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On soft equality

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ABSTRACT

Molodtsov introduced the concept of soft sets, which can be seen as a new mathematical tool for dealing with uncertainty. In this paper, we deal with the algebraic structure of soft sets. The lattice structures of soft sets are constructed. The concept of soft equality is introduced and some related properties are derived. It is proved that soft equality is a congruence relation with respect to some operations and the soft quotient algebra is established.

Contents

1. Introduction

To solve complicated problems in economics, engineering, environmental science and social science, methods in classical mathematics are not always successful because of various types of uncertainties present in these problems. While probability theory, fuzzy set theory [1], rough set theory [2], and other mathematical tools are well-known and often useful approaches to describing uncertainty, each of these theories has its inherent difficulties as pointed out in [3]. In 1999, Molodtsov [3] introduced the concept of soft sets, which can be seen as a new mathematical tool for dealing with uncertainties. This so-called soft set theory is free from the difficulties affecting the existing methods.

Presently, works on soft set theory are progressing rapidly. Soft set theory has a rich potential for applications in several directions, few of which had been shown in [3]. Maji et al. [4] described the application of soft set theory to a decision making

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problem. Chen et al. [5] presented a new definition of soft set parameterization reduction, and compared this definition to the related concept of attributes reduction in rough set theory. In theoretical aspects, Maji et al. [6] defined several operations on soft sets and made a theoretical study on the theory of soft sets. Aktas and Cagman [7] compared soft sets to the related concepts of fuzzy sets and rough sets. They also defined the notion of soft groups and derived some related properties. Jun [8] introduced the notion of soft BCK/BCI-algebras. Jun and Park [9] discussed the applications of soft sets in ideal theory of BCK/BCI-algebras. Feng et al. [10] initiated the study of soft semirings. Furthermore, based on [6], Irfan et al. [11] introduced some new operations on soft sets and improved the notion of complement of soft set. They proved that certain De Morgan's laws hold in soft set theory.

Theoretical foundations of soft computing techniques stem from purely mathematical concepts. Cabrera et al. [12] made a survey on the theoretical and practical development of the theory of fuzzy logic and soft computing. In this paper, we make a theoretical study of the soft set theory and concentrate on the algebraic structure of soft sets. The lattice structures of soft sets are constructed. The concept of soft equality is introduced and some related properties are derived. Some equivalent conditions for soft sets being soft equality are given. It is proved that soft equality is a congruence relation with respect to some operations and the soft quotient algebra is established.

2. Preliminaries

This section presents a review of some fundamental notions of soft sets. We refer to [3,6,11] for details.

Let U be an initial universe set and E the set of all possible parameters under consideration with respect to U. The power set of U (i.e., the set of all subsets of U) is denoted by $P(U)$. Usually, parameters are attributes, characteristics, or properties of objects in U. Molodtsov defined the notion of a soft set in the following way:

Definition 1 (*[3]*). A pair (F, A) is called a soft set over U, where $A \subseteq E$ and F is a mapping given by $F : A \to P(U)$.

In other words, a soft set over U is a parameterized family of subsets of U. For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A). For illustration, Molodtsov considered several examples in [3].

Example 2 (*[3]*). Suppose that there are six houses in the universe U given by $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ and $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ is the set of parameters. Where $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ and e_8 stand for the parameters 'expensive', 'beautiful', 'wooden', 'cheap', 'in the green surroundings', 'modern', 'in good repair', and 'in bad repair' respectively.

In this case, to define a soft set means to point out expensive houses, beautiful houses, and so on. The soft set (F, A) may describe the 'attractiveness of the houses' which Mr. X is going to buy. Suppose that $A = \{e_1, e_2, e_3, e_4\}$, $F(e_1) = \{h_2, h_4\}$, $F(e_2) = \{h_1, h_3\}$, $F(e_3) = \{h_3, h_4, h_5\}$, and $F(e_4) = \emptyset$. Then the soft set (F, A) is a parameterized family $\{F(e_i); 1 \le i \le 4\}$ of subsets of U and give us a collection of approximate descriptions of an object. $F(e_1) = \{h_2, h_4\}$ means houses h_2 and h_4 are 'expensive'.

Example 3 (*[3]*). Zadeh's fuzzy set may be considered as a special case of the soft set. Let A be a fuzzy set, and μ_A be the membership function of A, that is μ_A is a mapping of U into [0, 1]. For $\alpha \in [0, 1]$, let

$$F(\alpha) = \{x \in U; \mu_A(x) \ge \alpha\}$$

be the α-level set. If we know the family $\{F(\alpha); \alpha \in [0, 1]\}$, we can calculate $\mu_A(x)$ by means of the formulae $\mu_A(x) = \sup_{\alpha\in[0,1], x\in F(\alpha)} \alpha$. Thus, fuzzy set A may be considered as the soft set $(F, [0, 1])$.

Maji et al. [6] and Irfan et al. [11] introduced some operations on soft sets.

Definition 4 (*[6]*). The union of two soft sets (F, A) and (G, B) over a common universe U, denoted by $(F, A) \widetilde{\cup} (G, B)$, is the soft set (H, C), where $C = A \cup B$, and $\forall e \in C$, if $e \in A - B$, then $H(e) = F(e)$; if $e \in B - A$, then $H(e) = G(e)$; if $e \in A \cap B$, then $H(e) = F(e) \cup G(e)$.

Definition 5 (*[11]*). The extended intersection of two soft sets (F, A) and (G, B) over a common universe U, denoted by $(F, A) \sqcap_\varepsilon (G, B)$, is the soft set (H, C), where $C = A \cup B$, and $\forall e \in C$, if $e \in A - B$, then $H(e) = F(e)$; if $e \in B - A$, then $H(e) = G(e)$; if $e \in A \cap B$, then $H(e) = F(e) \cap G(e)$.

Definition 6 (*[11]*). The restricted intersection of two soft sets (F, A) and (G, B) over a common universe U, denoted by $(F, A) \Cap (G, B)$, is the soft set (H, C), where $C = A \cap B$, and $\forall e \in C$, $H(e) = F(e) \cap G(e)$.

Definition 7 (*[11]*). The restricted union of two soft sets (F, A) and (G, B) over a common universe U, denoted by $(F, A) \cup_{\Re} (G, B)$, is the soft set (H, C), where $C = A \cap B$, and $\forall e \in C$, $H(e) = F(e) \cup G(e)$.

Definition 8 (*[11]*).

(a) (F, A) is called a relative null soft set (with respect to the parameter set A), denoted by $\emptyset_A$, if $F(e) = \emptyset$ for all $e \in A$.
(b) (G, A) is called a relative whole soft set (with respect to the parameter set A), denoted by U_A, if $F(e) = U$ for all $e \in A$.

Definition 9 (*[11]*). The relative complement of a soft sets (F, A) is denoted by $(F, A)^r$ and is defined by $(F, A)^r = (F^r, A)$ where $F^r : A \to P(U)$ is a mapping given by $F^r(e) = U - F(e)$ for all $e \in A$.

Clearly, $((F, A)^r)^r = (F, A)$ hold.

The following types of De Morgan's laws hold in soft set theory.

Theorem 10 (*[11]*). *Let (F, A) and (G, B) be two soft sets over the same universe U such that $A \cap B \neq \emptyset$. Then*

(1) $((F, A) \cup_{\Re} (G, B))^r = (F, A)^r \Cap (G, B)^r$,
(2) $((F, A) \Cap (G, B))^r = (F, A)^r \cup_{\Re} (G, B)^r$.

3. The lattice structure of soft sets

Algebraic structures play a fundamental role in many fields of mathematics. The lattice structure of some fuzzy algebraic systems such as fuzzy groups, G-fuzzy groups, some fuzzy ordered algebras and fuzzy hyperstructures were discussed in [13]. It is proved that under suitable conditions, these structures form a distributive or modular lattice. In this section, we discuss the lattice structure of soft sets.

Theorem 11. *Let (F, A), (G, B) and (H, C) be soft sets over the same universe U. Then*

(1) $(F, A) \widetilde{\cup} (F, A) = (F, A)$,
(2) $(F, A) \widetilde{\cup} (G, B) = (G, B) \widetilde{\cup} (F, A)$,
(3) $((F, A) \widetilde{\cup} (G, B)) \widetilde{\cup} (H, C) = (F, A) \widetilde{\cup} ((G, B) \widetilde{\cup} (H, C))$.

Proof. (1) and (2) are trivial. We only prove (3). Suppose that

$$((F, A) \widetilde{\cup} (G, B)) \widetilde{\cup} (H, C) = (K, A \cup B \cup C),$$
$$(F, A) \widetilde{\cup} ((G, B) \widetilde{\cup} (H, C)) = (L, A \cup B \cup C).$$

For all $e \in A \cup B \cup C$, it follows that $e \in A$, or $e \in B$, or $e \in C$. Without loss of generality, we suppose that $e \in C$.

(a) If $e \notin A$ and $e \notin B$, then $K(e) = H(e) = L(e)$;
(b) If $e \in A$ and $e \notin B$, then $K(e) = F(e) \cup H(e) = L(e)$;
(c) If $e \notin A$ and $e \in B$, then $K(e) = G(e) \cup H(e) = L(e)$;
(d) If $e \in A$ and $e \in B$, then $K(e) = (F(e) \cup G(e)) \cup H(e) = F(e) \cup (G(e) \cup H(e)) = L(e)$.

Since K and L are indeed the same set-valued mappings, we conclude that $((F, A) \widetilde{\cup} (G, B)) \widetilde{\cup} (H, C) = (F, A) \widetilde{\cup} ((G, B) \widetilde{\cup} (H, C))$ as required. □

This theorem shows that the operation $\widetilde{\cup}$ is idempotent, associative and commutative. The following three theorems can be proved similarly:

Theorem 12. *Let (F, A), (G, B) and (H, C) be soft sets over the same universe U. Then*

(1) $(F, A) \sqcap_{\varepsilon} (F, A) = (F, A)$,
(2) $(F, A) \sqcap_{\varepsilon} (G, B) = (G, B) \sqcap_{\varepsilon} (F, A)$,
(3) $((F, A) \sqcap_{\varepsilon} (G, B)) \sqcap_{\varepsilon} (H, C) = (F, A) \sqcap_{\varepsilon} ((G, B) \sqcap_{\varepsilon} (H, C))$.

Theorem 13. *Let (F, A), (G, B) and (H, C) be soft sets over the same universe U. Then*

(1) $(F, A) \cup_{\Re} (F, A) = (F, A)$,
(2) $(F, A) \cup_{\Re} (G, B) = (G, B) \cup_{\Re} (F, A)$,
(3) $((F, A) \cup_{\Re} (G, B)) \cup_{\Re} (H, C) = (F, A) \cup_{\Re} ((G, B) \cup_{\Re} (H, C))$.

Theorem 14. *Let (F, A), (G, B) and (H, C) be soft sets over the same universe U. Then*

(1) $(F, A) \Cap (F, A) = (F, A)$,
(2) $(F, A) \Cap (G, B) = (G, B) \Cap (F, A)$,
(3) $((F, A) \Cap (G, B)) \Cap (H, C) = (F, A) \Cap ((G, B) \Cap (H, C))$.

The following theorem shows that the absorption law with respect to operations $\widetilde{\cup}$ and $\Cap$ holds.

Theorem 15. *Let (F, A) and (G, B) be two soft sets over the same universe U. Then*

(1) $((F, A) \widetilde{\cup} (G, B)) \Cap (F, A) = (F, A)$,
(2) $((F, A) \Cap (G, B)) \widetilde{\cup} (F, A) = (F, A)$.

Proof. (1) Suppose that $(F, A) \tilde{\cup} (G, B) = (H, A \cup B)$ and $((F, A) \tilde{\cup} (G, B)) \Cap (F, A) = (K, (A \cup B) \cap A) = (K, A)$. For all $e \in A$,
(a) if $e \in B$, then $K(e) = H(e) \cap F(e) = (F(e) \cup G(e)) \cap F(e) = F(e)$;
(b) if $e \notin B$, then $K(e) = H(e) \cap F(e) = F(e) \cap F(e) = F(e)$.
Hence $((F, A) \tilde{\cup} (G, B)) \Cap (F, A) = (F, A)$.
(2) Suppose that $(F, A) \Cap (G, B) = (H, A \cap B)$ and $((F, A) \Cap (G, B)) \tilde{\cup} (F, A) = (K, (A \cap B) \cup A) = (K, A)$. For all $e \in A$,
(a) if $e \in B$, then $K(e) = H(e) \cup F(e) = (F(e) \cap G(e)) \cup F(e) = F(e)$;
(b) if $e \notin B$, then $e \notin A \cap B$. It follows that $K(e) = F(e)$.
Hence $((F, A) \Cap (G, B)) \tilde{\cup} (F, A) = (F, A)$. □

We denote by $S(U, E)$ the set of all soft sets over the universe U and the parameter set E, that is

$$S(U, E) = \{(F, A); A \subseteq E, F : A \to P(U)\}.$$

Theorem 16. (1) $(S(U, E), \tilde{\cup}, \Cap)$ *is a distributive lattice.*
(2) *Let* $\leq_1$ *be the ordering relation in lattice* $(S(U, E), \tilde{\cup}, \Cap)$ *and* $(F, A), (G, B) \in S(U, E)$. $(F, A) \leq_1 (G, B)$ *if and only if:* $A \subseteq B$ *and* $F(e) \subseteq G(e)$ *for all* $e \in A$.

Proof. (1) By Theorems 11, 14 and 15, $(S(U, E), \tilde{\cup}, \Cap)$ is a lattice. We prove that the following distributive law

$$(F, A) \tilde{\cup} ((G, B) \Cap (H, C)) = ((F, A) \tilde{\cup} (G, B)) \Cap ((F, A) \tilde{\cup} (H, C))$$

holds for all $(F, A), (G, B), (H, C) \in S(U, E)$.
Suppose that

$$(F, A) \tilde{\cup} ((G, B) \Cap (H, C)) = (K, A \cup (B \cap C)),$$
$$((F, A) \tilde{\cup} (G, B)) \Cap ((F, A) \tilde{\cup} (H, C)) = (L, (A \cup B) \cap (A \cup C)) = (L, A \cup (B \cap C)).$$

For each $e \in A \cup (B \cap C)$,
(a) if $e \notin A$, then $e \in B$ and $e \in C$,it follows that $K(e) = G(e) \cap H(e) = L(e)$;
(b) if $e \in A, e \notin B, e \notin C$, then $K(e) = F(e) = F(e) \cap F(e) = L(e)$;
(c) if $e \in A, e \in B, e \notin C$, then $K(e) = F(e) = (F(e) \cup G(e)) \cap F(e) = L(e)$;
(d) if $e \in A, e \notin B, e \in C$, then $K(e) = F(e) = F(e) \cap (F(e) \cup H(e)) = L(e)$;
(e) if $e \in A, e \in B, e \in C$, then $K(e) = F(e) \cup (G(e) \cap H(e)) = (F(e) \cup G(e)) \cap (F(e) \cup H(e)) = L(e)$.
Hence $(F, A) \tilde{\cup} ((G, B) \Cap (H, C)) = ((F, A) \tilde{\cup} (G, B)) \Cap ((F, A) \tilde{\cup} (H, C))$.
(2) Suppose that $(F, A) \leq_1 (G, B)$. Then $(F, A) \tilde{\cup} (G, B) = (G, B)$. So by definition, we have $A \cup B = B$ and hence $A \subseteq B$. For all $e \in A$, by $F(e) \cup G(e) = G(e)$, it follows that $F(e) \subseteq G(e)$.
Conversely, suppose that $A \subseteq B$ and $F(e) \subseteq G(e)$ for all $e \in A$. It is trivial to verify that $(F, A) \tilde{\cup} (G, B) = (G, B)$ and hence $(F, A) \leq_1 (G, B)$. □

Clearly, $(S(U, E), \tilde{\cup}, \Cap)$ is a bounded lattice, U_E and $\emptyset_\emptyset$ are the upper bound and lower bound respectively. Now we consider soft sets over a definite parameter set. Let $A \subseteq E$ and

$$S_A = \{(F, A); F : A \to P(U)\}$$

be the set of soft sets over the universe U and the parameter set A. It is trivial to verify that $(F, A) \tilde{\cup} (G, A), (F, A) \Cap (G, A) \in S_A$ for all $(F, A), (G, A) \in S_A$. That is to say

Corollary 17. S_A *is a sublattice of* $(S(U, E), \tilde{\cup}, \Cap)$.

In $(S_A, \tilde{\cup}, \Cap)$, U_A and $\emptyset_A$ are the greatest element and the least element respectively.
In what follows, we consider another lattice structure of soft sets.

Theorem 18. *Let* (F, A) *and* (G, B) *be two soft sets over the same universe* U. *Then*

(1) $((F, A) \cup_{\Re} (G, B)) \sqcap_{\varepsilon} (F, A) = (F, A)$,
(2) $((F, A) \sqcap_{\varepsilon} (G, B)) \cup_{\Re} (F, A) = (F, A)$.

Proof. (1) Suppose that $(F, A) \cup_{\Re} (G, B) = (H, A \cap B)$ and $((F, A) \cup_{\Re} (G, B)) \sqcap_{\varepsilon} (F, A) = (K, (A \cap B) \cup A) = (K, A)$. For all $e \in A$,
(a) if $e \in B$, then $e \in A \cap B$ and $K(e) = H(e) \cap F(e) = (F(e) \cup G(e)) \cap F(e) = F(e)$;
(b) if $e \notin B$, then $e \notin A \cap B$ and $K(e) = F(e)$.
Hence $((F, A) \cup_{\Re} (G, B)) \sqcap_{\varepsilon} (F, A) = (F, A)$.
(2) Suppose that $(F, A) \sqcap_{\varepsilon} (G, B) = (H, \cup B)$ and $((F, A) \sqcap_{\varepsilon} (G, B)) \cup_{\Re} (F, A) = (K, (A \cup B) \cap A) = (K, A)$. For all $e \in A$,
(a) if $e \in B$, then $e \in A \cap B$ and $K(e) = H(e) \cup F(e) = (F(e) \cap G(e)) \cup F(e) = F(e)$;
(b) if $e \notin B$, then $K(e) = H(e) \cup F(e) = F(e) \cup F(e) = F(e)$.
Hence $((F, A) \sqcap_{\varepsilon} (G, B)) \cup_{\Re} (F, A) = (F, A)$. □

Theorem 19. *For all* $(F, A), (G, B), (H, C) \in S(U, E)$,

$$(F, A) \cup_{\mathfrak{R}} ((G, B) \sqcap_{\varepsilon} (H, C)) = ((F, A) \cup_{\mathfrak{R}} (G, B)) \sqcap_{\varepsilon} ((F, A) \cup_{\mathfrak{R}} (H, C)).$$

Proof. Suppose that

$$(F, A) \cup_{\mathfrak{R}} ((G, B) \sqcap_{\varepsilon} (H, C)) = (K, A \cap (B \cup C)),$$
$$((F, A) \cup_{\mathfrak{R}} (G, B)) \sqcap_{\varepsilon} ((F, A) \cup_{\mathfrak{R}} (H, C)) = (L, (A \cap B) \cup (A \cap C)) = (L, A \cap (B \cup C)).$$

For each $e \in A \cap (B \cup C)$, it follows that $e \in A$ and $e \in B \cup C$,
(a) if $e \in A, e \notin B, e \in C$, then $K(e) = F(e) \cup H(e) = L(e)$;
(b) if $e \in A, e \in B, e \notin C$, then $K(e) = F(e) \cup G(e) = L(e)$;
(c) if $e \in A, e \in B, e \in C$, then $K(e) = F(e) \cup (G(e) \cap H(e)) = (F(e) \cup G(e)) \cap (F(e) \cup H(e)) = L(e)$.
Hence $(F, A) \cup_{\mathfrak{R}} ((G, B) \sqcap_{\varepsilon} (H, C)) = ((F, A) \cup_{\mathfrak{R}} (G, B)) \sqcap_{\varepsilon} ((F, A) \cup_{\mathfrak{R}} (H, C))$. □

Theorem 20. (1) *$(S(U, E), \cup_{\mathfrak{R}}, \sqcap_{\varepsilon})$ is a distributive lattice.*
(2) *Let $\leq_2$ be the ordering relation in lattice $(S(U, E), \cup_{\mathfrak{R}}, \sqcap_{\varepsilon})$ and $(F, A), (G, B) \in S(U, E)$. $(F, A) \leq_2 (G, B)$ if and only if: $B \subseteq A$ and for all $e \in B$, $F(e) \subseteq G(e)$.*

Proof. (1) This can be deduced from Theorems 12, 13, 18 and 19.

(2) Suppose that $(F, A) \leq_2 (G, B)$. Then $(F, A) \cup_{\mathfrak{R}} (G, B) = (G, B)$. By definition, $A \cap B = B$ and hence $B \subseteq A$. For all $e \in B$, by $F(e) \cup G(e) = G(e)$, it follows that $F(e) \subseteq G(e)$.

Conversely, suppose that $B \subseteq A$ and $F(e) \subseteq G(e)$ for all $e \in B$. It is trivial to verify that $(F, A) \cup_{\mathfrak{R}} (G, B) = (G, B)$ and hence $(F, A) \leq_2 (G, B)$. □

Corollary 21. *S_A is a sublattice of $(S(U, E), \cup_{\mathfrak{R}}, \sqcap_{\varepsilon})$.*

We notice that the lattice structure $(S(U, E), \widetilde{\cup}, \Cap)$ is different from that of $(S(U, E), \cup_{\mathfrak{R}}, \sqcap_{\varepsilon})$.

4. The soft equality relation $\approx_S$

In this section, we introduce the concept of soft equality and study its related properties.

Let (F, A) be a soft set over the universe U. We consider the soft set (G, E), where $G(e) = F(e)$ for all $e \in A$ and $G(e) = \emptyset$ for all $e \in E - A$. The parameter sets of (F, A) and (G, E) are different. It follows that F and G are different set-valued mappings and hence (F, A) and (G, E) are different soft sets. But for any common parameter $e \in A$, the sets of e-approximation elements with respect to (F, A) and (G, E) are equal, and for all $e \in E - A$, $G(e) = \emptyset$. Thus (F, A) and (G, E) may be considered to represent almost same approximations and to some extent they are equal.

As illustration, we consider the soft set (F, A) given by Mr. X in Example 2. This soft set describes 'attractiveness of the houses'. It is worth noting that A is a proper subset of E. For a parameter $e \in E - A$, we may think that Mr. X does not care about the attribute of e, or he does not take e into account. In this case, we may suppose that Mr. X thinks that $F(e) = \emptyset$.

Based on these observations, we introduce the concept of 'soft equality' as below.

Definition 22. Let (F, A), (G, B) be two soft sets over the universe U. (F, A) is called soft equal to (G, B), denoted by $(F, A) \approx_S (G, B)$, if for all $e \in A \cup B$, $e \in A \cap B$ implies $F(e) = G(e)$, $e \in A - B$ implies $F(e) = \emptyset$, and $e \in B - A$ implies $G(e) = \emptyset$.

Example 23. We consider the soft set (F, A) given in Example 2. Let (G, B) be a soft set, where $B = \{e_1, e_2, e_3, e_5\}$, $G(e_1) = F(e_1)$, $G(e_2) = F(e_2)$, $G(e_3) = F(e_3)$, and $G(e_5) = \emptyset$. Then $(F, A) \approx_S (G, \{e_1, e_2, e_3, e_5\})$.

Theorem 24. *Let (F, A) and (G, B) be two soft sets over the universe U. Then $(F, A) \approx_S (G, B)$ if and only if $(F, A) \widetilde{\cup} (G, B) \approx_S (F, A) \Cap (G, B)$.*

Proof. Let $(F, A) \widetilde{\cup} (G, B) = (H, A \cup B)$ and $(F, A) \Cap (G, B) = (T, A \cap B)$.

Suppose that $(F, A) \approx_S (G, B)$. For all $e \in A \cap B$, by definition, we have $F(e) = G(e)$, and hence $H(e) = F(e) \cup G(e) = F(e) \cap G(e) = T(e)$. For all $e \in A \cup B - A \cap B$, (a) if $e \in A - B$, then $F(e) = \emptyset$ and hence $H(e) = F(e) = \emptyset$; (b) if $e \in B - A$, then $G(e) = \emptyset$ and hence $H(e) = G(e) = \emptyset$. Consequently, $(F, A) \widetilde{\cup} (G, B) \approx_S (F, A) \Cap (G, B)$.

Conversely, suppose that $(F, A) \widetilde{\cup} (G, B) \approx_S (F, A) \Cap (G, B)$. For all $e \in A \cap B$, we have $F(e) \cup G(e) = F(e) \cap G(e)$ and hence $F(e) = G(e)$. For all $e \in A - B$, it follows that $e \in A \cup B$, $e \notin A \cap B$, and hence $F(e) = H(e) = \emptyset$. For all $e \in B - A$, $G(e) = \emptyset$ can be proved similarly. Hence $(F, A) \approx_S (G, B)$. □

Theorem 25. *Let (F, A), (G, B) be two soft sets over the universe U and $(F, A) \approx_S (G, B)$. Then*

(1) $(F, A) \widetilde{\cup} (G, B) = (F, A) \sqcap_{\varepsilon} (G, B)$,
(2) $(F, A) \cup_{\mathfrak{R}} (G, B) = (F, A) \Cap (G, B)$.

Proof. (1) Suppose that $(F, A) \widetilde{\cup} (G, B) = (H, A \cup B)$, $(F, A) \sqcap_{\varepsilon} (G, B) = (T, A \cup B)$, and $e \in A \cup B$. If $e \in A \cap B$, then $F(e) = G(e)$ and hence $H(e) = F(e) \cup G(e) = F(e) \cap G(e) = T(e)$. If $e \in A - B$ or $e \in B - A$, then $H(e) = T(e)$ by definition.

(2) can be proved similarly. □

Theorem 26. *Let (F, A) and (G, B) be two soft sets over the universe U. Then $(F, A) \approx_S (G, B)$ if and only if $(F, A) \cup_{\Re} (G, B) \approx_S (F, A) \sqcap_{\varepsilon} (G, B)$.*

Proof. This can be deduced from Theorems 24 and 25. □

We denote by $S(U, E)$ the set of all soft sets over the universe U and the parameter set E, that is

$$S(U, E) = \{(F, A); A \subseteq E, F : A \to P(U)\}.$$

Theorem 27. *$\approx_S$ is an equivalence relation on $S(U, E)$.*

Proof. It is trivial to verify that $\approx_S$ is reflexive and symmetric.

Suppose that $(F, A) \approx_S (G, B)$ and $(G, B) \approx_S (H, C)$. For all $e \in A \cap C$, if $e \in B$, then $e \in A \cap B$ and $e \in B \cap C$, it follows that $F(e) = G(e) = H(e)$; if $e \notin B$, then $e \in A - B$ and $e \in C - B$, it follows that $F(e) = \emptyset = H(e)$.

For all $e \in A - C$, it follows that $e \in A$ and $e \notin C$. If $e \in B$, then $e \in A \cap B$ and $e \in B - C$. Consequently, $F(e) = G(e) = \emptyset$; if $e \notin B$, then $e \in A - B$ and $F(e) = \emptyset$.

For all $e \in C - A$, $H(e) = \emptyset$ can be proved similarly. Hence $(F, A) \approx_S (H, C)$. We conclude that $\approx_S$ is a transitive relation as required. □

Theorem 28. *$\approx_S$ is a congruence relation with respect to operations $\Cap$ and $\widetilde{\cup}$. That is to say, if $(F, A) \approx_S (G, B)$, $(H, C) \approx_S (L, D)$, then*

(1) $(F, A) \Cap (H, C) \approx_S (G, B) \Cap (L, D)$,
(2) $(F, A) \widetilde{\cup} (H, C) \approx_S (G, B) \widetilde{\cup} (L, D)$.

Proof. (1) Suppose that $(F, A) \Cap (H, C) = (M_1, A \cap C)$, $(G, B) \Cap (L, D) = (M_2, B \cap D)$.

(a) If $e \in (A \cap C) \cap (B \cap D)$, then $e \in (A \cap B)$ and $e \in (C \cap D)$. It follows that $F(e) = G(e)$, $H(e) = L(e)$ and hence

$$M_1(e) = F(e) \cap H(e) = G(e) \cap L(e) = M_2(e).$$

(b) If $e \in (A \cap C) - (B \cap D)$, then $e \in A$, $e \in C$ and $e \notin B \cap D$. It follows that $e \notin B$ or $e \notin D$. If $e \notin B$, then $e \in A - B$ and hence $F(e) = \emptyset$. Consequently, $M_1(e) = F(e) \cap H(e) = \emptyset$. If $e \notin D$, then $e \in C - D$ and hence $H(e) = \emptyset$. It follows that $M_1(e) = F(e) \cap H(e) = \emptyset$.

(c) If $e \in (B \cap D) - (A \cap C)$, then $M_2(e) = \emptyset$ can be proved similarly.

We conclude that $(F, A) \Cap (H, C) \approx_S (G, B) \Cap (L, D)$.

(2) Suppose that $(F, A) \widetilde{\cup} (H, C) = (T_1, A \cup C)$, $(G, B) \widetilde{\cup} (L, D) = (T_2, B \cup D)$.

For all $e \in (A \cup C) \cap (B \cup D)$, we have $e \in A \cup C$ and $e \in B \cup D$. Without loss of generality, we suppose that $e \in A$ and $e \in D$.

(a) If $e \in B$ and $e \in C$, then $e \in A \cap B$ and $e \in C \cap D$. It follows that $F(e) = G(e)$, $H(e) = L(e)$ and hence $T_1(e) = F(e) \cup H(e) = G(e) \cup L(e) = T_2(e)$.

(b) If $e \notin B$ and $e \in C$, then $e \in A - B$ and $e \in C \cap D$. It follows that $F(e) = \emptyset$, $H(e) = L(e)$ and hence $T_1(e) = F(e) \cup H(e) = H(e) = L(e) = T_2(e)$.

(c) If $e \in B$ and $e \notin C$, then $e \in A \cap B$ and $e \in D - C$. It follows that $F(e) = G(e)$, $L(e) = \emptyset$ and hence $T_1(e) = F(e) = G(e) = G(e) \cup L(e) = T_2(e)$.

(d) If $e \notin B$ and $e \notin C$, then $e \in A - B$ and $e \in D - C$. It follows that $F(e) = \emptyset$, $L(e) = \emptyset$ and hence $T_1(e) = F(e) = \emptyset = L(e) = T_2(e)$.

For all $e \in A \cup C - B \cup D$, we have $e \in A \cup C$, $e \notin B$ and $e \notin D$.

(a) If $e \in A$ and $e \in C$, then $e \in A - B$ and $e \in C - D$. It follows that $F(e) = H(e) = \emptyset$, and hence $T_1(e) = F(e) \cup H(e) = \emptyset$.

(b) If $e \in A$ and $e \notin C$, then $e \in A - B$. It follows that $F(e) = \emptyset$, and hence $T_1(e) = F(e) = \emptyset$.

(c) If $e \notin A$ and $e \in C$, then $e \in C - D$. It follows that $H(e) = \emptyset$, and hence $T_1(e) = H(e) = \emptyset$.

For all $e \in B \cup D - A \cup C$, $T_2(e) = \emptyset$ can be proved similarly.

Hence $(F, A) \widetilde{\cup} (H, C) \approx_S (G, B) \widetilde{\cup} (L, D)$. □

The following example shows that $\approx_S$ is not a congruence relation with respect to $\cup_{\Re}$, or $\sqcap_{\varepsilon}$.

Example 29. Let $U = \{x_1, x_2, x_3, x_4\}$ be the universe and $E = \{e_1, e_2, e_3, e_4\}$ the set of parameters. Suppose that (F, A), (G, B), (H, C) and (L, D) are soft sets over U defined by:

(1) $A = \{e_1, e_2, e_3\}$, $F(e_1) = \{x_1, x_2\}$, $F(e_2) = \{x_2, x_3, x_4\}$, $F(e_3) = \emptyset$.
(2) $B = \{e_1, e_2, e_4\}$, $G(e_1) = \{x_1, x_2\}$, $G(e_2) = \{x_2, x_3, x_4\}$, $G(e_4) = \emptyset$.
(3) $C = \{e_1, e_2\}$, $H(e_1) = \emptyset$, $H(e_2) = \{x_2, x_3\}$.
(4) $D = \{e_2, e_3\}$, $L(e_2) = \{x_2, x_3\}$, $L(e_3) = \emptyset$.

It is trivial to verify that $(F, A) \approx_S (G, B)$, $(H, C) \approx_S (L, D)$. On the other hand, $(F, A) \cup_{\Re} (H, C) \approx_S (G, B) \cup_{\Re} (L, D)$ does not hold and neither does $(F, A) \sqcap_{\varepsilon} (H, C) \approx_S (G, B) \sqcap_{\varepsilon} (L, D)$.

Let $(F, A)_{\approx_S} = \{(G, B);\ (G, B) \approx_S (F, A)\}$ be the congruence class including (F, A) and

$$S(U, E)/\approx_S = \{(F, A)_{\approx_S};\ (F, A) \in S(U, E)\}.$$

We define operations $\widetilde{\cup}_S$ and $\Cap_S$ on $S(U, E)/\approx_S$ as follows:

$$(F, A)_{\approx_S} \widetilde{\cup}_S (G, B)_{\approx_S} = ((F, A)\, \widetilde{\cup}\, (G, B))_{\approx_S},$$
$$(F, A)_{\approx_S} \Cap_S (G, B)_{\approx_S} = ((F, A) \Cap (G, B))_{\approx_S}.$$

These two operations are well defined by Theorem 28.

Definition 30. We call $(S(U, E)/\approx_S, \widetilde{\cup}_S, \Cap_S)$ the soft quotient algebra (with respect to $\approx_S$) over the universe U and the parameter set E.

Theorem 31. *The soft quotient algebra $(S(U, E)/\approx_S, \widetilde{\cup}_S, \Cap_S)$ is a distributive lattice.*

Proof. We demonstrate that the distributive law hold in $S(U, E)/\approx_S$. In fact, for all $(F, A), (G, B), (H, C) \in S(U, E)$,

$$\begin{aligned}(F, A)_{\approx_S} \widetilde{\cup}_S ((G, B)_{\approx_S} \Cap_S (H, C)_{\approx_S}) &= (F, A)_{\approx_S} \widetilde{\cup}_S ((G, B) \Cap (H, C))_{\approx_S} \\ &= ((F, A)\, \widetilde{\cup}\, ((G, B) \Cap (H, C)))_{\approx_S} = (((F, A)\, \widetilde{\cup}\, (G, B)) \Cap ((F, A)\, \widetilde{\cup}\, (H, C)))_{\approx_S} \\ &= ((F, A)_{\approx_S} \widetilde{\cup}_S (G, B)_{\approx_S}) \Cap_S ((F, A)_{\approx_S} \widetilde{\cup}_S (H, C)_{\approx_S}).\end{aligned}$$

By using similar techniques, we can prove that $\widetilde{\cup}_S$ and $\Cap_S$ are idempotent, associative and commutative. Furthermore, the absorption law with respect to $\widetilde{\cup}_S$ and $\Cap_S$ holds. Hence, we conclude that $(S(U, E)/\approx_S, \widetilde{\cup}_S, \Cap_S)$ is a distributive lattice as required. $\square$

5. The soft equality relation $\approx^S$

In this section, we discuss the soft equality $\approx^S$ and establish the soft quotient algebra with respect to operations $\cup_{\Re}$ and $\sqcap_{\varepsilon}$.

Definition 32. Let $(F, A), (G, B) \in S(U, E)$. $(F, A) \approx^S (G, B)$ if for all $e \in A \cup B$, $e \in A \cap B$ implies $F(e) = G(e)$, $e \in A - B$ implies $F(e) = U$, and $e \in B - A$ implies $G(e) = U$.

Theorem 33. *Let (F, A) and (G, B) be two soft sets over the universe U. Then $(F, A) \approx^S (G, B)$ if and only if $(F, A)^r \approx_S (G, B)^r$.*

Proof. Suppose that $(F, A) \approx^S (G, B)$. For all $e \in A \cap B$, by definition, we have $F(e) = G(e)$, and hence $F^r(e) = U - F(e) = G^r(e)$. For all $e \in A - B$, we have $F(e) = U$ and hence $F^r(e) = \emptyset$. Similarly, $G^r(e) = \emptyset$ for all $e \in B - A$. We conclude that $(F, A)^r \approx_S (G, B)^r$ as required.

Conversely, suppose that $(F, A)^r \approx_S (G, B)^r$. For all $e \in A \cap B$, we have $F^r(e) = G^r(e)$, and hence $F(e) = G(e)$. For all $e \in A - B$, we have $F^r(e) = \emptyset$ and hence $F(e) = U$. Similarly, $G(e) = U$ for all $e \in B - A$. Hence $(F, A) \approx^S (G, B)$. $\square$

Theorem 34. *Let (F, A) and (G, B) be two soft sets over the universe U.*

(1) $(F, A) \approx^S (G, B)$ *if and only if* $(F, A)\, \widetilde{\cup}\, (G, B) \approx^S (F, A) \Cap (G, B)$.
(2) $(F, A) \approx^S (G, B)$ *if and only if* $(F, A) \cup_{\Re} (G, B) \approx^S (F, A) \sqcap_{\varepsilon} (G, B)$.

Theorem 35. *Let (F, A), (G, B) be two soft sets over the universe U and $(F, A) \approx^S (G, B)$. Then*

(1) $(F, A)\, \widetilde{\cup}\, (G, B) = (F, A) \sqcap_{\varepsilon} (G, B)$,
(2) $(F, A) \cup_{\Re} (G, B) = (F, A) \Cap (G, B)$.

The proofs of these two theorems are similar to those of Theorems 24 and 25.

Theorem 36. *$\approx^S$ is an equivalence relation on $S(U, E)$.*

Proof. It is trivial to verify that $\approx_S$ is reflexive and symmetric.

Suppose that $(F, A) \approx^S (G, B)$ and $(G, B) \approx^S (H, C)$. It follows that $(F, A)^r \approx_S (G, B)^r$, $(G, B)^r \approx_S (H, C)^r$ and hence $(F, A)^r \approx_S (H, C)^r$. Consequently, $(F, A) = ((F, A)^r)^r \approx^S ((H, C)^r)^r = (G, B)$. We conclude that $\approx^S$ is a transitive relation as required. $\square$

The following type of De Morgan's law hold in soft set theory.

Theorem 37. *Let (F, A) and (G, B) be two soft sets over the same universe U. Then*

(1) $((F, A)\, \widetilde{\cup}\, (G, B))^r = (F, A)^r \sqcap_{\varepsilon} (G, B)^r$,
(2) $((F, A) \sqcap_{\varepsilon} (G, B))^r = (F, A)^r\, \widetilde{\cup}\, (G, B)^r$.

Proof. (1) Let $(F,A)\mathbin{\widetilde{\cup}}(G,B)=(H,A\cup B)$, $(F,A)^r\sqcap_\varepsilon(G,B)^r=(K,A\cup B)$. It follows that $((F,A)\mathbin{\widetilde{\cup}}(G,B))^r=(H^r,A\cup B)$. For all $e\in A\cup B$,

(a) if $e\in A-B$, then $H^r(e)=U-H(e)=U-F(e)=K(e)$;
(b) if $e\in B-A$, then $H^r(e)=U-H(e)=U-G(e)=K(e)$;
(c) if $e\in A\cap B$, then $H^r(e)=U-H(e)=U-F(e)\cup G(e)=(U-F(e))\cap(U-G(e))=K(e)$.
Hence $((F,A)\mathbin{\widetilde{\cup}}(G,B))^r=(F,A)^r\sqcap_\varepsilon(G,B)^r$.

(2) By $((F,A)^r)^r=(F,A)$ and (1);

$$((F,A)^r\mathbin{\widetilde{\cup}}(G,B)^r)^r=((F,A)^r)^r\sqcap_\varepsilon((G,B)^r)^r=(F,A)\sqcap_\varepsilon(G,B).$$

It follows that

$$((F,A)\sqcap_\varepsilon(G,B))^r=(((F,A)^r\mathbin{\widetilde{\cup}}(G,B)^r)^r)^r=(F,A)^r\mathbin{\widetilde{\cup}}(G,B)^r.\quad\square$$

Theorem 38. *$\approx^S$ is a congruence relation with respect to operations $\cup_{\Re}$ and $\sqcap_\varepsilon$. That is to say, if $(F,A)\approx^S(G,B)$, $(H,C)\approx^S(L,D)$, then*

(1) $(F,A)\cup_{\Re}(H,C)\approx^S(G,B)\cup_{\Re}(L,D)$,
(2) $(F,A)\sqcap_\varepsilon(H,C)\approx^S(G,B)\sqcap_\varepsilon(L,D)$.

Proof. (1) By $(F,A)\approx^S(G,B)$ and $(H,C)\approx^S(L,D)$, it follows that $(F,A)^r\approx_S(G,B)^r$ and $(H,C)^r\approx_S(L,D)^r$. So, by Theorem 28(1), we have $(F,A)^r\Cap(H,C)^r\approx_S(G,B)^r\Cap(L,D)^r$. Consequently, by Theorem 10(1), we have

$$(F,A)\cup_{\Re}(H,C)=((F,A)^r\Cap(H,C)^r)^r\approx^S((G,B)^r\Cap(L,D)^r)^r=(G,B)\cup_{\Re}(L,D).$$

(2) By $(F,A)\approx^S(G,B)$ and $(H,C)\approx^S(L,D)$, it follows that $(F,A)^r\approx_S(G,B)^r$ and $(H,C)^r\approx_S(L,D)^r$. Hence, by Theorem 28(2), we have $(F,A)^r\mathbin{\widetilde{\cup}}(H,C)^r\approx_S(G,B)^r\mathbin{\widetilde{\cup}}(L,D)^r$. Consequently, by Theorem 37(2), we have

$$(F,A)\sqcap_\varepsilon(H,C)=((F,A)^r\mathbin{\widetilde{\cup}}(H,C)^r)^r\approx^S((G,B)^r\mathbin{\widetilde{\cup}}(L,D)^r)^r=(G,B)\sqcap_\varepsilon(L,D).\quad\square$$

Let $(F,A)_{\approx^S}=\{(G,B);\,(G,B)\approx^S(F,A)\}$ be the congruence class (with respect to $\approx^S$) including (F,A) and

$$S(U,E)/\approx^S=\{(F,A)_{\approx^S};\,(F,A)\in S(U,E)\}.$$

We define operations $\cup^S_{\Re}$ and $\sqcap^S_\varepsilon$ on $S(U,E)/\approx^S$ as follows:

$$(F,A)_{\approx^S}\cup^S_{\Re}(G,B)_{\approx^S}=((F,A)\cup_{\Re}(G,B))_{\approx^S},$$
$$(F,A)_{\approx^S}\sqcap^S_\varepsilon(G,B)_{\approx^S}=((F,A)\sqcap_\varepsilon(G,B))_{\approx^S}.$$

These two operations are well defined by Theorem 38.

Definition 39. We call $(S(U,E)/\approx^S,\cup^S_{\Re},\sqcap^S_\varepsilon)$ the soft quotient algebra (with respect to $\approx^S$) over the universe U and the parameter set E.

Theorem 40. *The soft quotient algebra $(S(U,E)/\approx^S,\cup^S_{\Re},\sqcap^S_\varepsilon)$ is a distributive lattice.*

The proof is similar to that of Theorem 31.

6. Conclusions

This paper deals with the algebraic structure of soft sets. The concept of soft equality is introduced and some related properties are derived. It is proved that soft equality is a congruence relation with respect to some operations on soft sets and the soft quotient algebra is established. Based on these results, we can further probe the applications of soft sets. The quotient structure of soft sets is another important and interesting issue to be addressed.

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