B$ where $\delta^-(b'') = 0$ leads to $b' \to b''$. Since $b' \to b''$ for all $b'' \in B \setminus \{b', b^*\}$ and $\{b', b^*\} \to b$, $d_F(b, b') \geq 3$, again a contradiction.

When $|A'| = 3$ and $m = 4$, let $A' = \{y_2, y_3, y_4\}$. In this case, $|C| = p + 1$ and $\delta^-(y_1) + \delta^-(y_2) = p - 1$, $\delta^-(y_3) = 1$ for $j = 3, 4$, and $\delta^-(b) = 0$ for all $b \in B$. Now $d_F(x_1, b) \geq 3$, a contradiction. If $m = 5$, then $|C| = p + 2$, and $d_F(x_1, b) \leq 2$ implies $x_1 \to b' \to b$ for some $b' \in B$. In this case, we have $\delta^-(y_1) + \delta^-(y_2) = p - 1$, $\delta^-(y_3) = 1$ for $j = 3, 4$, $\delta^-(b') = 1$ and $\delta^-(b'') = 0$ for all $b'' \in B \setminus \{b'\}$. As $d_F(x_1, b'') \leq 2$ for all $b'' \in B \setminus \{b', b''\}$, $d_F(b, b') \geq 3$, a contradiction.

When $|A'| = 4$, we only need to consider $m = 5$. In this case, let $A' = \{y_2, y_3, y_4, y_5\}$ then we have $\delta^-(y_1) + \delta^-(y_2) = p - 1$, $\delta^-(y_3) = 1$ for $j = 3, 4, 5$, $\delta^-(b) = 0$ for all $b \in B$. Now $d_F(x_1, b) \geq 3$, a contradiction.

So we may now assume $\delta^-(b) \geq 1$ for all $b \in B$. As $1 \leq |A'| \leq p - 1$, $|B| \geq m$. If $|A'| = 1$, then $|B| \geq p + m - 2$, which implies $\sum_{b \in B} \delta^-(b) \geq p + m - 2$. This is a contradiction as $|C| = p + m - 3$. If $|A'| = 2$, let $A' = \{y_2, y_3\}$. In this case, $|B| \geq p + m - 3$, and since $\delta^-(b) \geq 1$ for all $b \in B$, we only need to consider $|B| = p + m - 3$, that is, $q = p + m, m = 4, 5$. In this case, $\delta^-(b) = 1$ for all $b \in B$ and $\delta^-(y_j) = 0$ for $j = 1, 2, 3$. As $\delta^+(y_j) \leq p - 1$ and $\delta^+(y_3) \leq p - 2$, we have:

(i) when $m = 4$, $|C| = p + 1$, $\sum_{x_j y_2 \to x_j} \delta^+(x_j) \leq p < |B|$ and $\sum_{x_k y_3 \to x_k} \delta^+(x_k) \leq p - 1 < |B|$; or

(ii) when $m = 5$, $|C| = p + 2$, $\sum_{x_j y_2 \to x_j} \delta^+(x_j) \leq p + 1 < |B|$ and $\sum_{x_k y_3 \to x_k} \delta^+(x_k) \leq p < |B|$.

As $d_F(y_2, b) \leq 2$ and $d_F(y_3, b) \leq 2$ for all $b \in B$, $\{y_2, y_3\} \to y_1$. As $d_F(y_1, x_i) \leq 2$ for all $i = 2, \ldots, p$, $x_1 \to \{x_2, \ldots, x_p\}$. Since $\delta^-(y_j) = 0$ for all $j = 1, 2, 3$, $d_F(x_2, x_1) \geq 3$, a contradiction.

Now consider $|A'| = k \geq 3$. Since $3 \leq k \leq p - 1$, $|B| \geq p + m - 1 - k$. If there exists $y_j \in A'$ such that $y_1 \to y_j$, then

(i) $d_F(y_j, b) \leq 2$ for all $b \in B$, which implies $\delta^+(y_j) \geq p + m - 2 - k = p + 2 - k$ when $m = 4$ and $|C| = p + 1$; or

(ii) $d_F(y_j, b) \leq 2$ for all $b \in B$, which implies $\delta^+(y_j) \geq p + m - 3 - k = p + 2 - k$ when $m = 5$ and $|C| = p + 2$.

However, we have

$$\sum_{y_j \in A'} \delta^+(y_j) \geq (p + 2 - k) + (k - 1) = p + 1,$$

and so $\delta^+(y_1) + \sum_{y_j \in A'} \delta^+(y_j) \geq p + 2$,

a contradiction since $|D| = p + 1$. Thus $y_j \to y_1$ for all $y_j \in A'$, that is $A' \to y_1$. As $d_F(y_1, x_i) \leq 2$ for all $i = 2, \ldots, p$, $x_1 \to \{x_2, \ldots, x_p\}$; and as $d_F(x_1, x_i) \leq 2$, $x_i \to y_1$ or $y_2$. Thus $\delta^-(y_1) + \delta^-(y_2) \geq p - 1$. However, as $|B| \geq m$, we have $\sum_{b \in B} \delta^-(b) \geq m$ and $\sum_{i=1}^q \delta^-(y_i) \geq p - 1 + m$. This is a contradiction since $|C| = p + m - 3$.

(2.3) Suppose both $y_1$ and $y_2$ do not belong to $A'$. In this case, $\delta^+(y_1) = \delta^+(y_2) = 1$. Similar to the discussions in the previous two cases, that $d_F(b, x_i) \leq 2$ for all $b \in B$ and $i = 2, \ldots, p$ implies $B \to A'$. Let $y_3 \in A'$. If there
exists $b \in B$ such that $\delta^-(b) = 0$, then as $d_F(y_3, b) \leq 2$, $y_3 \to y_1 \to b$ or $y_3 \to y_2 \to b$. By symmetry, we assume $y_3 \to y_2 \to b$. As $d_F(b, x_1) \leq 2, b \to y_1$. Since $d_F(y, b) \leq 2$ for all $y \in A'$, we have $y_j \to y_2$. Thus $A' \to y_2$. As $d_F(y_2, x_i) \leq 2$ for all $i = 2, \ldots, p$, we have $x_1 \to \{x_2, \ldots, x_p\}$; and as $d_F(x_i, x_1) \leq 2$ for all $i = 2, \ldots, p, x_i \to y_1$ or $y_2$. Thus $\delta^-(y_1) + \delta^-(y_2) \geq 2 - 1$.

If $|A'| \geq m - 1$, as $|C| = p + m - 3$, there exists $y_j \in A'$ such that $\delta^-(y_j) = 0$. Let $x_k \in \{x_2, \ldots, x_p\}$ such that $\delta^+(x_k) = 1$. As $d_F(x_k, x_1) \leq 2, x_k \to y_1$ or $y_2$. If $x_k \to y_1$, then $d_F(x_k, b) \geq 2$, a contradiction. Thus $x_k \to y_2$. Now $d_F(x_2, y_j) \geq 3$, again a contradiction. We shall consider $|A'| = 1, 2, \ldots, m - 2$ separately for $m = 4, 5, 6, \ldots$.

When $|A'| = 1$, let $A' = \{y_2\}$. Then $y_2 \to \{x_2, \ldots, x_p\}$. Let $x_k \in \{x_2, \ldots, x_p\}$ such that $\delta^+(x_k) = 1$. As $d_F(x_2, x_k) \leq 2, x_k \to y_2$ or $y_1$. If $x_k \to y_1$, then $d_F(x_k, b) \geq 3$, a contradiction. On the other hand, if $x_k \to y_2$, then $d_F(x_2, x_k) \geq 3$, again a contradiction.

When $|A'| = 2$ and $m = 4$, let $A' = \{y_3, y_4\}$. Since $|C| = p + 1$, we must have $\delta^-(y_1) + \delta^-(y_2) = p - 1$, $\delta^-(y_3) = 1$ for $j = 3, 4$ and $\delta^-(b) = 0$ for all $b \in B$. Now $d_F(x_1, b) \geq 3$, a contradiction. If $m = 5$ and $|C| = p + 2$, let $A' = \{y_3, y_4\}$. As $d_F(x_1, b) \leq 2, x_1 \to b' \to b$ for some $b' \in B$. We now have $\delta^-(y_1) + \delta^-(y_2) = p - 1, \delta^-(y_3) = 1$ for $j = 3, 4, \delta^-(b') = 1$ and $\delta^-(b'') = 0$ for all $b'' \in B \setminus \{b'\}$. As $d_F(x_1, b'') \leq 2$ for all $b'' \in B \setminus \{b'\}, b' \to B \setminus \{b'\}$. As $d_F(b', b'') \leq 2, y_1 \to b'$; and as $d_F(b', x_1) \leq 2, b' \to y_2$. Now let $x_k \in \{x_2, \ldots, x_p\}$ be such that $\delta^+(x_k) = 1$. As $d_F(x_k, b) \leq 2, x_k \to y_2$. Now $d_F(x_k, b') \geq 3$, a contradiction.

When $|A'| = 3$, we only need to consider $m = 5$ and $|C| = p + 2$. Let $A' = \{y_3, y_4, y_5\}$. In this case, $\delta^-(y_1) + \delta^-(y_2) = p - 1, \delta^-(y_3) = 1$ for $j = 3, 4, 5$ and $\delta^-(b) = 0$ for all $b \in B$. Now $d_F(x_1, b) \geq 3$, a contradiction.

So we may now assume that $\delta^-(b) \geq 1$ for all $b \in B$. As $1 \leq |A'| \leq p - 1$, we have $|B| \geq m - 1$. If $|A'| = 1$, then $|B| \geq p + m - 3$. In this case, as $\delta^-(b) \geq 1$ for all $b \in B$, we only need to consider $|B| = p + m - 3$, that is, $q = p + m$. Thus $\delta^-(y_j) = 0$ for $j = 1, 2, 3$.

(i) When $m = 4$, we have $|B| = |C| = p + 1$. Since $\sum_{j=2}^p \delta^+(x_j) \leq p < |B|$, there exists $b \in B$ such that $x_j \not\to b$ for all $j = 2, \ldots, p$. As $d_F(y_3, b) \leq 2, y_3 \to y_2 \to b$ or $y_3 \to y_1 \to b$. By symmetry, we may assume $y_3 \to y_2 \to b$. Now as $d_F(y_2, x_i) \leq 2$ for all $i = 2, \ldots, p, x_i \to \{x_2, \ldots, x_p\}$. Since $\delta^-(y_j) = 0$ for all $j = 1, 2, 3, d_F(x_2, x_1) \geq 3$, a contradiction.

(ii) When $m = 5$, $|B| = p + 2$ and $|C| = p + 2, \delta^-(y_3) = p - 1$ and $\sum_{j=2}^p \delta^+(x_j) \leq p + 1$. Again there exists $b \in B$ such that $x_j \not\to b$ for all $j = 2, \ldots, p$. Similar arguments as shown in (i) above lead to another contradiction.

Now let $|A'| = k$, where $2 \leq k \leq p - 1$ and $|B| \geq p + m - k - 2$. Consider $y_j \in A'$ and suppose $\{y_1, y_2\} \to y_j$.

(i) Suppose $m = 4$ and $|C| = p + 1$. As $d_F(y_j, y) \leq 2$ for all $b \in B$, we have $\delta^+(y_j) \geq p + m - k - 3 = p - k - 1$.

However this is not possible since $\sum_{y \in A'} \delta^+(y_j) \geq p - 1$ and $\max_{y \in A'} \delta^+(y_j) \leq p - 1 - (k - 1) = p - k$.

(ii) Suppose $m = 5$ and $|C| = p + 2$. As $d_F(y_j, y) \leq 2$ for all $b \in B$, we have $\delta^+(y_j) \geq p + m - k - 4 = p - k$.

Similar to the discussion in (i), this is not possible as $\max_{y \in A'} \delta^+(y_j) \leq p - 1 - (k - 1) = p - k$.

Thus for all $y_j \in A'$, we have $y_2 \to y_j \to y_1$ or $y_1 \to y_j \to y_2$ or $y_j \to \{y_1, y_2\}$. Suppose $y_j \to y_1$ and $y_j \to X' \subset \{x_2, \ldots, x_p\}$. As $d_F(y_1, x') \leq 2$ for all $x' \in X' \setminus x_1$ $x \to x'$. On the other hand, if $y_j \to y_2$, then that $d_F(y_2, x') \leq 2$ for all $x' \in X'$ implies $x_1 \to x'$. Hence we have $x_1 \to \{x_2, \ldots, x_p\}$. Now as $d_F(x_i, x_1) \leq 2$ for all $i = 2, \ldots, p, x_i \to y_1$ or $y_2$, which implies $\delta^-(y_1) + \delta^-(y_2) \geq 2 - 1$. However, as $|B| \geq m - 1$ and $\delta^-(b) \geq 1$ for all $b \in B$, $\sum_{b \in B} \delta^-(b) \geq m - 1$. Now $\sum_{y \in A'} \delta^-(y_j) \geq (p - 1) + (m - 1) = p + m - 2$. This is a contradiction since $|C| = p + m - 3$.

This concludes the proof of the proposition. □

3. Existence of graphs in $G(p, p + 4; 2p + 3)$ with orientation number equal to 2

Proposition 2. There exists a graph in $G(p, p + 4; 2p + 3)$ such that $\overrightarrow{\Delta}(G) = 2$.

Proof. Assume that $p$ is odd. We first provide an orientation of a graph $G$ in $G_{2p+3}$ with diameter 2. Let $V(K_p) = \{a_1, a_2, \ldots, a_p\}, V(K_p) = \{b_1, b_2, \ldots, b_{2p+3}, b_{p+4}\}$ and $G_1$ be the subgraph of $K_p$ induced by $\{b_1, \ldots, b_p\}$. Noting that $K_p$ and $G_1$ are both complete graphs of odd order $p$, we define the following orientation $F$ of a graph $G$ in $G_{2p+3}$:

(i) orient the edges in $E(K_p)$ and $E(G_1)$ as follows:

1. when $i$ is odd, orient $a_i \to \{a_{i+1}, a_{i+3}, \ldots, a_{p-1}\} \cup \{a_j|j < i, j = 1, 3, 5, \ldots, i - 2\}$;
(2) when $i$ is even, orient $a_i \rightarrow \{a_{i+1}, a_{i+2}, \ldots, a_p\} \cup \{a_j\} j < i, j = 2, 4, \ldots, i - 2$;
(3) for $i \neq j$, if $a_i \not\rightarrow a_j$ in the orientation defined in (1) and (2), then orient $a_j \rightarrow a_i$;
(4) for $b_i, b_j$, if $a_i \rightarrow a_j$, then orient $b_j \rightarrow b_i$;

(ii) orient
(a) $b_{p+1} \rightarrow \{b_1, b_2, \ldots, b_p, b_{p+2}, b_{p+3}, b_{p+4}\}$;
(b) $\{b_1, b_3, \ldots, b_{p-2}, b_p\} \rightarrow b_{p+2} \rightarrow \{b_2, b_4, \ldots, b_{p-1}, b_{p+4}\}$;
(c) $\{b_2, b_4, \ldots, b_{p-3}, b_{p-1}, b_p\} \rightarrow b_{p+3} \rightarrow \{b_1, b_3, \ldots, b_{p-2}, b_{p+2}\}$;
(d) $\{b_1, b_3, \ldots, b_{p-4}, b_{p-2}, b_{p-1}\} \rightarrow b_{p+4} \rightarrow \{b_2, b_4, \ldots, b_{p-3}, b_p, b_{p+3}\}$;

(iii) add the $2p + 3$ arcs $\{a_1, a_2, \ldots, a_p\} \rightarrow b_{p+1}$, $b_i \rightarrow a_i$ for $i = 1, \ldots, p$, $b_{p+2} \rightarrow a_p$, $b_{p+3} \rightarrow a_{p-1}$ and $b_{p+4} \rightarrow a_{p-2}$ between $K_p$ and $K_q$.

It is easy to verify that $d(F) = 2$. An illustrative example is given in Fig. 1 where $p = 5$.

For the purpose of clarity not all arcs are shown, and for all $b_i b_j \in E(K_q)$, $1 \leq i \leq 5$, $7 \leq j \leq 9$, if $b_j \not\rightarrow b_i$ in Fig. 1, then $b_i \rightarrow b_j$ in $F$.

We next consider the case when $p$ is even and provide an orientation $F^*$ for a graph $H$ in $G_{2p+3}$ with diameter 2. Let $V(K_p) = \{a_1, a_2, \ldots, a_{p-1}, a_p\}$ and $V(K_q) = \{b_1, b_2, \ldots, b_{p+3}, b_{p+4}\}$. Let $G_1$ and $G_2$ be the subgraphs of $K_p$ and $K_q$ induced by $\{a_1, \ldots, a_{p-1}\}$ and $\{b_1, \ldots, b_{p-1}\}$ respectively. Noting that $G_1$ and $G_2$ are both complete graphs of odd order $p - 1$, we define the following orientation $F^*$ of a graph $H$ in $G_{2p+3}$:

(i) orient the edges in $E(G_1)$ and $E(G_2)$ as described above in (1) to (4);
(ii) orient
(a) $\{a_{2g+1}, \ldots, a_{p-1}\} \rightarrow a_p \rightarrow \{a_1, \ldots, a_q\}$;
(b) $\{b_1, \ldots, b_g\} \rightarrow b_p \rightarrow \{b_{2g+1}, \ldots, b_{p-1}\}$;
(c) $b_{p+1} \rightarrow \{b_1, b_2, \ldots, b_p, b_{p+2}, b_{p+3}, b_{p+4}\}$;
(d) $\{b_1, b_3, \ldots, b_{p-1}, b_p\} \rightarrow b_{p+2} \rightarrow \{b_2, b_4, \ldots, b_{p-2}, b_{p+3}\}$;
(e) $\{b_1, b_3, \ldots, b_g, b_p\} \rightarrow b_{p+3} \rightarrow \{b_{2g+1}, \ldots, b_{p-1}, b_{p+4}\}$;
(f) $\{b_2, b_4, b_6, \ldots, b_{p-1}\} \rightarrow b_{p+4} \rightarrow \{b_1, b_3, b_5, \ldots, b_{p-2}, b_{p+2}\}$;

(iii) add the $2p + 3$ arcs $\{a_1, a_2, \ldots, a_p\} \rightarrow b_{p+1}$, $b_i \rightarrow a_i$ for $i = 1, \ldots, p$, $b_{p+2} \rightarrow a_{p-1}$, $b_{p+3} \rightarrow a_p$ and $b_{p+4} \rightarrow a_2$ between $K_p$ and $K_q$.
Fig. 2. Orientation for $G$ in $G_{15}$ where $p = 6, q = p + 4 = 10$.

It is easy to verify that $d(F^*) = 2$. An illustrative example is given in Fig. 2 where $p = 6$. For clarity not all arcs are shown, and for all $b_i b_j \in E(K_q), 1 \leq i \leq 6, 8 \leq j \leq 10$, if $b_j \not\rightarrow b_i$ in Fig. 2, then $b_i \rightarrow b_j$ in $F^*$.

This concludes the proof of the proposition.  □

The above proposition, together with the orientation provided in [6] for graphs in $G_{2p+4}$, shows that the lower bound on $\alpha$ in Proposition 1 is attainable.

In conclusion, we present the following theorem which completely determines the value of $\alpha$.

**Theorem 3.** Given two integers $p$ and $q$ with $q \geq p \geq 5$ and an integer $n$ with $2 \leq n \leq pq$, let $G_n$ denote the family of graphs that are obtained from the disjoint union of two complete graphs $K_p$ and $K_q$ by adding $n$ edges linking them in an arbitrary manner. Suppose $d^+(n) = \min\{d^+(G) | G \in G_n\}$ and $\alpha = \min\{n | d^+(n) = 2\}$, then we have

$$\alpha = \begin{cases} 2p & \text{if } p \leq q \leq p + 1, \\ 2p + k - 1 & \text{if } q = p + k, k = 2, 3, 4, \\ 2p + 4 & \text{if } q \geq p + 5. \end{cases}$$

□

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**References**