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On hamiltonian line graphs and connectivity

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Abstract

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A well-known conjecture of Thomassen says that every 4-connected line graph is hamiltonian. In this paper we prove that every 7-connected line graph is hamiltonian-connected.

1. Introduction

Large connectivity of graph cannot always guarantee the graph to be hamiltonian in following sense: For any given positive integer n there exists a nonhamiltonian graph with the connectivity at least n. For instance, $K_{n,n+1}$ is an n-connected nonhamiltonian graph. The example in [11] shows that there exists a 3-connected nonhamiltonian claw-free graph, which is also a line graph of some graph. Moreover, one can immediately get infinitely many 3-connected non-hamiltonian line graphs $L(G)$ by setting $r=3$ and $\varepsilon<3/4$ in following results.

Theorem 1 (Harary and Nash-Williams [8]). *If G is a graph with at least 4 vertices, then its line graph $L(G)$ is hamiltonian if and only if G has a closed trail which includes at least one end-vertex of each edge of G or G is isomorphic to $K_{1,s}$, for some integer $s \geqslant 3$.*

Theorem 2 (Jackson and Parsons [10]). *For a given integer $r \geqslant 3$ and any real $\varepsilon>0$, there exists an integer $N(r, \varepsilon)>0$ such that if r is even and $p \geqslant N(r, \varepsilon)$, or if r is odd and p is even and $p \geqslant N(r, \varepsilon)$, then there exists an r-regular r-connected graph with p vertices such that the length of the longest cycle in G is less than εp.*

For line graph, C. Thomassen [1] made the following conjecture.

Conjecture. Every 4-connected line graph is hamiltonian.

Thomassen [3] announced that he had verified the conjecture in the special case that G is 4-edge-connected. Furthermore, in [14] we prove that if G is 4-edge-connected then its line graph $L(G)$ is hamiltonian-connected. The main result of this paper is the following theorem.

Theorem 3. *Every 7-connected line graph is hamiltonian-connected.*

2. Notation and terminology

Let $G = (V, E)$ be a finite undirected graph with vertex set $V(G)$ and edge set $E(G)$—we allow G to have multiple edges but no loops. Let $\kappa(G)$, $\lambda(G)$, $\omega(G)$ and $\delta(G)$ denote the connectivity, edge-connectivity, the number of components and the minimum degree of G respectively.

If V^* is a subset of the vertex set $V(G)$, then we use $G - V^*$ denote the induced subgraph $G[V\backslash V^*]$ (i.e., $V(G - V^*) = V - V^*$ and $E(G - V^*) = \{uv \in E(G)\colon u, v \in V\backslash V^*\}$). If E^* is a subset of the edge set $E(G)$, then we use $G - E^*$ denote the spanning subgraph $G[E\backslash E^*]$ (i.e., $V(G - E^*) = V(G)$ and $E(G - E^*) = E(G) - E^*$).

A subset D of the vertex set $V(G)$ is a *dominating set* if every edge has at least one end-vertex in D.

Let uTv be a trail T with end-vertices u and v. We write xTy when we wish to emphasize the end-edges x and y of the trail T. We also use xTx denote a closed trail T containing the edge x. A trail is a *dominating trail*, denoted uT_dv (or xT_dy), if each edge of G is incident with at least one internal vertex of the trail. A trail is a *spanning trail*, denoted uT_sv (or xT_sy), if it is a dominating trail which contains all the vertices of G. A graph is *dominating trailable* if for each pair x and y of edges of G there exists a dominating trail xT_dy with end-edges x and y. Similarly one can define the *spanning trailable* graph. A graph is *hamiltonian-connected* if for each pair u and v of vertices of G there exists a hamiltonian path with end-vertices u and v. For other definitions, we refer the reader to [4].

3. Reduction

It is trivial to prove the following lemma by a slight modification of the proof of Theorem 1.

Lemma 4. *Let G be a graph with at least 4 vertices. Then the line graph $L(G)$ is hamiltonian-connected if and only if G is dominating trailable.*

Let G be a graph (possibly with multiple edges). We define operations R1 and R2 on G as follows:

R1: delete a vertex, which has degree at most 3 but is adjacent to at most one vertex, and delete its incident edges;

R2: delete a vertex u with degree 2 and its incident edges uv and uw while $v \neq w$ and add a new edge vw.

Example:

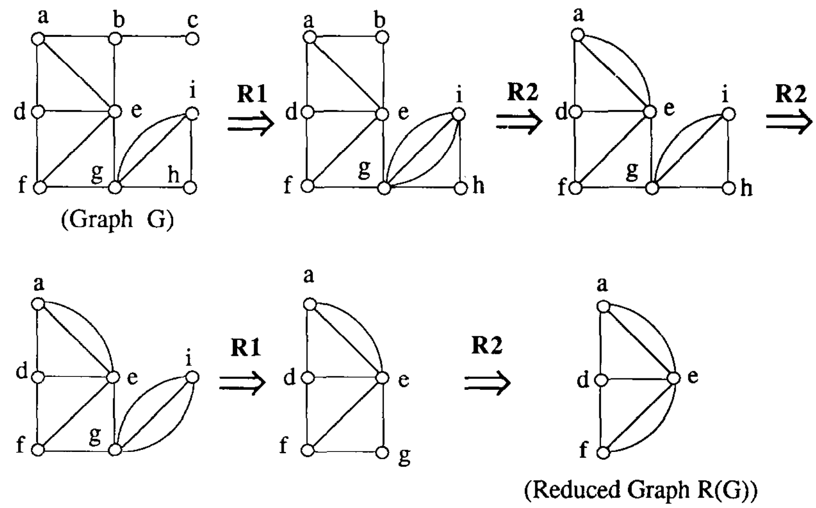

Fig. 1.

For convenience, a graph G is called a *multi-star* if it is obtained from some star $K_{1,s}$ by adding some multiple edges. The *edge multiplicity* of a graph G is the maximum number of multi-edges joining two vertices in G.

Lemma 5. *If G is a graph, which is not a multi-star with the edge multiplicity at most* 3 *and, if its line graph $L(G)$ has connectivity at least* 4, *then there is a unique graph (up to an isomorphism) $R(G)$, so called reduced graph of G, obtained by applying a sequence of operations* R1 *and* R2 *from G such that*:

(i) $\delta(R(G)) \geq 3$;

(ii) $\kappa(L(R(G))) \geq \kappa(L(G))$;

(iii) *$V(R(G))$ is a dominating set of G.*

Proof. First we prove $D = \{v \in V(G)\colon \deg_G(v) \geq 3$ and v is adjacent to at least two vertices in $G\}$ is a dominating set of G. If not, there must be an edge $u'v'$ of G which is not incident with any vertex of D. Since $L(G)$ is 4-connected, we can assume that v' is adjacent to a vertex v of D. Since u' and v' are not in D, $\{vv'\}$ must be a cut set of $L(G)$ which contradicts the assumption on the connectivity of $L(G)$.

Now we prove that $D = V(R(G))$. It is obvious that $V(R(G)) \subseteq D$. In the process of carrying out the reductions R1 and R2 from G, if we delete a vertex v of D in some step from G' to G'', then the set of edges incident with v in G which is correpondent to the set of edges incident with v in G' is a cut set of $L(G)$ and has cardinality at most three. This contradicts to the connectivity of $L(G)$. So $D = V(R(G))$.

Therefore edge uv is in $R(G)$ only if the edge uv is in G or there is a vertex w with degree 2 such that uw and vw are both in G. Hence $R(G)$ is unique and non-empty and (i), (ii) and (iii) follows immediately. □

Lemma 6. *If G is a graph, which is not a multi-star with the edge multiplicity at most* 3 *and, if its line graph $L(G)$ has connectivity at least* 4, *then G is dominating trailable if its reduced graph $R(G)$ is spanning trailable.*

Proof. Let $x = uv$ and $y = st$ be any two edges of G. We choose edges x' and y' in $R(G)$ as follows: If x is in $R(G)$, then choose x' to be x; If x is incident with a vertex v of degree 2 in G while uv and vw are two edges of G and $u \neq w$, then choose x' to be uw in $R(G)$ and if x is incident with a vertex v of degree 1 in G, then choose any other edge x' in $R(G)$ incident with u. Choose y' similarly ($y' \neq x'$). Since $R(G)$ is spanning trailable, there is a spanning trail $x'T_s^*y'$ in $R(G)$. Let T be a trail in G corresponding to the trail $x'T_s^*y'$. Now one can naturally extend T to a dominating trail xT_dy in G by Lemma 5(iii). □

If G is a multi-star, then Theorem 3 is true obviously. So from Lemma 6 it suffices to show that if a graph G has minimum degree at least 3 and its line graph $L(G)$ has connectivity at least 7 then G is spanning trailable. Hence we can always suppose that G satisfies $\delta(G) \geqslant 3$ and $\kappa(L(G)) \geqslant 7$ in the remaining sections except in Theorem 7.

4. Packing trees

In order to find a spanning trail xT_sy for any edges x and y in G, we decompose the graph G into two connected factors (or say, pack two spanning trees into G). The following theorem of Nash-Williams and Tutte [12–13] will be used in our proof.

Theorem 7 (Nash-Williams and Tutte [12–13]). *In order that a finite graph G shall be decomposable into n connected factors it is necessary and sufficient that*

$$|S| \geqslant n(\omega(G - S) - 1)$$

for each subset S of the edge set $E(G)$.

Let $G_1, \ldots, G_r, \ldots, G_t, \ldots, G_\omega$ be all the components of $G-S$, where $G_1, \ldots, G_r$ are the (possibly empty set of) components consisting of a single vertex of degree 3 in G and $G_{r+1}, \ldots, G_t$ are the (possibly empty set of) components containing at least one vertex which is adjacent to some vertices of $\bigcup_{i=1}^r V(G_i)$ and $G_{t+1}, \ldots, G_\omega$ are the remaining (possibly empty set of) components of $G-S$ for some $1 \leqslant r \leqslant s \leqslant \omega = \omega(G-S)$. Let $M(H)$ denote the set of edges of G which have precisely one end-vertix in $V(H)$ and $m(H)$ be the cardinality of $M(H)$ for a subgraph H of G.

Lemma 8. *If $\omega(G-S) \geqslant 3$ for a subset S of the edge set $E(G)$, then*
(i) $m(G_i) = 3$, *for* $1 \leqslant i \leqslant r$;
(ii) $m(G_i) \geqslant 6$, *for* $r+1 \leqslant i \leqslant t$;
(iii) $m(G_i) \geqslant 4$, *for* $t+1 \leqslant i \leqslant \omega$;
(iv) $\sum_{i=r+1}^t m(G_i) \geqslant \sum_{i=1}^r m(G_i)$;
(v) $\bigcup_{i=1}^\omega M(G_i) \subseteq S$.

Proof. Parts (i), (iv) and (v) are obvious from the definition of G_i and $m(G_i)$. In part (ii), let G_a be the component of $G-S$ having a vertex adjacent to a vertex of G_i in G for some $1 \leqslant a \leqslant r$. If $M(G_i \cup G_a)$ is a vertex cut of $L(G)$, then $m(G_i \cup G_a) \geqslant 7$ as $\kappa(L(G)) \geqslant 7$ which immediately implies $m(G_i) \geqslant 6$. If $M(G_i \cup G_a)$ is not a vertex cut of $L(G)$, then $M(G_i \cup G_a)$ must separate $G_i \cup G_a$ from some single vertex components in G and, since $\omega(G-S) \geqslant 3$ then $m(G_i) \geqslant 6$. Part (iii) follows directly from the definition of G_i and $k(L(G) \geqslant 7$. □

Lemma 9. *If S is a subset of $E(G)$, then*

$$|S| \geqslant 2\omega(G-S) - 1, \quad \text{if } r = 1 \text{ and } \omega = 2;$$
$$|S| \geqslant 2\omega(G-S), \quad \text{otherwise.}$$

Proof. It is easy to verify if $r = 0$ or $\omega \leqslant 2$. So we may assume that $r \neq 0$ and $\omega \geqslant 3$. From (i), (iii) and (v) of Lemma 8 we have

$$|S| \geqslant \left| \bigcup_{i=1}^{\omega} M(G_i) \right| \geqslant \tfrac{1}{2} \sum_{i=1}^{\omega} m(G_i)$$
$$= \tfrac{1}{2} \sum_{i=1}^{r} m(G_i) + \tfrac{1}{2} \sum_{i=r+1}^{t} m(G_i) + \tfrac{1}{2} \sum_{i=t+1}^{\omega} m(G_i)$$
$$\geqslant \tfrac{1}{2}3r + \tfrac{1}{2} \sum_{i=r+1}^{t} m(G_i) + \tfrac{1}{2}4(\omega - t).$$

If $r \leqslant 2(t-r)$, then, by (ii) of Lemma 8,

$$|S| \geqslant \tfrac{1}{2}3r + \tfrac{1}{2}6(t-r) + \tfrac{1}{2}4(\omega - t) = 2\omega + (t-r) - \tfrac{1}{2}r \geqslant 2\omega.$$

If $r > 2(t-r)$, then, by (iv) of Lemma 8,

$$|S| \geqslant \tfrac{1}{2}3r + \tfrac{1}{2}3r + \tfrac{1}{2}4(\omega - t) = 2\omega + r - 2(t-r) > 2\omega. \qquad \square$$

From Theorem 7 and Lemma 9 we have the following corollary.

Corollary 10. *For every pair x and y of edges of G, the subgraph $G - \{x, y\}$, or $G - \{x\}$ if x and y have an end-vertex of degree 3 in common, can be decomposed into two connected factors F_1 and F_2.*

Proof. If x and y are incident with a common vertex u of degree 3, then $|S| \geqslant 2\omega(G - S) - 1$ by Lemma 9 for any subset S of the edge set $E(G)$ and hence

$$|S \cup \{x\}| \geqslant 2\omega(G - [\{x\} \cup S]) - 1$$

for any subset S of the edge set $E(G - \{x\})$, i.e.,

$$|S| \geqslant 2\omega([G - \{x\}] - S) - 2.$$

So the subgraph $G - \{x\}$ can be decomposed into two connected factors by the Theorem 7.

If x and y are not incident with a common vertex of degree 3, then either $r \neq 1$ or $\omega \neq 2$ for the components of $G - [\{x, y\} \cup S]$ $(=[G - \{x, y\}] - S)$ of G and for any subset S of the edge set $E(G - \{x, y\})$. By Theorem 9,

$$|S \cup \{x, y\}| \geqslant 2\omega(G - [\{x, y\} \cup S]),$$

i.e.,

$$|S| \geqslant 2\omega([G - \{x, y\}] - S) - 2.$$

So the subgraph $G - \{x, y\}$ can be decomposed into two connected factors by the Theorem 7.

5. Proof of Theorem 3

Lemma 11. *Let x, y and z be edges of G. If x and y are incident with a common vertex of degree 3, then there is a spanning closed trail containing y and z but not containing x; If x and y are not incident with a common vertex of degree 3, then there is a spanning closed trail containing z but not containing x or y.*

Proof. Let F_1 and F_2 be the two factors in Corollary 10. So z must be in one of them, say, in F_1. Let B be the set of odd degree vertices in F_1. Then $|B|$ must be even, say, $|B| = 2k$. Pair off the vertices of B arbitrary and let $P_1, P_2, \ldots, P_k$ be the paths joining the two correspondent vertices of each pair in F_2. Let D be the set of all edges which appear an odd number of the P_i in F_2. Then $F_1 + D$ must be eulerian. Regarded as a closed trail of G, $F_1 + D$ is the trail we need. □

Lemma 12. *G is spanning trailable.*

Proof. Let x and y be any two edges of G. If x and y are incident with a common vertex of degree 3 in G, by Lemma 11, there is a spanning closed trail $yT_s y$

containing y but not containing x (where z can be any edge except x). Then xT_sy, by adding the edge x in the trail yT_sy, is a spanning trail in G with end-edges x and y; If x and y are incident with a common vertex u which is not of degree 3, by Lemma 11, there is a spanning closed trail zT_sz in G containing z but not containing x or y, where z is an edge sharing the common vertex u with x and y in G. Then xT_szy is a spanning trail in G; Otherwise if x and y are non-adjacent in G and $z = uv$ where u and v are end-vertices of x and y respectively, then by Lemma 11 there is a spanning closed trail zT_sz which contains z but does not contain x or y (if z is not in G, then we take $G + \{z\}$, which also has $\delta(G + \{z\}) \geqslant 3$ and $\kappa(L(G + \{z\})) \geqslant 7$, instead of G in Lemma 11). Then xT_sy is a spanning trail. □

Proof of Theorem 3. The proof follows from Lemma 4, Lemma 6 and Lemma 12 immediately. □

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