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Gröbner bases of syzygies and Stanley depth

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ABSTRACT

Let $F_\bullet$ be any free resolution of a finitely generated $\mathbb{Z}^n$-graded module over the polynomial ring $K[x_1,\dots,x_n]$. We show that for a suitable term order on $F_\bullet$, then for $0 \leqslant p < n$ the initial module of the $p+1$'th syzygy module $Z_p \subseteq F_p$ is generated by terms $m_i e_i$ where the m_i are monomials in $K[x_{p+1},\dots,x_n]$. Also for a large class of free resolutions $F_\bullet$, encompassing Eliahou–Kervaire resolutions, we show that a Gröbner basis for Z_p is given by the boundaries of generators of F_{p+1}.

We apply the above to give lower bounds for the Stanley depth of the $p+1$'th syzygy module Z_p, in particular showing it is greater or equal to $p+1$. We also show that if I is any squarefree ideal in $K[x_1,\dots,x_n]$, the Stanley depth of I is at least of order $\sqrt{2n}$.

Introduction

Let K be a field and $S = K[x_1,\dots,x_n]$ the polynomial ring in n variables over K. We study Gröbner bases of syzygies of finitely generated $\mathbb{Z}^n$-graded modules over S, and apply this to give lower bounds for the Stanley depth of syzygy modules.

Fix any monomial order $<$ on S and let F be a free $\mathbb{Z}^n$-graded S-module with a homogeneous basis $\mathcal{F} = e_1,\dots,e_m$. We define a monomial order on F by setting $ue_i > ve_j$ if $i < j$, or $i = j$ and $u > v$, where u and v are monomials of S. If M is a $\mathbb{Z}^n$-graded submodule of F, a basic observation is that the initial module $\mathrm{in}(M)$ does not depend on the monomial order $<$ on S but only on the basis $\mathcal{F}$. Therefore we denote the initial module of M with respect to this monomial order by $\mathrm{in}_{\mathcal{F}}(M)$. We have $\mathrm{in}_{\mathcal{F}}(M) = \bigoplus_{j=1}^m I_j e_j$, where each I_j is a monomial ideal.

We call the basis $\mathcal{F}$ of F *lex-refined*, if $\deg(e_1) \geqslant \deg(e_2) \geqslant \cdots \geqslant \deg(e_m)$ in the lexicographical order. Here $\deg(m)$ denotes the $\mathbb{Z}^n$-degree of a homogeneous element m of a $\mathbb{Z}^n$-graded module.

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Recall that the lexicographic order on $\mathbb{Z}^n$ is defined as follows: let $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$. Then $\mathbf{a} > \mathbf{b}$ if for some integer k we have $a_1 = b_1, \dots, a_k = b_k, a_{k+1} > b_{k+1}$.

Our first main result, Theorem 1.1, shows that the initial modules of syzygy modules, when choosing a lex-refined basis, have a simple and natural property : let M be a finitely generated $\mathbb{Z}^n$-graded module with free resolution $\cdots \to F_1 \to F_0 \to M \to 0$. For $0 \leqslant p < n$ let $Z_p \subseteq F_p$ be the $p+1$'th syzygy module. Then the initial module $\mathrm{in}_{\mathcal{F}}(Z_p)$ is $\bigoplus_{j=1}^m I_j e_j$, where the minimal set of monomial generators of each I_j belongs to $K[x_{p+1}, \dots, x_n]$.

This theorem may remind the reader of a well-known result of F.-O. Schreyer, see Section 15.5 of [5], who showed that for any finitely generated module M one can find a free resolution and suitable monomial orders on the free modules of the resolution such that the initial modules of the syzygies enjoy the same nice property as described above. The point here is that no assumption is made on the $\mathbb{Z}^n$-graded resolution on M. In particular, the theorem is valid for the graded minimal free resolution of M.

In general of course it is not so easy to compute the initial module of a syzygy module in a free resolution $F_\bullet$ of a module M. But for certain classes of resolutions this may be done in a pleasant way. We say that the resolution has *boundary Gröbner bases* if for each p there exists a basis $\mathcal{F}_p$ of F_p such that $\mathrm{in}_{\mathcal{F}_p}(Z_p(F_\bullet))$ is generated by the initial terms of $\partial_{p+1}(e_i)$ where $\partial_\bullet$ denotes the differential of $F_\bullet$ and e_i ranges over $\mathcal{F}_{p+1}$. If $F_\bullet$ has such bases, then the initial modules of the syzygies can easily be read off from the matrices describing $\partial_\bullet$ with respect to these bases. We show that the Taylor resolution as well as the Eliahou–Kervaire resolution has boundary Gröbner bases.

We then apply the first result on syzygies to give lower bounds for the Stanley depth of syzygies. A *Stanley decomposition* of a finitely generated $\mathbb{Z}^n$-graded S-module M is a direct sum decomposition $M = \bigoplus_{i=1}^m u_i K[Z_i]$ of M as a $\mathbb{Z}^n$-graded K-vector space, where each u_i is a homogeneous element of M, $K[Z_i]$ is a polynomial ring in a set of variables $Z_i \subset \{x_1, \dots, x_n\}$, and each $u_i K[Z_i]$ is a free $K[Z_i]$-submodule of M. The minimum of the numbers $|Z_i|$ is called the Stanley depth of this decomposition. The *Stanley depth* of M, denoted $\operatorname{sdepth} M$, is the maximal Stanley depth of a Stanley decomposition of M. In his paper [10] Stanley conjectured that $\operatorname{sdepth} M \geqslant \operatorname{depth} M$. This conjecture is widely open. In the papers listed in the references in this paper and the references therein, the reader can inform himself about the present status of the conjecture.

Naively one could expect, that like for the ordinary depth, the Stanley depth of the first syzygy module $Z_0(M)$ of a finitely generated $\mathbb{Z}^n$-graded module M is one more than that of M, as along as M is not free. This of course would immediately imply Stanley's conjecture. However this is not the case. For example, if we let $\mathfrak{m} = (x_1, \dots, x_n)$ be the graded maximal ideal of S. Then $\operatorname{sdepth} S/\mathfrak{m} = 0$, while $\operatorname{sdepth} \mathfrak{m} = \lceil n/2 \rceil$, as shown in [1]. Nevertheless it might be true that one always has $\operatorname{sdepth} Z_0(M) \geqslant \operatorname{sdepth} M$. But at the moment even the inequality $\operatorname{sdepth} I \geqslant \operatorname{sdepth} S/I$ for a monomial ideal I is unknown. However as one of the main results of this paper, we show in Theorem 2.2 that if M is a finitely generated $\mathbb{Z}^n$-graded module, then for $p \geqslant 0$ the $p+1$'th syzygy module of M with respect to any (not necessarily minimal) $\mathbb{Z}^n$-graded free resolution of M, is either free or has Stanley depth at least $p+1$.

One problem in proving such a result as stated in Theorem 2.2 is the fact that at present no method is known to compute the Stanley depth of a $\mathbb{Z}^n$-graded module in a finite number of steps. So far this can be done only for modules of the form I/J where $J \subset I \subset S$ are monomial ideals, see [8]. It seems not even to be known that for a monomial ideal $I \subset S$ one has $\operatorname{sdepth} I \oplus S = \operatorname{sdepth} I$, as one would expect. The only method known to get a lower bound for the Stanley depth of a $\mathbb{Z}^n$-graded module M is to find a suitable filtration of the module whose factors are of the form I or S/I where I is a monomial ideal. The Stanley depth of M is then just the minimum of the Stanley depth of the factors of the filtration. This enables us to give lower bounds for the Stanley depth of a syzygy module by using that if the initial module $\mathrm{in}_{\mathcal{F}}(Z_p)$ is $\bigoplus_{j=1}^m I_j e_j$, then the monomial ideals I_j are the factors of a suitable filtration of Z_p. Therefore the Stanley depth of M is greater or equal to the minimum of the Stanley depths of the I_j.

We also apply the second result on Gröbner basis of syzygies to show that when $I \subset S$ is a monomial complete intersection minimally generated by m elements, then for $p \geqslant 0$ the $p+1$'th syzygy module of S/I is either free or has Stanley depth at least $n - \lfloor \frac{m-p}{2} \rfloor$, see Proposition 2.4. This indicates that our general lower bounds for the Stanley depth of syzygy modules are far from being optimal.

Somewhat independently of the above, we show that if I is a squarefree ideal over the polynomial ring in n variables, the Stanley depth of M is at least of order $2\sqrt{n}$, Theorem 3.4. This is quite in contrast to ordinary monomial ideals. In fact it is known by Cimpoeaş [3] that sufficiently high powers of $\mathfrak{m}$ have Stanley depth 1. The proof of Theorem 3.4 is based on a construction of interval partitions [7] by M. Keller et al. which is further refined M. Ge et al. in [6]. Applying this we give lower bounds, Theorem 2.6, for the Stanley depth of syzygy modules of a squarefree submodule of a free module. This bound is considerably better than what we have for arbitrary $\mathbb{Z}^n$-graded submodules.

The organisation of the paper is as follows. In Section 1 we consider resolutions of finitely generated $\mathbb{Z}^n$-graded S-modules and initial modules of their syzygy modules determined by choosing ordered multihomogeneous bases for the terms in the resolutions. We first prove that for lex-refined orders, the generators of the initial module of the $p+1$'th syzygy module does not involve the first p variables. We then give classes of resolutions which have boundary Gröbner bases. In Section 2 we give the lower bounds on Stanley depth of syzygies. These are a consequence of the results in Section 1, and in the squarefree case, a consequence of the result in the next Section 3. In this last section, we show that the Stanley depth of any squarefree monomial ideal in n variables is at least of order $2\sqrt{n}$.

1. Gröbner basis of syzygies of multigraded modules

We consider term orders on $\mathbb{Z}^n$-graded free modules over a polynomial ring in n variables which are determined by fixing a multihomogeneous basis $e_1, \dots, e_m$ of the free module and comparing terms ue_i and ve_j by first comparing their basis elements. For such term orderings the initial term of any multihomogeneous element will be determined solely by the ordering of the e_i's, and so also the initial module of any multihomogeneous submodule.

A natural ordering of the e_i's is by lexicographic ordering of their multidegrees. We show then that the syzygies of a free resolution of a finitely generated $\mathbb{Z}^n$-graded module have the nice and natural property that for each successive syzygy module we miss an extra variable in their generating set.

Let K be a field, $S = K[x_1, \dots, x_n]$ the polynomial ring over K in n indeterminates, M a finitely generated $\mathbb{Z}^n$-graded S-module and

$$F_\bullet : \cdots \to F_p \to F_{p-1} \to \cdots \to F_0 \to M \to 0 \tag{1}$$

a $\mathbb{Z}^n$-graded (not necessarily minimal) free S-resolution of M with all F_i finitely generated. Let $Z_p \subset F_p$ be the $p+1$'th syzygy module of M (with respect to this resolution). We are interested in the initial module of the syzygy module Z_p with respect to a monomial order on F_p. In this paper we restrict our attention to monomial orders of the following type: we denote by $\mathrm{Mon}(S)$ the set of monomials of S, fix a multihomogeneous basis $\mathcal{F} = e_1, \dots, e_m$ of F_p and a monomial order $<$ on S, and define a monomial order on F_p by setting

$$ue_i > ve_j, \quad \text{if } i < j, \text{or } i = j \text{ and } u > v. \tag{2}$$

Here $u, v \in \mathrm{Mon}(S)$. We denote by $\mathrm{in}_<(Z_p)$ the monomial submodule of F_p which is generated by all the initial monomials of elements of Z_p. Notice that

$$\mathrm{in}_<(Z_p) = \bigoplus_{j=1}^{m} I_j e_j, \tag{3}$$

where each I_j is a monomial ideal.

Since Z_p is $\mathbb{Z}^n$-graded, $\mathrm{in}_<(Z_p)$ is generated by the initial monomials of multihomogeneous elements of Z_p. Let $z = \sum_{i=1}^m f_i e_i$ be a multihomogeneous element in Z_p. Then each f_i is a term $a_i u_i$ with $a_i \in K$ and u_i a monomial. Thus $\mathrm{in}_<(z) = u_j e_j$, where j is the smallest number i such that $a_i \neq 0$. This consideration shows that for the above monomial order, the initial module of a syzygy module of a finitely generated $\mathbb{Z}^n$-graded module *depends only on the given basis* not on the chosen monomial order on S. Thus we write $\mathrm{in}_{\mathcal{F}}(Z_p)$ to denote the initial module of Z_p with respect to this monomial order induced by $\mathcal{F}$.

In general, for a given resolution $F_\bullet$ there are several equally natural choices of multihomogeneous bases for the F_p. Here we will choose for each F_p a basis compatible with the lexicographical order of the multidegrees of the basis elements. We call such a basis of F_p *lex-refined*. Thus a basis $\mathcal{F} = e_1, \dots, e_m$ of F_p of multihomogeneous elements is lex-refined, if $\deg(e_1) \geqslant \deg(e_2) \geqslant \cdots \geqslant \deg(e_m)$ in the lexicographical order.

Theorem 1.1. *Suppose M is a finitely generated $\mathbb{Z}^n$-graded S-module, and $F_\bullet$ a resolution of M. Let $0 \leqslant p < n$ be an integer, and $\mathcal{F}$ a lex-refined basis of F_p. Then the initial module of the $p+1$'th syzygy module, $\mathrm{in}_{\mathcal{F}}(Z_p)$, is $\bigoplus_{j=1}^m I_j e_j$, where the minimal set of monomial generators of each ideal I_j belongs to $K[x_{p+1}, \dots, x_n]$.*

This theorem is an immediate consequence of the following

Proposition 1.2. *Let $\varphi : F \to G$ be a homomorphism of finitely generated $\mathbb{Z}^n$-graded free S-modules with $M = \operatorname{Im} \varphi$ and $N = \operatorname{Ker} \varphi$. Let $\mathcal{G} = e'_1, \dots, e'_d$ be a lex-refined basis of G and $\mathcal{F} = e_1, \dots, e_c$ a lex-refined basis of F, and assume that for some $p \leqslant n$, $\mathrm{in}_{\mathcal{G}}(M)$ is generated by monomials not divisible by $x_1, \dots, x_{p-1}$. Then $\mathrm{in}_{\mathcal{F}}(N)$ is generated by monomials not divisible by $x_1, \dots, x_p$.*

Proof. 1. Let $s \in N$ be a multihomogeneous element, and let r be least index such that x_r divides $\mathrm{in}(s)$. To demonstrate the desired property of $\mathrm{in}_{\mathcal{F}}(N)$, it will be sufficient to show that there exists an element $\tilde{s} \in N$ such that

(1) $\mathrm{in}(s) = \mathrm{in}(\tilde{s})$;
(2) x_r divides $\tilde{s}$, if $r \leqslant p$.

Indeed, if (1) and (2) are satisfied, then $t = \tilde{s}/x_r \in N$ and $\mathrm{in}(s) = x_r \,\mathrm{in}(t)$. Thus $\mathrm{in}(s)$ is not a generator of $\mathrm{in}_{\mathcal{F}}(N)$.

2. In order to show the existence of $\tilde{s}$ with these properties, we write $s = s' + s''$, where s' is the sum of all terms of s which are not divisible by any of the variables $x_1, \dots, x_{r-1}$, and s'' the sum of the other terms in s.

Let $s = \sum_{i=1}^c a_i u_i e_i$ with $a_i \in K$ and $u_i \in \mathrm{Mon}(S)$. For simplicity we may assume that $\mathrm{in}(s) = u_1 e_1$. Then, since s is multihomogeneous and since $\mathcal{F}$ is a lex-refined basis of F, it follows that $u_i \leqslant u_j$ in the lexicographical order for all $i \leqslant j$ in the support of s. Since $\mathrm{in}(s') = \mathrm{in}(s) = u_1 e_1$, it follows that $u_1 \leqslant u_i$ for all $i \in \mathrm{supp}(s')$. The monomial u_1 is divisible by x_r, and no u_i with $i \in \mathrm{supp}(s')$ is divisible by any x_j for $j < r$. Hence the inequality $u_1 \leqslant u_i$ implies that x_r divides all u_i with $i \in \mathrm{supp}(s')$. In other words, x_r divides s'.

3. Since $s \in N$, it follows that $\varphi(s') = -\varphi(s'')$. We denote this element by z. Since $z = \varphi(s')$, we see that x_r divides z, and since $z = -\varphi(s'')$ it follows that each of the terms of z is divisible by at least one x_i with $i < r$.

Let $\mathrm{in}(z/x_r) = w e'_k$. Since $\deg(z/x_r) = \deg(s) - \varepsilon_r$, where ε_r is the rth canonical basis vector of $\mathbb{Z}^n$, it follows that $\deg(e'_k)_j \leqslant \deg(s)_j$ for $j = 1, \dots, r-1$, and since each term of z/x_r is divisible at least one x_j with $j < r$, we conclude that $\deg(e'_k)_j < \deg(s)_j$ for at least one $j < r$.

4. Now we may write $z/x_r = \sum b_i v_i g_i$ with $b_i \in K$ and $v_i \in \mathrm{Mon}(S)$, where the g_i form a reduced Gröbner basis of M (with respect to the monomial order induced by $\mathcal{G}$) and with the additional property that $\mathrm{in}(v_i g_i) \leqslant \mathrm{in}(z/x_r) = w e'_k$ for all i. Let $\mathrm{in}(g_i) = w_{j_i} e'_{j_i}$. Then $j_i \geqslant k$, and hence $\deg(e'_{j_i}) \leqslant \deg(e'_k)$ in the lexicographic order. Thus for all i it follows that $\deg(e'_{j_i})_j \leqslant \deg(s)_j$ for $j = 1, \dots, r-1$ with strict inequality for at least one j.

5. According to our hypothesis, we may assume that none of the variables $x_1, \dots, x_{p-1}$ divides any of the w_{j_i}. Thus, together with what we have shown in the last paragraph we see that for all i we

have that $\deg(g_i)_j \leqslant \deg(s)_j$ for $j = 1, \ldots, r-1$, but with strict inequality for at least one $j < r$. We may now lift each g_i to an element f_i of F with the same multidegree, and set

$$s''' = \sum b_i v_i f_i \quad \text{and} \quad \tilde{s} = s' - x_r s'''.$$

Obviously, x_r divides $\tilde{s}$ and $\tilde{s} \in N$. We claim that $\mathrm{in}(x_r s''') < \mathrm{in}(s')$. Since $\mathrm{in}(s) = \mathrm{in}(s')$, the claim will then imply that $\mathrm{in}(\tilde{s}) = \mathrm{in}(s)$, as desired.

In order to prove the claim, assume to the contrary that $\mathrm{in}(x_r s''') \geqslant \mathrm{in}(s') = u_1 e_1$. Since $\mathrm{in}(x_r v_i f_i) \geqslant \mathrm{in}(x_r s''')$ for some i, it follows for this index i that $\mathrm{in}(x_r v_i f_i) \geqslant u_1 e_1$ which implies that $\mathrm{in}(f_i) = v e_1$ for some $v \in \mathrm{Mon}(S)$. In particular, $\deg(e_1)_j \leqslant \deg(f_i)_j$ for all j. Since $\deg(f_i) = \deg(g_i)$, and since there exists $j < r$ such that $\deg(g_i)_j < \deg(s)_j = \deg(u_1 e_1)_j$, it follows that $\deg(e_1)_j < \deg(u_1 e_1)_j$ for some $j < r$. This implies that x_j divides u_1, a contradiction. $\square$

Let F be a finitely generated $\mathbb{Z}^n$-graded S-module with multigraded basis $\mathcal{F} = e_1, \ldots, e_m$, and M a $\mathbb{Z}^n$-graded submodule of F. In general it is not easy to compute $\mathrm{in}_{\mathcal{F}}(M)$ explicitly. In the following we describe resolutions where for suitable bases the initial modules of the syzygies can be simply determined. These bases are however not necessarily lex-refined.

Definition 1.3. Let $F_\bullet$ be the resolution (1) with differential $\partial_\bullet$. It has *boundary Gröbner bases* if for each $p \geqslant 0$ there exists a basis $\mathcal{F}_p$ of F_p such that

$$\mathrm{in}_{\mathcal{F}_p}\big(Z_p(F_\bullet)\big) = \big(\mathrm{in}_{\mathcal{F}_p}\big(\partial_{p+1}(e_i)\big) : e_i \in \mathcal{F}_{p+1}\big).$$

Resolutions with boundary Gröbner bases have the pleasant property that the initial modules of the syzygies can be immediately read off from the matrices describing the differential maps with respect to these bases.

The resolutions we have in mind arise as iterated mapping cones. So let $I \subset S$ be a monomial ideal with monomial generators $u_1, \ldots, u_m$. The iterated mapping cone resolution is constructed inductively by using induction on the number of generators. For $m = 1$ it is just the complex $F_\bullet^{(1)} : S(-\mathbf{a}_1) \to S$, where $\mathbf{a}_1$ is the multidegree of u_1 and the differential of the complex is multiplication by u_1. Let I_j be the ideal generated by $u_1, \ldots, u_j$, and suppose for some $j < m$ we have already constructed the resolution $F_\bullet^{(j)}$ of S/I_j. Consider the exact sequence

$$0 \to I_{j+1}/I_j \to S/I_j \to S/I_{j+1} \to 0.$$

Observe that

$$I_{j+1}/I_j \cong \big(S/\big(I_j : (u_{j+1})\big)\big)(-\mathbf{a}_{j+1}), \tag{4}$$

where $\mathbf{a}_{j+1} = \deg u_{j+1}$. Let $G_\bullet^{(j)}$ be a $\mathbb{Z}^n$-graded free S-resolution of this cyclic module and $\varphi^{(j)} : G_\bullet^{(j)} \to F_\bullet^{(j)}$ a complex homomorphism of $\mathbb{Z}^n$-graded complexes extending the inclusion map $I_{j+1}/I_j \to S/I_j$. Then we define $F_\bullet^{(j+1)}$ as the mapping cone of $\varphi^{(j)}$.

The free resolution obtained by iterated mapping cones is not at all unique. It depends on the choice of the free resolutions $G_\bullet^{(j)}$ as well as on the complex homomorphisms $\varphi^{(j)}$.

Here are a few prominent examples of resolutions which arise as iterated mapping cones.

Examples 1.4. (a) The Taylor complex (cf. [5]) is an iterated mapping cone. Let $u_1, \ldots, u_m$ be a sequence of monomials. Assuming the Taylor complex for any sequence of monomials of length $m-1$ is already constructed, one constructs the Taylor complex for $u_1, \ldots, u_m$ by choosing for $F_\bullet^{(m-1)}$ the Taylor complex of the sequence $u_1, \ldots, u_{m-1}$, and for $G_\bullet^{(m-1)}$ one takes the Taylor complex for the sequence

$$u_1/\gcd(u_1, u_m), \ldots, u_{m-1}/\gcd(u_{m-1}, u_m)$$

which is a system of generators of $(u_1, \dots, u_{m-1}) : (u_m)$. The map φ_{m-1} can be defined in a canonical way.

The Taylor complex provides a $\mathbb{Z}^n$-graded free S-resolution of $S/(u_1, \dots, u_m)$ (which in general is not minimal). In case $u_1, \dots, u_m$ is a regular sequence, the Taylor complex coincides with the Koszul complex of this sequence and is minimal.

(b) A monomial ideal $I \subset S$ is said to have linear quotients, if I is generated by homogeneous polynomials $u_1, \dots, u_m$ with $\deg u_1 \leqslant \deg u_2 \leqslant \dots \leqslant \deg u_m$ such that each of the colon ideals $L_j = (u_1, \dots, u_j) : (u_{j+1})$ is generated by a subset of the variables. For such an ideal we can use in the construction of the iterated mapping cone for each j the Koszul complex as $G_\bullet^{(j)}$ to resolve S/L_j. Considering the degrees of the resolutions at each step we see that $\varphi_j(G_\bullet^{(j)}) \subset \mathfrak{m}F_\bullet^{(j)}$, so that in this case the iterated mapping cone provides a minimal free $\mathbb{Z}^n$-graded resolution of S/I.

An important special case is that of a stable ideal. Recall that a monomial ideal I is called stable if for all monomial $u \in I$ the monomial $x_j(u/x_{m(u)}) \in I$ for all $j < m(u)$. Here $m(u)$ is the largest index with the property that $x_{m(u)}$ divides u. Let the minimal set of monomial generators $u_1, \dots, u_m$ of I be ordered in such a way that for $i < j$ either $\deg u_i < \deg u_j$, or $\deg u_i = \deg u_j$ and $u_j < u_i$ in the lexicographic order. Then with respect to this sequence of generators, I has linear quotients. The corresponding iterated mapping cone yields the so-called Eliahou–Kervaire resolution of S/I, provided the complex homomorphisms φ_j at each step are chosen properly.

The next lemma is a direct consequence of the following two observations:

(i) Let F be a finitely generated $\mathbb{Z}^n$-graded S-module with multigraded basis $\mathcal{F} = e_1, \dots, e_m$, and M a $\mathbb{Z}^n$-graded submodule of F with $\operatorname{in}_{\mathcal{F}}(M) = \bigoplus_{j=1}^m I_j e_j$. For $j = 1, \dots, m$ set $F\langle j\rangle = \bigoplus_{i=j}^m Se_i$ and let $F\langle m+1\rangle = 0$. Then for the factors of the induced filtration

$$M = M \cap F\langle 1\rangle \supset M \cap F\langle 2\rangle \supset \dots \supset M \cap F\langle m\rangle \supset M \cap F\langle m+1\rangle = (0) \tag{5}$$

we have

$$M \cap F\langle j\rangle / M \cap F\langle j+1\rangle \cong I_j\bigl(-\deg(e_j)\bigr). \tag{6}$$

(ii) Let M be a finitely generated $\mathbb{Z}^n$-graded module and $N \subset M$ a $\mathbb{Z}^n$-graded submodule of M, $F_\bullet$ a $\mathbb{Z}^n$-graded free S-resolution of M, $G_\bullet$ a $\mathbb{Z}^n$-graded free S-resolution of N and $\varphi_\bullet : G_\bullet \to F_\bullet$ a $\mathbb{Z}^n$-graded complex homomorphism which extends the inclusion map $N \to M$. The mapping cone $C_\bullet$ of $\varphi_\bullet$ is a $\mathbb{Z}^n$-graded free resolution of M/N. For the syzygies of these complexes we have for all $i \geqslant 0$ the following exact sequences

$$0 \to Z_i(F_\bullet) \to Z_i(C_\bullet) \to Z_{i-1}(G_\bullet) \to 0, \tag{7}$$

where we set $Z_{-1}(G_\bullet) = N$.

Lemma 1.5. *With the notation introduced in (ii), let $\mathcal{G} = g_1, \dots, g_r$ be a basis of G_{i-1}, $\mathcal{F} = f_1, \dots, f_s$ a basis of F_i and $\mathcal{C}$ the basis of C_i which is obtained by composing $\mathcal{G}$ with $\mathcal{F}$, that is, $\mathcal{C} = g_1, \dots, g_r, f_1, \dots, f_s$. Then*

$$\operatorname{in}_{\mathcal{C}} Z_i(C_\bullet) = \operatorname{in}_{\mathcal{G}} Z_{i-1}(G_\bullet) \oplus \operatorname{in}_{\mathcal{F}} Z_i(F_\bullet).$$

In the following corollary we consider for the syzygy modules appearing in the preceding lemma, Gröbner bases with respect to monomial orders induced by the given bases.

Corollary 1.6. *A Gröbner basis of $Z_i(C_\bullet)$ is obtained by composing a Gröbner basis of $Z_i(F_\bullet)$ with the preimages of the elements of a Gröbner bases of $Z_{i-1}(G_\bullet)$ with respect to the epimorphism $Z_i(C_\bullet) \to Z_{i-1}(G_\bullet)$. In particular if $F_\bullet$ and $G_\bullet$ have boundary Gröbner bases, then $C_\bullet$ has boundary Gröbner bases.*

Now Corollary 1.6 yields

Corollary 1.7. *Let the resolution $F_\bullet$ of $S/(u_1, \dots, u_n)$ be an iterated mapping cone by resolutions $G^{(j)}$ of* (4) *for $j = 1, \dots, m-1$. If each $G^{(j)}$ has boundary Gröbner bases, then $F_\bullet$ has boundary Gröbner bases.*

As an application of these observations we obtain

Proposition 1.8. *The Taylor complex and the iterated mapping cone of an ideal with linear quotients have boundary Gröbner bases. In particular, the Koszul complex attached to a regular sequence as well as the Eliahou–Kervaire resolution for stable ideals have boundary Gröbner bases.*

Proof. Let $F_\bullet$ be the Taylor complex on a sequence of monomials of length m. The complexes $G_\bullet^{(j)}$ are Taylor complexes on sequences of length $m-1$. Thus by using induction on m, it follows from Corollary 1.7 that $F_\bullet$ has boundary Gröbner bases. On the other hand, if $F_\bullet$ is an iterated mapping cone for an ideal with linear quotients, then all $G_\bullet^{(j)}$ are Koszul complexes, which are special Taylor complexes, so that all $G_\bullet^{(j)}$ have boundary Gröbner bases. Hence the desired result follows again by applying Corollary 1.7. $\square$

To be more concrete let $T_\bullet$ be the Taylor complex attached with the sequence $u_1, \dots, u_m$. For each p, T_p has the following basis: $e_F = e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_p}$ with $F = 1 \leqslant i_1 < i_2 < \dots < i_p \leqslant m$, and the differential is given by

$$\partial_p(e_F) = \sum_{j=1}^{p} (-1)^{j+1} \frac{u_F}{u_{F \setminus \{i_j\}}} e_{F \setminus \{i_j\}},$$

where for any subset $G \subset [n]$ we let u_G be the least common multiple of the monomials u_i with $i \in G$. If we order the basis elements iteratively as described in Lemma 1.5, then $e_m > e_{m-1} > \dots > e_1$ and more generally $e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_p} > e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_p}$ if for some k one has $i_p = j_p, \dots, i_{k+1} = j_{k+1}$ and $i_k > j_k$. With this order, the elements e_F with $F \subset [n]$ and $|F| = p$ form boundary Gröbner bases. Thus we obtain

$$\operatorname{in}\bigl(Z_p(T_\bullet)\bigr) = \bigoplus_{F \subset [m],\, |F| = p} I_F e_F,$$

with

$$I_F = \left(\frac{u_{F \cup \{i\}}}{u_F}\right)_{i \in [m],\, i < \min(F)}. \tag{8}$$

2. Stanley depth of syzygies

In this section we consider lower bounds for the Stanley depth of syzygies. First we give a lower bound in general for syzygies of $\mathbb{Z}^n$-graded submodules of free modules. Then, in the case of squarefree modules we can give a considerably better bound. The lower bounds have a form which is natural for syzygies. They essentially increase by one for each successive syzygy. However the actual behavior of Stanley depth of successive syzygy modules is probably far from the lower bound.

Our tool to obtain lower bounds for the Stanley depth is the following simple observation.

Lemma 2.1. *Let F be a finitely generated $\mathbb{Z}^n$-graded S-module with multigraded basis $\mathcal{F} = e_1, \dots, e_m$, M a $\mathbb{Z}^n$-graded submodule of F, and $\operatorname{in}_{\mathcal{F}}(M) = \bigoplus_{j=1}^m I_j e_j$. Then*

$$\operatorname{sdepth} M \geqslant \min\{\operatorname{sdepth} I_1, \dots, \operatorname{sdepth} I_m\}.$$

Proof. Let $M = M_0 \supset M_1 \supset \cdots \supset M_r = 0$ be any $\mathbb{Z}^n$-graded filtration of M. Since, as we already observed, for a short exact sequence of $\mathbb{Z}^n$-graded modules

$$0 \to M' \to M \to M'' \to 0$$

one has $\operatorname{sdepth} M \geqslant \min\{\operatorname{sdepth} M', \operatorname{sdepth} M''\}$, we deduce that

$$\operatorname{sdepth} M \geqslant \max_i \{\operatorname{sdepth} M_i/M_{i+1}\}.$$

Applying this general fact to the filtration (5) induced by $\mathcal{F}$, the result follows from (6). □

Now we present our main results concerning Stanley depth of syzygies.

Theorem 2.2. *Let M be a finitely generated $\mathbb{Z}^n$-graded module, and let $F_\bullet$ be a free resolution as in* (1)*. Then for $p \geqslant 0$ the $p+1$'th syzygy module Z_p has Stanley depth greater than or equal to $p+1$, or it is a free module.*

Proof of Theorem 2.2. Let $\mathcal{F}$ be a lex-refined basis for F_p. If $p \geqslant n$ then Z_p is free, so suppose $p < n$. By Theorem 1.1, $\operatorname{in}_{\mathcal{F}}(Z_p) = \bigoplus_{j=1}^m I_j e_j$, where the minimal set of monomial generators of each of the monomial ideals I_j belongs to $K[x_{p+1}, \ldots, x_n]$. But then $\operatorname{sdepth} I_j \geqslant p+1$. In fact, Cimpoeaş [4, Corollary 1.5] showed that the Stanley depth of any finitely generated $\mathbb{Z}^n$-graded torsionfree S-module is at least 1. Hence the asserted inequality for the Stanley depth of I_j follows from [8, Lemma 3.6]. Now the desired inequalities for the Stanley depths of the syzygy modules follow from (2.1). □

Remark 2.3. In general the lower bound p is probably far too small. W. Bruns, C. Krattenthaler, and J. Uliczka consider in [2] syzygies of the Koszul complex. They conjecture that the last half of these syzygies always have Stanley depth equal to $n-1$.

On the other hand, the bound is sharp for the second syzygy module of any monomial ideal $I \subset S = K[x_1, x_2, x_3]$ with $\dim S/I = 0$. Indeed, the predicted Stanley depth of $Z_1(I)$ is at least 2. It cannot be three, because otherwise $Z_1(I)$ would be free.

That indeed our lower bound for the Stanley depth is in general far too small can be seen in the following special case.

Proposition 2.4. *Let $I \subset S$ be a monomial complete intersection minimally generated by m elements, and Z_p the $p+1$'th syzygy module of S/I. Then either Z_p is free or*

$$\operatorname{sdepth} Z_p \geqslant n - \left\lfloor \frac{m-p}{2} \right\rfloor.$$

Proof. Let $u_1, \ldots, u_m$ be the regular sequence generating I. The Taylor complex associated with this sequence, which in this case is the Koszul complex, is a minimal free resolution of S/I. With the notation of (8) we have $I_F = (u_i)_{i \in [m],\, i < \min(F)}$, so that $\operatorname{sdepth} Z_p \geqslant \min\{\operatorname{sdepth}(u_i\colon\ i < \min(F))\}$. By a result of Shen [9] one has $\operatorname{sdepth} J = n - \lfloor m/2 \rfloor$ for a monomial ideal $J \subset K[x_1, \ldots, x_n]$ generated by a regular sequence of length m. This yields the desired conclusion. □

In the case where M is a squarefree ideal or more generally a squarefree module, we also get better bounds. Recall that a $\mathbb{Z}^n$-graded S-module M is *squarefree* (defined by K. Yanagawa [11]), if it fulfils the following.

- $M_{\mathbf{a}}$ is nonzero only if $\mathbf{a} \in \mathbb{N}^n$.

- When $\mathbf{a}$ is in $\mathbb{N}^n$, with nonzero i'th coordinate, and ε_i is the i'th unit coordinate vector, the multiplication map

$$M_{\mathbf{a}} \xrightarrow{\cdot x_i} M_{\mathbf{a}+\varepsilon_i}$$

is an isomorphism of vector spaces.

Squarefree modules form an abelian category with squarefree projective covers. In particular kernels of morphisms of squarefree modules are squarefree, and so syzygies modules in a squarefree resolution of a squarefree module, are squarefree.

Lemma 2.5. *Let M be a finitely generated squarefree module. Then for any term ordering on F, the initial module* $\mathrm{in}(M)$ *is a squarefree module.*

Proof. Let $g_1, \dots, g_p$ be a basis for $M_{\mathbf{a}}$, such that the $\mathrm{in}(g_r) = u_r e_{i_r}$ are a basis for $\mathrm{in}(M)_{\mathbf{a}}$, and suppose $a_i \neq 0$. Then $x_i g_1, \dots, x_i g_p$ are a basis for $M_{\mathbf{a}+\varepsilon_i}$, and their initial terms are the $x_i u_r e_{i_r}$ which form a basis for $\mathrm{in}(M_{\mathbf{a}+\varepsilon_i})$. $\square$

In the proof of Theorem 2.2 we use that a lower bound for the Stanley depth of a monomial ideal is 1. In the last section we show that for squarefree ideals there is a considerably better lower bound for the Stanley depth than 1. Using this we may sharpen Theorem 2.2 when the resolution is of a squarefree module.

Theorem 2.6. *Let M be a finitely generated squarefree module, with d the smallest degree of a generator of M. Let s and s' be the largest integers such that*

$$(2s+1)(s+1) \leqslant n+1-d-p, \quad 2s'\bigl(s'+1\bigr) \leqslant n+1-d-p.$$

Then for $p \geqslant 0$ the $p+1$'th syzygy module in a squarefree resolution of M is either free, or it has Stanley depth greater or equal to the maximum of $2s+1+d+p$ and $2s'+d+p$.

Proof. For $p \geqslant 0$ use a term order as in (2) on the p'th term F_p in the resolution. We get

$$\mathrm{in}(Z_p) = \bigoplus_{j=1}^{m} I_j e_j, \tag{9}$$

where I_j is a squarefree monomial ideal. Since $\mathrm{in}(Z_p)$ is a squarefree module, each generator $g_{ji} e_j$ of $I_j e_j$ is a squarefree term.

Suppose first that the resolution is minimal. Then the total degree of e_j is at least $d+p$ where $d+p \leqslant n$. Hence the generators of I_j involve no more than $n-d-p$ variables, corresponding to the coordinates of the multidegree of e_j which are zero. The result then follows from Lemma 2.1 and Theorem 3.4.

In the case that the resolution is not necessarily minimal, the syzygy Z'_p of such a resolution differ from the syzygy Z_p of the minimal free resolution by a free summand, that is, $Z'_p = Z_p \oplus F$, so either Z'_p is free or it has Stanley depth greater or equal to the Stanley depth of Z_p. $\square$

3. Stanley depth of squarefree monomial ideals

In this section we show that the Stanley depth of any squarefree monomial ideal in n variables, is bounded below by a bound of order $\sqrt{2n}$. This is quite in contrast to ordinary depth where for instance the maximal ideal $(x_1, \dots, x_n)$ has depth one.

Our argument is based on a construction of interval partitions [7] by M. Keller et al. which is further refined M. Ge, J. Lin, and Y.-H. Shen in [6]. The argument is an application of Proposition 3.5 in [6].

We recall the construction of [7]. Let $[n] = \{1, 2, \dots, n\}$. For subsets A and B of $[n]$, the interval $[A, B]$ consists of the subsets C of $[n]$ such that $A \subseteq C \subseteq B$. We think of the elements of $[n]$ arranged clockwise around the circle and for i, j in $[n]$ let the block $[i, j]$ be the set of points starting with i, going clockwise, and ending with j. Now given $A \subseteq [n]$, and a real number $\delta \geqslant 1$, called a *density*, the *block structure of* A with respect to δ is a partition of the elements of $[n]$ into connected blocks $B_1, G_1, B_2, G_2, \dots, B_p, G_p$ fulfilling the following.

- The first (going clockwise) element of b_i of B_i is in A.
- Each G_i is disjoint from A.
- For each B_i we have

$$\delta \cdot |A \cap B_i| - 1 < |B_i| \leqslant \delta \cdot |A \cap B_i|.$$

- For each y such that $[b_i, y]$ is a proper subset of B_i, we have

$$\big|[b_i, y]\big| + 1 \leqslant \delta \cdot \big|[b_i, y] \cap A\big|.$$

For $1 \leqslant \delta \leqslant \frac{(n-1)}{|A|}$ the block structure for a subset A exists and is unique by Lemma 2.7 in [7]. Let $\mathscr{G}_\delta$ be the union $G_1 \cup \cdots \cup G_p$. We then define the set $f_\delta(A)$ to be $A \cup \mathscr{G}_\delta(A)$. The intervals we shall study will now be of the form $[A, f_\delta(A)]$ or closely related.

For certain values of n and cardinalities of A, the intervals fulfil some very nice properties. The following are basic facts from [7]. It is a synopsis of Lemma 3.1, Lemma 3.2 and Lemma 3.5 there.

Lemma 3.1. *Let $n = as + a + s$.*

1. *If $A \subseteq [n]$ is an a-set, then $f_{s+1}(A)$ is and $(a + s)$-set.*
2. *The intervals $[A, f_{s+1}(A)]$ are disjoint when A varies over the a-sets.*

We are interested in getting disjoint intervals, but we need a way to adopt the above lemma to the case of arbitrary n, and to be able to vary a and s. The following still fixes s and a but allows n to be arbitrary above a bound. It is Proposition 3.3 in [6].

Proposition 3.2. *Let $n \geqslant as + a + s$ and $A \subseteq [n]$ an a-set. Consider $\tilde{A} = A \cup \{n + 1, \dots, n + n - a\}$ as a subset of $[ns + n + s]$.*

1. *The intersection $f_{s+1}(\tilde{A}) \cap [n]$ is an $(a + s)$-set.*
2. *The intervals $[A, f_{s+1}(\tilde{A}) \cap [n]]$ are disjoint as A varies over the a-sets.*

Finally we need to be more flexible with a and s, and still have disjoint intervals. The following is Proposition 3.5 of [6] specialized to the case when $d = 1, d + q = a$ and $d + l = b$.

Proposition 3.3. *Let A and B be subsets of $[n]$ of cardinalities $a \leqslant b$. Suppose $s' \leqslant s$ are non-negative integers such that*

$$n + 1 \geqslant (b + 1)\big(s' + 1\big) \geqslant (a + 1)(s + 1).$$

Consider $\tilde{A}$ as a subset of $[ns + n + s]$ and $\tilde{B}$ as a subset of $[ns' + n + s']$. Then if B is not in $[A, f_{s+1}(\tilde{A}) \cap [n]]$, this interval is disjoint from the interval $[B, f_{s'+1}(\tilde{B}) \cap [n]]$.

We are now ready to prove our theorem.

Theorem 3.4. *Let s and s' be the largest integers such that*

$$n+1 \geqslant (2s+1)(s+1), \quad n+1 \geqslant 2s'\big(s'+1\big).$$

Then the Stanley depth of any squarefree monomial ideal in n variables is greater or equal to the maximum of $2s+1$ and $2s'$. Explicitly these lower bounds are

$$2\left\lfloor \frac{\sqrt{2n+2.25}+0.5}{2} \right\rfloor - 1, \quad 2\left\lfloor \frac{\sqrt{2n+3}-1}{2} \right\rfloor.$$

Remark 3.5. For $n=5$, the above bound says that the Stanley depth is greater than or equal to 3 which is best possible, since this is the Stanley depth of the maximal ideal $(x_1,\dots,x_5)$.

Remark 3.6. In [4] it is shown that the Stanley depth of the squarefree Veronese ideal generated by squarefree monomials of degree d has Stanley depth less or equal to $\lfloor \frac{n+1}{d+1} \rfloor + d - 1$. With $d+1$ approximately $\sqrt{n+1}$ this is approximately $2\sqrt{n+1}-2$. Thus our lower bound is right up to a constant.

Proof of Theorem 3.4. The squarefree ideal I corresponds to an order filter P_I of the poset consisting of subsets of $[n]$, by taking supports of the squarefree monomials in I. By J. Herzog et al. [8], if we have a partition $\mathscr{P}$ of P_I, the Stanley depth of I is greater or equal to the minimum cardinality of any subset B of $[n]$ such that $[A,B]$ is an interval in $\mathscr{P}$. We shall therefore construct a suitable partition to give the lower bound.

Given positive integers n, r and s with $r > s$ and

$$(n+1) \geqslant (r+1)(s+1).$$

Define the sequence $\sigma : [r] \to \mathbb{Z}$ by

$$\sigma(i) = \begin{cases} s, & i \geqslant s+1, \\ s+k, & i = s+1-k \leqslant s+1. \end{cases}$$

Note that if $u \leqslant v$ then $(u+1)v \geqslant u(v+1)$. Therefore the expression $(i+1)(\sigma(i)+1)$ weakly decreases as i decreases, enabling us to apply Proposition 3.3.

We now construct a partition $\mathscr{P}$ of P_I as follows. Let $\mathscr{P}_1$ consist of all intervals

$$\big[\{i\}, f_{\sigma(1)+1}\big(\widetilde{\{i\}}\big) \cap [n]\big]$$

where $\{i\}$ is in P_I. Let $\mathscr{P}_2$ consist of $\mathscr{P}_1$ together with all intervals

$$\big[\{i_1,i_2\}, f_{\sigma(2)+1}\big(\widetilde{\{i_1,i_2\}}\big) \cap [n]\big]$$

where $\{i_1,i_2\}$ is in P_I but not in any of the intervals in $\mathscr{P}_1$. By Proposition 3.3, $\mathscr{P}_2$ will consist of disjoint intervals. Having constructed $\mathscr{P}_{a-1}$ we construct $\mathscr{P}_a$ by adding to $\mathscr{P}_{a-1}$ all intervals $[A, f_{\sigma(a)+1}(\tilde{A}) \cap [n]]$ where A is an a-set in P_I not in any interval of $\mathscr{P}_{a-1}$. Having reached $\mathscr{P}_r$ we obtain $\mathscr{P}$ by adding all trivial intervals $[B,B]$ where B is in P_I but not in any of the intervals in $\mathscr{P}_r$. Note that each such B has cardinality greater or equal to $r+1$.

The Stanley depth of the partition $\mathscr{P}$ will be the smallest of the numbers $r+1$ and $i+\sigma(i)$ for $i=1,\dots,r$, which is the minimum of $r+1$ and $2s+1$. Choose $r=2s$ and s to be the largest integer such that $n+1 \geqslant (2s+1)(s+1)$. Then the Stanley depth of $\mathscr{P}$ is $2s+1$. We get

$$
\begin{aligned}
(2s+1)(s+1) &\leqslant n+1, \\
4s^2+6s+2 &\leqslant 2n+2, \\
(2s+1.5)^2 &\leqslant 2n+2.25, \\
2s+2 &\leqslant \sqrt{2n+2.25}+0.5
\end{aligned}
$$

which gives

$$
\begin{aligned}
s &\leqslant \left\lfloor \frac{\sqrt{2n+2.25}+0.5}{2} \right\rfloor - 1, \\
2s+1 &\leqslant 2\left\lfloor \frac{\sqrt{2n+2.25}+0.5}{2} \right\rfloor - 1.
\end{aligned}
$$

The largest value of $2s+1$ is then given by the right side above. Alternatively (when an even number gives a better lower bound), we may choose $r = 2s-1$ (when $s \geqslant 2$ so $r > s$), and s to be the largest integer such that $n+1 \geqslant 2s(s+1)$. Then the Stanley depth of $\mathscr{P}$ is $2s$. The explicit formula in this case is derived as above. That this also works when $s = 1$ is easily verified. $\square$

References

[1] C. Biró, D. Howard, M. Keller, W. Trotter, S. Young, Interval partitions and Stanley depth, J. Combin. Theory Ser. A 117 (4) (2010) 475–482.
[2] W. Bruns, C. Krattenthaler, J. Ulizka, Stanley depth and Hilbert decompositions in the Koszul complex, arXiv:0909.0686.
[3] M. Cimpoeaş, Some remarks on the Stanley depth for multigraded modules, Le Matematiche 63 (2008) 165–171.
[4] M. Cimpoeaş, Stanley depth of squarefree Veronese ideals, arXiv:0907.1232.
[5] D. Eisenbud, Commutative Algebra with a View Towards Algebraic Geometry, Grad. Texts in Math., Springer-Verlag, 1994.
[6] M. Ge, J. Lin, Y.-H. Shen, On a conjecture of Stanley depth of squarefree Veronese ideals, arXiv:0911.5458.
[7] M. Keller, Y.-H. Shen, N. Streib, S. Young, On the Stanley depth of squarefree Veronese ideals, arXiv:0910.4645.
[8] J. Herzog, M. Vladiou, X. Zheng, How to compute the Stanley depth of a monomial ideal, J. Algebra 322 (2009) 3151–3169.
[9] Y. Shen, Stanley depth of complete intersection monomial ideals and upper-discrete partitions, J. Algebra 321 (2009) 1285–1292.
[10] R. Stanley, Linear Diophantine equations and local cohomology, Invent. Math. 68 (1982) 175–193.
[11] K. Yanagawa, Alexander duality for Stanley–Reisner rings and squarefree $\mathbb{N}^n$-modules, J. Algebra 225 (2000) 630–645.