Topological stability for conservative systems

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A B S T R A C T

We prove that the C^1 interior of the set of all topologically stable C^1 incompressible flows is contained in the set of Anosov incompressible flows. Moreover, we obtain an analogous result for the discrete-time case.

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1. Introduction: basic definitions and statement of the results

We consider an n-dimensional ($n \geqslant 3$) closed and connected C^∞ Riemannian manifold M endowed with a volume-form ω. Let μ denote the measure associated to ω, that we call Lebesgue measure, and let $d(\cdot,\cdot)$ denote the metric induced by the Riemannian structure. We say that a vector field $X : M \to TM$ is *divergence-free* if $\nabla \cdot X = 0$ or equivalently if the measure μ is invariant for the associated flow, $X^t : M \to M$, $t \in \mathbb{R}$. In this case we say that the flow is *incompressible* or *volume-preserving*. We denote by $\mathfrak{X}^r_\mu(M)$ ($r \geqslant 1$) the space of C^r divergence-free vector fields on M topologized with the usual C^r Whitney topology.

Given $X \in \mathfrak{X}^1_\mu(M)$ let $Sing(X)$ denote the set of *singularities* of X and $\mathcal{R} = \mathcal{R}(M) := M \setminus Sing(X)$ the set of *regular* points. We say that $\sigma \in Sing(X)$ is a *hyperbolic singularity* if DX_σ has no eigenvalue with null real part.

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Let $X, Y \in \mathfrak{X}^1_\mu(M)$; Y is *semiconjugated* to X if the flow associated to Y is semiconjugated to the one of X, i.e., there exists a continuous and onto map $h : M \to M$ and a continuous real map $\tau : M \times \mathbb{R} \to \mathbb{R}$ such that

a) for any $x \in M$, $\tau_x : \mathbb{R} \to \mathbb{R}$ is an orientation preserving homeomorphism where $\tau(x, 0) = 0$ and
b) for all $x \in M$ and $t \in \mathbb{R}$ we have $h(Y^t(x)) = X^{\tau(x,t)}(h(x))$.

We say that $X \in \mathfrak{X}^1_\mu(M)$ is *topologically stable* in $\mathfrak{X}^1_\mu(M)$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $Y \in \mathfrak{X}^1_\mu(M)$ δ–C^0-close to X, there exists a semiconjugacy from Y to X, i.e., there exists $h : M \to M$ and $\tau : M \times \mathbb{R} \to \mathbb{R}$ satisfying a) and b) above, and $d(h(x), x) < \epsilon$ for all $x \in M$. It is worth to emphasize that our definition of topological stability is restricted to the conservative setting and not to the broader space of dissipative flows. We denote the set of topological stable incompressible flows by $\mathcal{TS}_\mu(M)$.

A vector field is said to be *Anosov* if the tangent bundle TM splits into three continuous DX^t-invariant nontrivial subbundles $E^0 \oplus E^1 \oplus E^2$ where E^0 is the flow direction, the sub-bundle E^2 is uniformly contracted and the sub-bundle E^1 is uniformly expanded by DX^t for $t > 0$. Of course that, for an Anosov flow, we have $Sing(X) = \emptyset$ which follows from the fact that the dimensions of the subbundles are constant on the entire manifold.

For analogous definitions in the volume-preserving diffeomorphisms context see Section 4.

The concept of topological stability was first introduced by Walters. In [20] he proved that Anosov diffeomorphisms are topologically stable. In [18] Nitecki proved that topological stability was a necessary condition to get Axiom A plus strong transversality. Later, in [15], Robinson proved that Morse–Smale flows are topologically stable. In the mid 1980's, Hurley obtained necessary conditions for topological stability (see [10–12]). It was proved by Moriyasu [16] that the C^1-interior of the set of all topologically stable diffeomorphisms is characterized as the set of all C^1-structurally stable diffeomorphisms. About ten years ago it was proved by Moriyasu, Sakai and Sumi (see [17]) that, if X is a vector field in the C^1 interior of the set of topologically stable vector fields (in the broader space of dissipative flows) then X satisfies the Axiom A and the strong transversality properties. Our main result (Theorem 1) is a generalization of the main theorem in [17] for divergence-free vector fields. Although this result is expectable, its proof uses perturbations techniques that only recently become available.

Given a set $A \subset \mathfrak{X}^1_\mu(M)$ let $int_{C^1}(A)$ denote the interior of A in $\mathfrak{X}^1_\mu(M)$ with respect to the C^1-topology.

Theorem 1. *If $X \in int_{C^1}(\mathcal{TS}_\mu(M))$ then X is Anosov.*

Due to well-know results (see [14]) about the restrictions of the existence of manifolds supporting Anosov flows, we obtain that, for general manifolds, the C^1-interior of topological stable incompressible flows must be empty. It is also interesting to note that, in the geodesic flow context, Anosov systems are not necessarily topological stable (see [19]).

Given $x \in \mathcal{R}(X)$ we consider its normal bundle $N_x = X(x)^\perp \subset T_x M$ and define the *linear Poincaré flow* by $P^t_X(x) := \Pi_{X^t(x)} \circ DX^t_x$ where $\Pi_{X^t(x)} : T_{X^t(x)} M \to N_{X^t(x)}$ is the projection along the direction of $X(X^t(x))$. Let $\Lambda \subset R$ be an X^t-invariant and compact set. We say that Λ is a *hyperbolic* set for the linear Poincaré flow if there exist constants $\lambda \in (0, 1)$, $C > 0$ and a splitting $N_x = N^u_x \oplus N^s_x$ such that for all $x \in \Lambda$ we have:

$$\left\| \left(P^t_X(x)^{-1} \right) \big|_{N^u_x} \right\| < C\lambda^t \quad \text{and} \quad \left\| P^t_X(x) \big|_{N^s_x} \right\| < C\lambda^t.$$

We say that $X \in \mathcal{G}^1_\mu(M)$ if there exists a neighborhood $\mathcal{V}$ of X in $\mathfrak{X}^1_\mu(M)$ such that any $Y \in \mathcal{V}$, has all the closed orbits and all the singularities of hyperbolic-type.

The following result, which is important *per se*, will be crucial to obtain Theorem 1.

Theorem 2. *If $X \in int_{C^1}(\mathcal{TS}_\mu(M))$ then $X \in \mathcal{G}^1_\mu(M)$.*

The next result was proved recently by Ferreira [9] and is a generalization of a three-dimensional theorem by the authors [6].

Theorem 3. *(See Ferreira [9].) If $X \in \mathcal{G}_\mu^1(M)$ then X is Anosov.*

Theorem 1 is a direct consequence of Theorem 2 and Theorem 3 and for this reason we just have to concentrate on the proof of Theorem 2.

2. Perturbation lemmata

2.1. Perturbations near singularities

Some key results to perform perturbations in the conservative setting are available (see [2]). Nevertheless, neither [2, Theorem 3.1] nor [2, Theorem 3.2] are adequate to go on with the proof of our result. Therefore, we need to obtain an upgrade of these pasting lemmas and this is the content of Lemma 2.1. We believe that this lemma is interesting by itself and should be used in other situations where special C^1-perturbations of conservative vector fields are required. We also would like to thank Carlos Matheus for a valuable suggestion for the proof of it.

Lemma 2.1. *Let M be a compact and boundaryless Riemannian manifold of dimension $\geqslant 2$. Given $\epsilon > 0$, $X \in \mathfrak{X}_\mu^1(M)$, a compact $\mathcal{K} \subset M$ and an open neighborhood $\mathcal{U}$ of $\mathcal{K}$, there are $\delta > 0$ and an open set $\mathcal{K} \subset \mathcal{V} \subset \mathcal{U}$ such that, if $Y \in \mathfrak{X}_\mu^2(M)$ is δ–C^1-close to X in $\mathcal{U}$, then there exists $Z \in \mathfrak{X}_\mu^1(M)$ satisfying*

  a) $Z = Y$ in $\mathcal{V}$,
  b) Z is ϵ–C^1-close to X, and
  c) $Z = X$ outside $\mathcal{U}$.

Proof. We consider $\mathcal{V} \supset \mathcal{K}$ such that $\partial \mathcal{V}$ is C^∞, $\overline{\mathcal{V}} \subset \mathcal{U}$ and $\{\mathcal{U}, int(M \setminus \overline{\mathcal{V}})\}$ is an open covering of M. Let $\alpha : M \to [0, 1]$ be a C^∞ function such that $\alpha = 1$ in $\mathcal{V}$, $\alpha = 0$ outside $\mathcal{U}$ and $|\nabla \alpha| \leqslant K$, where K is a positive constant depending only on $\mathcal{U}$ and $\mathcal{V}$. Define

$$Z_0(x) := \alpha(x)Y(x) + \big(1 - \alpha(x)\big)X(x), \tag{2.1}$$

where $Y \in \mathfrak{X}_\mu^2(M)$ is δ–C^1-close to X on a small open neighborhood $\mathcal{U}$ of $\mathcal{K}$, where $\delta > 0$ will be fixed in (2.3). We observe that $Z_0 = Y$ inside $\mathcal{K}$ and $Z_0 = X$ outside $\mathcal{U}$, and thus $\nabla \cdot Z_0 = 0$ in the closed set $\overline{\mathcal{V}} \cup (M \setminus \mathcal{U})$. However, although $\nabla \cdot Z_0$ is close to zero in $\mathcal{U} \setminus \mathcal{V}$, in general $\nabla \cdot Z_0 \neq 0$. Actually,

$$\nabla \cdot Z_0 = (\nabla \alpha) \cdot Y + \alpha(\nabla \cdot Y) - (\nabla \alpha) \cdot X + (1 - \alpha)\nabla \cdot Y$$

$$= (\nabla \alpha) \cdot Y - (\nabla \alpha) \cdot X = (\nabla \alpha) \cdot (Y - X),$$

and we have $|\nabla \cdot Z_0| < K\delta$.

Now we will make use of [8, Theorem 2] in order to obtain $Z_1 \in \mathfrak{X}_\mu^2(M)$ (supported in $\mathcal{U} \setminus \mathcal{V}$) such that $\nabla \cdot Z_1 = -\nabla \cdot Z_0$ and $Z_1 = 0$ in $\partial(\mathcal{U} \setminus \mathcal{V})$. For more details we refer the reader to [2, §3].

Finally, we define

$$Z := Z_0 + Z_1. \tag{2.2}$$

Of course that, by construction, $\nabla \cdot Z = 0$ and c) holds.

In $\mathcal{U} \setminus \mathcal{V}$ we have

$$\|Z - Y\|_{C^1} = \|Z_0 + Z_1 - Y\|_{C^1} \leqslant \|Z_1\|_{C^1} + \|Z_0 - Y\|_{C^1}$$
$$\leqslant C\|\nabla \cdot Z_1\|_{C^0} + \|Z_0 - Y\|_{C^1},$$

where $C > 0$ is a constant given in Dacorogna–Moser theorem (see [8, Theorem 2.3]) that depends only on $\mathcal{U}$.

Going back to the beginning of the proof, we take

$$\delta < \min\left\{\epsilon, \frac{\epsilon}{2CK}\right\}. \tag{2.3}$$

Now, using $|\nabla \cdot Z_0| < K\delta$ we get,

$$\|Z - Y\|_{C^1} \leqslant C\|\nabla \cdot Z_1\|_{C^0} + \|Z_0 - Y\|_{C^1} \leqslant CK\delta + \|\alpha Y + (1-\alpha)X - Y\|_{C^1}$$
$$\leqslant \frac{\epsilon}{2} + \|(1-\alpha)X - (1-\alpha)Y\|_{C^1} \leqslant \frac{\epsilon}{2} + |1-\alpha|\|X - Y\|_{C^1}$$
$$\leqslant \frac{\epsilon}{2} + \|X - Y\|_{C^1} \leqslant \frac{\epsilon}{2} + \delta < \epsilon. \qquad \square$$

The next lemma is the divergence-free vector fields version of [17, Lemma 1.1].

Lemma 2.2. *Let σ be a singularity of $X \in \mathfrak{X}^1_\mu(M)$. For any $\epsilon > 0$ there exist $\delta_0 > 0$ and $\epsilon_0 > 0$ such that if $Y_\delta : T_\sigma M \to T_\sigma M$ is a traceless linear map $\frac{\delta}{2}$–C^0-close to DX_σ (with $\delta < \delta_0$) then there exists $Z_\delta \in \mathfrak{X}^1_\mu(M)$, such that $Z_\delta = Y_\delta$ in $B_{\epsilon_0/4}(\sigma)$, Z_δ is ϵ–C^1-close to X and $Z_\delta = X$ outside the set $B_{\epsilon_0}(\sigma)$.*

Proof. Let (U, ϕ) be a conservative chart given by Moser's theorem [13] such that $\sigma \in U$ and $\phi(\sigma) = \mathbf{0}$. Now we will work on the euclidean space $\mathbb{R}^n$.

We fix $\delta \in (0, \delta_0)$ where $\delta_0 > 0$ will be defined in the sequel. For simplicity we assume that $Y_\delta : \mathbb{R}^n \to \mathbb{R}^n$ is written in the canonical coordinates. Now, we consider the divergence-free linear vector field in $\mathbb{R}^n$ defined (in the canonical coordinates) by

$$(\dot{x}_1, \ldots, \dot{x}_n) = Y_\delta(x_1, \ldots, x_n). \tag{2.4}$$

Now, let $\mathcal{K} = \overline{B_{\epsilon_0/4}(\mathbf{0})}$ and $\mathcal{U} = B_{\epsilon_0}(\mathbf{0})$. Since, by hypothesis, the map Y_δ is $\frac{\delta}{2}$–C^0-close to $DX_\mathbf{0}$, if we choose ϵ_0 very small, then X is $\frac{\delta_0}{2}$–C^1-close to $DX_\mathbf{0}$ when restricted to $\mathcal{U}$. Therefore Y_δ (defined in (2.4)) and X are δ_0–C^1-close in $\mathcal{U}$. This conditions gives ϵ_0 depending on δ_0. So, by Lemma 2.1, fixed ϵ, there exist δ_0 and there are an open set $\mathcal{V}$; $\mathcal{K} \subset \mathcal{V} \subset \mathcal{U}$ and $Z_\delta \in \mathfrak{X}^1_\mu(M)$ such that $Z_\delta = Y_\delta$ in $\mathcal{V}$, Z_δ is ϵ–C^1-close to X and $Z_\delta = X$ outside $\mathcal{U}$ for $\delta < \delta_0$. The lemma is proved. $\quad \square$

2.2. Perturbations near closed orbits

Lemma 2.3 below is a *Franks' lemma* for incompressible flows.

Define $\Gamma(p, \tau) = \{X^t(p); \ t \in [0, \tau]\}$. Let $V, \tilde{V} \subset N_p$, $\dim(V) = j$, $2 \leqslant j \leqslant n - 1$, and $N_p = V \oplus \tilde{V}$. A *one-parameter linear family* $\{A_t\}_{t \in \mathbb{R}}$ associated to $\Gamma(p, \tau)$ and V is defined as follows:

- $A_t : N_p \to N_p$ is a linear map, for all $t \in \mathbb{R}$,
- $A_t = id$, for all $t \leqslant 0$, and $A_t = A_\tau$, for all $t \geqslant \tau$,

- $A_t|_V \in sl(j, \mathbb{R})$, and $A_t|_{\tilde{V}} \equiv id$, $\forall t \in [0, \tau]$, in particular we have $\det(A_t) = 1$, for all $t \in \mathbb{R}$, and
- the family A_t is C^∞ on the parameter t.

Lemma 2.3. *(See [4, Lemma 3.2].) Given $\epsilon > 0$ and a vector field $X \in \mathfrak{X}^4_\mu(M)$ there exists $\theta_0 = \theta_0(\epsilon, X)$ such that $\forall \tau \in [1, 2]$, for any periodic point p of period greater than 2, for any sufficient small flowbox $\mathcal{T}$ of $\Gamma(p, \tau)$ and for any one-parameter linear family $\{A_t\}_{t\in[0,\tau]}$ such that $\|\dot{A}_t A_t^{-1}\| < \theta_0$, $\forall t \in [0, \tau]$, there exists $Y \in \mathfrak{X}^1_\mu(M)$ satisfying the following properties*

(A) *Y is ϵ–C^1-close to X;*
(B) *$Y^t(p) = X^t(p)$, for all $t \in \mathbb{R}$;*
(C) *$P^\tau_Y(p) = P^\tau_X(p) \circ A_\tau$, and*
(D) *$Y|_{\mathcal{T}^c} \equiv X|_{\mathcal{T}^c}$.*

3. Proof of Theorem 2

In order to go on with our proof we observe that the main steps are based on the arguments in [17, Section 2]. From Section 3.1 and Section 3.2 below it follows that any $X \in int_{C^1}(\mathcal{TS}_\mu(M))$ has all its singularities and closed orbits of hyperbolic-type. Therefore, $int_{C^1}(\mathcal{TS}_\mu(M)) \subset \mathcal{G}^1_\mu(M)$.

3.1. Singularities

We are going to prove that, if $X \in int_{C^1}(\mathcal{TS}_\mu(M))$, then any singularity of X is hyperbolic. By contradiction let us assume that there exists a non-hyperbolic singularity $\sigma \in Sing(X)$. According to Lemma 2.2 we consider a family of divergence-free vector fields $\{Z_\delta\}_{\delta \geqslant 0}$, where Z_δ is linear and $Z_0 = DX_\sigma$ in $B_{\frac{\epsilon_0}{4}}(\sigma)$, and δ is sufficiently small to assure that this family is contained in $int_{C^1}(\mathcal{TS}_\mu(M))$. We observe that we can chose δ' arbitrarily small such that σ is a hyperbolic singularity for $Z_{\delta'}$ and such that $Z_{\delta'}$ is semiconjugated to Z_0, that is, there exists a continuous and onto map $h: M \to M$ (arbitrarily close to the identity depending on δ') and a continuous real map $\tau: M \times \mathbb{R} \to \mathbb{R}$ such that for any $x \in M$, $\tau_x: \mathbb{R} \to \mathbb{R}$ is an orientation preserving homeomorphism where $\tau(x, 0) = 0$ and for all $x \in M$ and $t \in \mathbb{R}$ we have

$$h(Z^t_{\delta'}(x)) = Z^{\tau(x,t)}_0(h(x)). \tag{3.5}$$

As σ is non-hyperbolic for Z_0, there exists $z \in M$ such that

$$\sigma \notin \{B_\epsilon(Z^t_0(z)): t \in \mathbb{R}\} \quad \text{and} \quad \{B_\epsilon(Z^t_0(z)): t \in \mathbb{R}\} \subset B_{\frac{\epsilon_0}{8}}(\sigma)$$

and $\epsilon < \frac{\epsilon_0}{16}$. Let $w \in h^{-1}(z)$. From (3.5) we get that $h(Z^t_{\delta'}(w)) = Z^{\tau(w,t)}_0(z)$ and, since h is arbitrarily close to the identity, we obtain that

$$\{Z^t_{\delta'}(w): t \in \mathbb{R}\} \subset \{B_\epsilon(Z^t_0(z)): t \in \mathbb{R}\} \subset B_{\frac{\epsilon_0}{8}}(\sigma),$$

which is a contradiction because σ is a hyperbolic singularity of $Z_{\delta'}$ and, when restricted to $B_{\frac{\epsilon_0}{8}}(\sigma)$, the vector field Z_0 is linear.

3.2. Closed orbits

Fix $X \in int_{C^1}(\mathcal{TS}_\mu(M))$. Now we are going to prove that all the closed orbits of X are hyperbolic.

Assume that X has a non-hyperbolic closed orbit p of period $\pi(p)$. In order to proceed with the arguments in [17, Section 2] we need to C^1-approximate the vector field X by a new one which is *linear* in a neighborhood of the closed orbit p. To perform this task, in the conservative setting, it is required more differentiability of the vector field (cf. Lemma 2.3).

If X is of class C^∞, take $Z = X$, otherwise we use [21] in order to obtain a C^∞ incompressible vector field $Y \in int_{C^1}(\mathcal{TS}_\mu(M))$, arbitrarily C^1-close to X, and such that Y has a closed orbit[1] q, close to p, and with period $\pi(q)$ close to $\pi(p)$. If q is not hyperbolic take $Z = Y$. If q is hyperbolic for $P_Y^{\pi(q)}(q)$, then, since Y is C^1-arbitrarily close to X, the distance between the spectrum of $P_Y^{\pi(q)}(q)$ and $\mathbb{S}^1$ can be taken arbitrarily close to zero (weak hyperbolicity). Now, we are in position to apply Lemma 2.3 to obtain a new vector field $Z \in \mathfrak{X}_\mu^\infty(M) \cap int_{C^1}(\mathcal{TS}_\mu(M))$, C^1-close to Y and having a non-hyperbolic closed orbit.[2]

Now, we argue as in [5, §3] in order to obtain $L \in int_{C^1}(\mathcal{TS}_\mu(M))$ such that L is linear (equal to P_Z^t) in a neighborhood of the closed non-hyperbolic orbit, Γ.

Finally, we C^1-approximate L by $W \in int_{C^1}(\mathcal{TS}_\mu(M))$ such that Γ is hyperbolic (for W). This is a contradiction because W is semiconjugated to L, although there is an L^t-orbit (different from Γ) contained in a small neighborhood of Γ and the same cannot occur for W^t because Γ is a hyperbolic closed orbit for W^t.

4. The volume-preserving diffeomorphisms case

Let $\mathrm{Diff}_\mu^1(M)$ denote the set of volume-preserving (or conservative) diffeomorphisms defined on M, and consider this space endowed with the C^1 Whitney topology. In this section we assume that $\dim(M) \geqslant 2$. We say that a diffeomorphism f is *Anosov* if, there exist $\lambda \in (0, 1)$ and $C > 0$ such that the tangent vector bundle over M splits into two Df-invariant subbundles $TM = E^u \oplus E^s$, with $\|Df^n|_{E^s}\| \leqslant C\lambda^n$ and $\|Df^{-n}|_{E^u}\| \leqslant C\lambda^n$.

We say that $f \in \mathcal{F}_\mu^1(M)$ if there exists a neighborhood $\mathcal{V}$ of f in $\mathrm{Diff}_\mu^1(M)$ such that any $g \in \mathcal{V}$ has all the periodic orbits hyperbolic. In [1] Arbieto and Catalan proved the following result.

Theorem 4. *If* $f \in \mathcal{F}_\mu^1(M)$ *then* f *is Anosov.*

Given $f, g \in \mathrm{Diff}_\mu^1(M)$ we say that g is *semiconjugated* to f if there exists a continuous and onto map $h: M \to M$ such that for all $x \in M$ one has $h(g(x)) = f(h(x))$.

We say that f is *topologically stable* in $\mathrm{Diff}_\mu^1(M)$ if, for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $g \in \mathrm{Diff}_\mu^1(M)$ δ–C^0-close to f, there exists a semiconjugacy from g to f, i.e., there exists $h: M \to M$ satisfying $h(g(x)) = f(h(x))$ and $d(h(x), x) < \epsilon$, for all $x \in M$. Once again we emphasize that our definition of topological stability is restricted to the conservative setting and not to the broader space of dissipative diffeomorphisms. We denote the set of topological stable volume-preserving diffeomorphisms by $\mathbf{TS}_\mu(M)$.

In this section we obtain the discrete-time version of Theorem 1.

Theorem 5. *If* $f \in int_{C^1}(\mathbf{TS}_\mu(M))$ *then* f *is Anosov.*

The proof is similar to the one done in Section 3.2 and, for this reason, we present a brief highlight of it. As before, given $f \in int_{C^1}(\mathbf{TS}_\mu(M))$, we prove that all its periodic orbits are hyperbolic; from this it follows that $int_{C^1}(\mathbf{TS}_\mu(M)) \subset \mathcal{F}_\mu^1(M)$. Then, using Theorem 4, we obtain that any $f \in int_{C^1}(\mathbf{TS}_\mu(M))$ is Anosov.

Once again we assume, by contradiction, that there is some non-hyperbolic orbit. Now, to argue as in the flow case, we make use of the following two ingredients:

(1) a way to linearize the diffeomorphism in a neighborhood of a periodic point and
(2) a "perturbation of the derivative" result in the vein of Lemma 2.3.

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[1] Notice that q may not be the analytic continuation of p. This is precisely the case when 1 is an eigenvalue of $P_X^{\pi(p)}(p)$.
[2] Observe that if the vector field Y in Lemma 2.3 is of class C^∞, then the resulting vector field Z is also of class C^∞.

The item (2) is available in the literature (see [7, Proposition 7.4]).

For (1) we just have to approximate $f \in int_{C^1}(\mathbf{TS}_\mu(M))$ by a diffeomorphism $g \in int_{C^1}(\mathbf{TS}_\mu(M)) \cap$ Diff$_\mu^\infty(M)$ using a recent result by Avila [3], and then we use the Pasting lemma [2, Theorem 3.6] to obtain

$$h \in int_{C^1}\left(\mathbf{TS}_\mu(M)\right) \cap \text{Diff}_\mu^\infty(M),$$

such that $h = Dg_p$ in a neighborhood of the periodic orbit p. This is precisely what we need to obtain a contradiction as we did in Section 3.2.

Acknowledgments

References

[1] A. Arbieto, T. Catalan, Hyperbolicity in the volume preserving scenario, arXiv:1004.1664, preprint, 2010.
[2] A. Arbieto, C. Matheus, A pasting lemma and some applications for conservative systems, with an appendix by David Diica and Yakov Simpson-Weller, Ergodic Theory Dynam. Systems 27 (2007) 1399–1417.
[3] A. Avila, On the regularization of conservative maps, Acta Math. 205 (2010) 5–18.
[4] M. Bessa, J. Rocha, On C^1-robust transitivity of volume-preserving flows, J. Differential Equations 245 (11) (2008) 3127–3143.
[5] M. Bessa, J. Rocha, Homoclinic tangencies versus uniform hyperbolicity for conservative 3-flows, J. Differential Equations 247 (11) (2009) 2913–2923.
[6] M. Bessa, J. Rocha, Three-dimensional conservative star flows are Anosov, Discrete Contin. Dyn. Syst. Ser. A 26 (3) (2010) 839–846.
[7] C. Bonatti, L.J. Díaz, E. Pujals, A C^1-generic dichotomy for diffeomorphisms: Weak forms of hyperbolicity or infinitely many sinks or sources, Ann. of Math. 158 (2005) 355–418.
[8] B. Dacorogna, J. Moser, On a partial differential equation involving the Jacobian determinant, Ann. Inst. Henri Poincaré 7 (1) (1990) 1–26.
[9] C. Ferreira, Stability properties of divergence-free vector fields, arXiv:1004.2893, preprint, 2010.
[10] M. Hurley, Fixed points of topologically stable flows, Trans. Amer. Math. Soc. 294 (2) (1986) 625–633.
[11] M. Hurley, Consequences of topological stability, J. Differential Equations 54 (1) (1984) 60–72.
[12] M. Hurley, Combined structural and topological stability are equivalent to Axiom A and the strong transversality condition, Ergodic Theory Dynam. Systems 4 (1) (1984) 81–88.
[13] J. Moser, On the volume elements on a manifold, Trans. Amer. Math. Soc. 120 (1965) 286–294.
[14] J.F. Plante, W. Thurston, Anosov flows and the fundamental group, Topology 11 (1972) 147–150.
[15] C. Robinson, Stability theorems and hyperbolicity in dynamical systems, Rocky Mountain J. Math. 7 (3) (1977) 425–437.
[16] K. Moriyasu, The topological stability of diffeomorphisms, Nagoya Math. J. 123 (1991) 91–102.
[17] K. Moriyasu, K. Sakai, N. Sumi, Vector fields with topological stability, Trans. Amer. Math. Soc. 353 (8) (2001) 3391–3408.
[18] Z. Nitecki, On semi-stability for diffeomorphisms, Invent. Math. 14 (1971) 83–122.
[19] R.O. Ruggiero, Topological stability and Gromov hyperbolicity, Ergodic Theory Dynam. Systems 19 (1999) 143–154.
[20] P. Walters, Anosov diffeomorphisms are topologically stable, Topology 9 (1970) 71–78.
[21] C. Zuppa, Regularisation C^∞ des champs vectoriels qui préservent l'elément de volume, Bol. Soc. Bras. Mat. 10 (2) (1979) 51–56.