Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 308 (2008) 5691–5701
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

An isoperimetric inequality in the universal cover of the punctured plane

Noga Alon[a], Adi Pinchasi[a], Rom Pinchasi[b]

[a] *Schools of Mathematics and Computer Science, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel*
[b] *Mathematics Department, Technion—Israel Institute of Technology, Haifa 32000, Israel*

Received 29 May 2007; received in revised form 16 October 2007; accepted 25 October 2007
Available online 20 February 2008

Abstract

We find the largest ϵ (approximately 1.71579) for which any simple closed path α in the universal cover $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ of $\mathbb{R}^2 \setminus \mathbb{Z}^2$, equipped with the natural lifted metric from the Euclidean two-dimensional plane, satisfies $L(\alpha) \geq \epsilon A(\alpha)$, where $L(\alpha)$ is the length of α and $A(\alpha)$ is the area enclosed by α. This generalizes a result of Schnell and Segura Gomis, and provides an alternative proof for the same isoperimetric inequality in $\mathbb{R}^2 \setminus \mathbb{Z}^2$.

Keywords: Isoperimetric inequality; Universal covering space

1. Introduction

A classical theorem of Jarnik in number theory asserts that for every embedded closed curve $\alpha \subset \mathbb{R}^2 \setminus \mathbb{Z}^2$, with no integer lattice points inside the domain bounded by α, $L(\alpha) \geq A(\alpha)$, where $L(\alpha)$ is the length of α and $A(\alpha)$ is the area enclosed by α, (see [4], p. 123). For a related work on *convex* simple curves in the plane see [1]. See also [8] where the simple curves in the universal cover of a space are considered.

We first bring an *elementary* argument showing the existence of such a linear isoperimetric inequality in a more general setting:

Theorem 1.1. *Let Z be a closed set in $\mathbb{R}^2$. Assume that there exists a constant M such that for every $x \in \mathbb{R}^2 \setminus Z$, $d(x, Z) \leq M$, where $d(x, Z)$ denotes the Euclidean distance from x to the set Z. Then there is a linear isoperimetric inequality in $\mathbb{R}^2 \setminus Z$. That is, there is an absolute constant $c_Z > 0$ such that for every simple contractible closed curve α in $\mathbb{R}^2 \setminus Z$ we have $L(\alpha) \geq c_Z A(\alpha)$.*

Proof. If α is contained in a disc of radius 1, then the theorem follows from the classic isoperimetric inequality in the plane $4\pi A(\alpha) \leq L^2(\alpha)$. Indeed, if $L(\alpha) \leq 1$, then $L(\alpha) \geq L^2(\alpha) \geq 4\pi A(\alpha)$, and if $L(\alpha) > 1$, then $L(\alpha) > 1 \geq \frac{1}{\pi} A(\alpha)$.

E-mail addresses: nogaa@post.tau.ac.il (N. Alon), pinhassi@post.tau.ac.il (A. Pinchasi), room@math.technion.ac.il (R. Pinchasi).

0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.10.033

Let $x_1, \ldots, x_n$ $(n > 1)$ be a maximal set of points on α such that for every $i \neq j$ $d(x_i, x_j) > 1$. Then clearly $L(\alpha) \geq n$. Observe that any point in α is at distance at most 1 from some x_i because of the maximality of n. Let x be any point in the region bounded by α. Let $z \in Z$ be a point such that $d(z, x) \leq M$. The line segment between x and z must cross α, for x is in the region bounded by α and z is not. It follows that x is at distance at most $M + 1$ from some point x_i. Therefore $A(\alpha) \leq n\pi(M + 1)^2$. We can now conclude that

$$L(\alpha) \geq n = n\pi(M+1)^2 \frac{1}{\pi(M+1)^2} \geq \frac{1}{\pi(M+1)^2} A(\alpha). \quad \square$$

Taking $Z = \mathbb{Z}^2$ in Theorem 1.1 we deduce a linear isoperimetric inequality in $\mathbb{R}^2 \setminus \mathbb{Z}^2$.

In [7] Schnell and Segura Gomis give a tight (best possible) linear isoperimetric inequality for the relation between the perimeter and the area of a simply connected region in $\mathbb{R}^2 \setminus \mathbb{Z}^2$. Their proof is very elegant and relies on Pick's formula in the Euclidean plane.

In this paper we generalize their result and show that the same (best possible) linear isoperimetric inequality holds in the more general space $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$, the universal covering of $\mathbb{R}^2 \setminus \mathbb{Z}^2$ equipped with the natural lifted metric from the two-dimensional Euclidean plane. This is somewhat a surprising example for a tight isoperimetric inequality in a base space X that can be lifted to be the same tight isoperimetric inequality in $\widetilde{X}$, the universal covering space of X. This is not the case for many other spaces. Indeed, consider for instance the space C, the infinite cylinder of radius 1. The universal covering space of C is $\mathbb{R}^2$. The area of a simple closed curve in $\mathbb{R}^2$ may depend quadratically on its perimeter. However, a simple contractible closed curve α on C of perimeter L may enclose an area of at most πL, as the difference between the heights of the highest and lowest points of α on C is at most $L/2$. Another such a natural example where the isoperimetric inequality in the base space is different in nature than the isoperimetric inequality in the universal covering space is the torus.

It follows from a general theorem of Bonk and Eremenko [2], that contractible closed curves satisfy a linear isoperimetric inequality in $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$. Polterovich and Sikorav [5] showed that for a generalized definition of the area of a closed contractible curve $\beta \subseteq \widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$, $(1 + \sqrt{2})L(\beta) \geq A(\beta)$. In fact, using methods in [5], one can show that for an embedded closed curve $\beta \subseteq \widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$, $(1+\sqrt{2})L(\beta) \geq A(\beta)$ [6]. In this paper we find the tight linear isoperimetric inequality in $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$. More specifically, we define a constant that we denote by ϵ and prove, in Section 4, the following main result of this paper:

Theorem 1.2. *Let* $\alpha : S^1 \to \widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ *be a simple closed curve. Then* $L(\alpha) \geq \epsilon A(\alpha)$. *The constant* ϵ *(approximately* 1.71579*) is best possible.*

The constant ϵ is defined in Section 3. There it will also be shown that ϵ can be obtained implicitly by the equations:

$$\epsilon = \frac{\frac{\pi-\alpha}{\sin\alpha}}{\frac{\pi-\alpha}{4\sin^2\alpha} + \frac{\cos\alpha}{4\sin\alpha} + \frac{1}{2}}, \qquad \sin\alpha = \frac{\epsilon}{2}, \qquad \frac{\pi}{2} \leq \alpha \leq \pi.$$

An approximate solution to this system is $\epsilon \approx 1.71579$.

We note that since the constant ϵ is the same constant found by Schnell and Segura Gomis, the fact that ϵ cannot be replaced by a larger constant in the statement of Theorem 1.2 follows from the tightness of the result in [7].

2. A description of $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$

Definition 2.1. A *basic square* S in $\mathbb{R}^2 \setminus \mathbb{Z}^2$ is a unit square in $\mathbb{R}^2$ that is closed, except for the four vertices, and is centered at a point $(m + \frac{1}{2}, n + \frac{1}{2})$, where $n, m \in \mathbb{Z}$.

Consider the grid $G \subset \mathbb{R}^2 \setminus \mathbb{Z}^2$ consisting of horizontal and vertical lines through the points $(m + \frac{1}{2}, n + \frac{1}{2})$, where $n, m \in \mathbb{Z}$. The universal cover $\widetilde{G}$ of $(G, \mathit{Euclidean})$ is an infinite tree for which the degree of every vertex is four and the length of every edge is one. Let $P : \widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2} \to \mathbb{R}^2 \setminus \mathbb{Z}^2$, be the covering map. G is a deformation retract of $\mathbb{R}^2 \setminus \mathbb{Z}^2$, hence $P^{-1}(G)$ is a deformation retract of $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$. Therefore, $\pi_1(P^{-1}(G)) = 1$ which implies that $P^{-1}(G)$ can be identified with $\widetilde{G}$. Let S be a basic square in $\mathbb{R}^2 \setminus \mathbb{Z}^2$. Then the contractibility of S implies that any connected component of $P^{-1}(S)$ can be identified with S. Hence, $P^{-1}(S)$, the lifting of every basic square S in $\mathbb{R}^2 \setminus \mathbb{Z}^2$ centered at a fixed point $(m_0 + \frac{1}{2}, n_0 + \frac{1}{2})$, is a set of infinitely many copies of S centered at the points of $P^{-1}(m_0 + \frac{1}{2}, n_0 + \frac{1}{2})$. One can think of the universal covering space $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ as a thick 4-regular tree of width one, that is, the tree $\widetilde{G}$ with infinitely many basic squares centered at its vertices.

Definition 2.2. A *fundamental square* in $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ is a connected component of $P^{-1}(S)$, where S is a basic square in $\mathbb{R}^2 \setminus \mathbb{Z}^2$.

An *edge* in $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ is a boundary edge of a fundamental square in $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$.

We now show a fundamental property of closed embedded curves in $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$.

Lemma 2.3. *Let $\beta : [0, 1] \to \widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ be an oriented simple closed curve. Assume that the curve β leaves fundamental square S by crossing an edge e of S at $\beta(t_1)$, and does not cross e again until t_2. Then there is no $t_1 < c < t_2$ such that $\beta(c)$ intersects S.*

In other words. If β leaves a fundamental square S through an edge e, the next time it intersects S is through the same edge e.

Proof. Assume not. Then there is c $(t_1 < c < t_2)$ such that $\beta(c) \in S$. We can assume that $\beta(c)$ is an intersection point of β with an edge $e' \neq e$, where e' is an edge of S. Denote by M the union of the subarc of β, $\{\beta(t) : t_1 \le t \le c\}$ with the straight line segment between $\beta(t_1)$ and $\beta(c)$. Then M is an embedded closed curve. Looking at M and e as curves in the completion of $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$, we see that the intersection number $M \circ e$, mod 2, equals one. This is a contradiction, because M is contractible, hence $M \circ e = 0$. □

3. Defining ϵ

We will now define a number that we denote by ϵ. This number satisfies a certain isoperimetric inequality and will play a crucial role in the following. Let $P_0 = (0, 0)$ and $Q_0 = (1, 0)$. Consider the family $\mathcal{H}$ of all simple paths β with P_0 and Q_0 as endpoints that lie in the planar region $\{(x, y) | 0 \le x \le 1, y \ge 0\}$.

For every such β let $L(\beta)$ denote the length of β and $A(\beta)$ denote the area enclosed by β and the interval P_0Q_0.

We define ϵ to be $\inf_{\beta \in \mathcal{H}} \frac{L(\beta)}{A(\beta)+1/2}$.

Claim 3.1. *ϵ is a minimum which is obtained for a curve in $\mathcal{H}$.*

Proof. First, observe that by considering β to be the half-circle $\{(x, y) \mid y = \sqrt{1 - x^2}, 0 \le x \le 1\}$ we conclude that $\epsilon < 1.8$. Moreover, it is enough to consider only the subfamily of $\mathcal{H}$ of all the curves β with the property that no y-coordinate of a point of β exceeds 100. Indeed, if the largest y-coordinate of a point on β is k, then $L(\beta) \ge 2k$ and $A(\beta) \le k$. Therefore,

$$\frac{L(\beta)}{A(\beta) + \frac{1}{2}} \ge \frac{2k}{k + \frac{1}{2}} > 2 - \frac{1}{k}.$$

If $k > 100$, then $2 - \frac{1}{k} > 1.8$. It now follows easily by the principles of compactness that there exists an optimal curve in $\mathcal{H}$. □

Claim 3.2. *The value ϵ, defined before Claim 3.1, is obtained for a curve β which is a circular arc.*

Claim 3.2 will be a consequence of the following theorem.

Theorem 3.3. *Let P and Q be two points on the x-axis of $\mathbb{R}^2$. Denote by $\mathcal{T}$ be the family of all curves $\beta \subset \mathbb{R}^2$ with the following two properties:*

1. *β is a simple curve with endpoints P and Q.*
2. *β lies above the segment PQ and in the region bounded by the lines through P and Q that are perpendicular to PQ.*

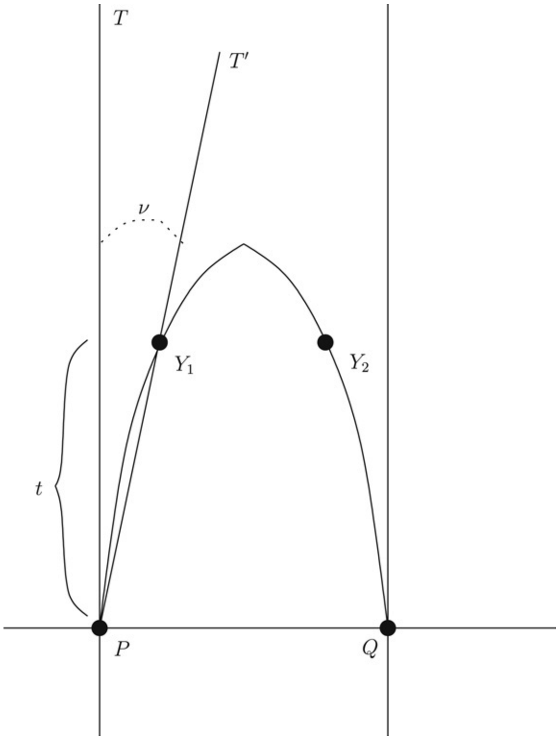

Fig. 1.

Let $c > 0$ be a positive constant such that $c|PQ| < 2$. Then the minimum over all $\beta \in \mathcal{T}$ of the expression $L(\beta) - cA(\beta)$, is obtained for $\beta \in \mathcal{T}$ which is a circular arc.

Proof. Without loss of generality we assume that P is to the left of Q. An optimal curve β must be a concave curve. We can also assume that the optimal curve β (that may not be unique) is symmetric with respect to the line which is the perpendicular bisector to the segment PQ. Here we exploit the well-known technique of Steiner-Symmetrization, due to J. Steiner (1838), applied in the classical isoperimetric inequality of balls in the Euclidean space (see [3] for a survey).

We need the following claim.

Claim 3.4. *Let T be a line which passes through P such that β, is fully contained in a closed half-plane bounded by T. Assume further that T has the smallest positive slope among all such lines. Then the slope of T (with respect to the segment PQ) is strictly smaller than $\pi/2$.*

Proof. Assume on the contrary that T is vertical. Let T' be a line through P whose slope equals $\pi/2 - \nu$, where ν is a small positive number to be determined later.

Let Y_1 be the point on $\beta \cap T'$ with the largest y coordinate. Denote the value of the y-coordinate of Y_1 by t. Let Y_2 be the symmetric point to Y_1 with respect to the vertical line through the midpoint of PQ. See Fig. 1.

Let β' be the subarc of β between Y_1 and Y_2. Observe that $L(\beta') \leq L(\beta) - 2t$ and $A(\beta') \geq A(\beta) - t|PQ|$.

Let $s = |PQ|/|Y_1Y_2|$ and let β'' be the curve obtained from β' by applying a similarity transformation with ratio s. The endpoints of β'' are at distance $|PQ|$ from each other. Therefore, by identifying them with P and Q we may regard β'' as a curve in $\mathcal{T}$. Moreover, $L(\beta'') = sL(\beta')$ and $A(\beta'') = s^2A(\beta')$. Denote $x = \frac{2t\tan\nu}{|PQ|-2t\tan\nu}$. Observe that $s = 1 + x$.

Therefore,

$$\begin{aligned} L(\beta'') - cA(\beta'') &= sL(\beta') - cs^2A(\beta') \\ &\leq s(L(\beta) - 2t) - cs^2(A(\beta) - t|PQ|) \\ &= (1+x)(L(\beta) - 2t) - c(1+x)^2(A(\beta) - t|PQ|) \end{aligned}$$

$$= L(\beta) - cA(\beta) + (c|PQ|t - 2t + x(L(\beta) - 2t) - c(2x + x^2)(A(\beta) - t|PQ|))$$

For ν small enough x is much smaller than t as it is of the order of $t \tan \nu$. We thus obtain a contradiction to the minimality of β, assuming that $c|PQ| - 2 < 0$. □

Let X be any point on the curve β. We will show that the angle $\measuredangle PXQ$ is independent of X. This will clearly show that β is a circular arc.

Let $\alpha_0 = \measuredangle PXQ$. For every α in a small neighborhood of α_0 we define a curve in $\mathcal{T}$ in the following way:

We think of β_{PX}, the subarc of β between P and X, as solid and similarly of β_{XQ}, the subarc of β between X and Q. On the other hand, we think of the point X as an axis about which the solid parts β_{PX} and β_{XQ} can rotate. We then rotate the parts β_{PX} and β_{XQ} in such a way that $\measuredangle PXQ$ becomes equal to α. We thus obtain a curve the distance between whose endpoints is $r_\alpha = \sqrt{|PX|^2 + |XQ|^2 - 2|PX||XQ|cos(\alpha)}$. We then apply a similarity transformation with ratio equal to $|PQ|/r_\alpha$ to obtain a new curve β_α whose endpoints (that are at distance $|PQ|$ from each other) we identify with P and Q. It follows from Claim 3.4 that if α is in a small enough neighborhood of α_0, then β_α belongs to the class $\mathcal{T}$ — the crucial point is that if α is very close to α_0, β_α is still contained in the region bounded by the vertical lines through P and Q.

Let $g(\alpha) = L(\beta_\alpha) - cA(\beta_\alpha)$. We know that $g(\alpha)$ is minimized for $\alpha = \alpha_0$.

We will now obtain a more direct formula for $g(\alpha)$ and interpret the condition $g'(\alpha_0) = 0$.

We start with $g(\alpha) = L(\beta_\alpha) - cA(\beta_\alpha)$. Clearly, $L(\beta_\alpha) = \frac{|PQ|}{r_\alpha} L(\beta)$. $A(\beta_\alpha)$ is given by $(\frac{|PQ|}{r_\alpha})^2 (A(\beta_{PX}) + A(\beta_{XQ}) + \frac{1}{2}|PX||XQ| \sin\alpha)$, where $A(\beta_{PX})$ is the area enclosed by β_{PX} and the line segment PX, and $A(\beta_{XQ})$ is enclosed by β_{XQ} and the line segment XQ.

We can now compute $g'(\alpha)$ and obtain

$$\begin{aligned} g'(\alpha) &= -L(\beta)\frac{|PQ|}{r_\alpha^3}|PX||XQ|\sin\alpha \\ &\quad - c\frac{|PQ|^2}{r_\alpha^4}(2|PX||XQ|\sin\alpha)\left(A(\beta_{PX}) + A(\beta_{XQ}) + \frac{1}{2}|PX||XQ|\sin\alpha\right) \\ &\quad - c\frac{|PQ|^2}{r_\alpha^2}\left(\frac{1}{2}|PX||XQ|\cos\alpha\right). \end{aligned} \tag{1}$$

We know that $g'(\alpha_0) = 0$. Moreover, $r_{\alpha_0} = |PQ|$ and $A(\beta_{PX}) + A(\beta_{XQ}) + \frac{1}{2}|PX||XQ| \sin\alpha_0 = A(\beta)$. Therefore, by plugging $\alpha = \alpha_0$ in (1), we obtain

$$0 = -L(\beta)\frac{|PX||XQ|\sin\alpha_0}{|PQ|^2} - \frac{c}{|PQ|^2}(2|PX||XQ|\sin\alpha_0 A(\beta)) - c\frac{1}{2}|PX||XQ|\cos\alpha_0. \tag{2}$$

After dividing (2) by $|PX||XQ|$, and some easy manipulations we obtain:

$$\tan\alpha_0 = \frac{\sin\alpha_0}{\cos\alpha_0} = \frac{-c|PQ|^2}{2L(\beta) + 4cA(\beta)}.$$

It is evident that $\tan\alpha_0$ does not depend on X which is what we wanted to prove. □

Proof of Claim 3.2. Let β_0 be an optimal curve. We know that for every $\beta \in \mathcal{H}$, $\frac{L(\beta)}{A(\beta)+\frac{1}{2}} \geq \epsilon$, or in other words $L(\beta) - \epsilon A(\beta) \geq \frac{\epsilon}{2}$. Since we have equality for β_0, it follows from Theorem 3.3 (with $|PQ| = 1$ and $c = \epsilon$) that β_0 is a circular arc. □

The following easy claim can be verified via direct calculations:

Claim 3.5. *Let β be a subarc of a circle. Let P and Q denote the endpoints of β and assume that $|PQ| = 1$. Let α denote the constant angle $\measuredangle PXQ$ for any point X on β. Then $\frac{L(\beta)}{A(\beta)+\frac{1}{2}}$ is given by*

$$F(\alpha) = \frac{\frac{\pi-\alpha}{\sin\alpha}}{\frac{\pi-\alpha}{4\sin^2\alpha} + \frac{\cos\alpha}{4\sin\alpha} + \frac{1}{2}}.$$

By definition, $F(\alpha) \geq \epsilon$, hence

$$\frac{\pi - \alpha}{\sin \alpha} - \epsilon \left(\frac{\pi - \alpha}{4 \sin^2 \alpha} + \frac{\cos \alpha}{4 \sin \alpha} \right) \geq \frac{\epsilon}{2}. \tag{3}$$

We will need the following theorem regarding isoperimetric inequality.

Theorem 3.6. *Let $0 \leq t \leq 1$ and let β be a simple curve whose endpoints are $(0, 0)$ and $(t, 0)$ and which is fully contained in the region $\{(x, y) | 0 \leq x \leq t, y \geq 0\}$. Then $L(\beta) - \epsilon A(\beta) \geq t\frac{\epsilon}{2}$.*

Remark. Observe that when $t = 1$ Theorem 3.6 follows immediately from the definition of ϵ.

Proof. By Theorem 3.3, the curve β for which the expression $L(\beta) - \epsilon A(\beta)$ is minimum, is a subarc of a circle.

Let $\frac{\pi}{2} \leq \alpha \leq \pi$ be the constant angle defined by the cord between $(0, 0)$ and $(t, 0)$. It can then be verified by direct calculations that the expression $L(\beta) - \epsilon A(\beta)$ is given by

$$g(\alpha) = t\frac{\pi - \alpha}{\sin \alpha} - t^2 \epsilon \left(\frac{\pi - \alpha}{4 \sin^2 \alpha} + \frac{\cos \alpha}{4 \sin \alpha} \right).$$

A direct calculation gives:

$$g'(\alpha) = \frac{\sin \alpha + (\pi - \alpha) \cos \alpha}{\sin^2 \alpha} \left(\frac{t^2 \epsilon}{2 \sin \alpha} - t \right).$$

It is easy to see that $\sin \alpha + (\pi - \alpha) \cos \alpha > 0$ for every $\frac{\pi}{2} < \alpha < \pi$ and that $\frac{t^2 \epsilon}{2 \sin \alpha} - t$ is an increasing function of α in the range $\frac{\pi}{2} < \alpha < \pi$. Therefore $g(\alpha)$ obtains a minimum when $\frac{t^2 \epsilon}{2 \sin \alpha} - t = 0$, that is, $\sin \alpha = \frac{t\epsilon}{2}$.

It follows from (3) that for every $\frac{\pi}{2} \leq \alpha \leq \pi$ we have

$$\epsilon \left(\frac{\pi - \alpha}{4 \sin^2 \alpha} + \frac{\cos \alpha}{4 \sin \alpha} \right) \leq \frac{\pi - \alpha}{\sin \alpha} - \frac{\epsilon}{2}.$$

Therefore, for every $\frac{\pi}{2} \leq \alpha \leq \pi$,

$$g(\alpha) \geq (t - t^2) \frac{\pi - \alpha}{\sin \alpha} + t^2 \frac{\epsilon}{2} \geq t\frac{\epsilon}{2}. \quad \square$$

Remark. We can now obtain an implicit equation for ϵ. That is,

$$\epsilon = \frac{\frac{\pi - \alpha}{\sin \alpha}}{\frac{\pi - \alpha}{4 \sin^2 \alpha} + \frac{\cos \alpha}{4 \sin \alpha} + \frac{1}{2}},$$

where $\sin \alpha = \frac{\epsilon}{2}$ and $\frac{\pi}{2} \leq \alpha \leq \pi$. An approximate solution to this equation is $1.71579\ldots$. Nevertheless, we will not make use of this observation through the rest of the paper.

4. Proof of the main theorem

We will now prove Theorem 1.2. We need to show that for every simple closed curve $\alpha : S^1 \to \widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ we have $L(\alpha) \geq \epsilon A(\alpha)$. We assume that the curves α in question are transverse to every edge. Indeed, this can be achieved by a small perturbation of the given curve α.

The following claim shows that in order to prove Theorem 1.2 it is enough to consider the curves that intersect every edge in $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ at exactly two points or none.

Claim 4.1. *Let $\alpha : S^1 \to \widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ be a simple closed curve, then there exists a simple closed curve $\beta : S^1 \to \widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ such that $L(\beta) \leq L(\alpha)$ and $A(\beta) \geq A(\alpha)$ and such that β intersects every edge at exactly two points or none.*

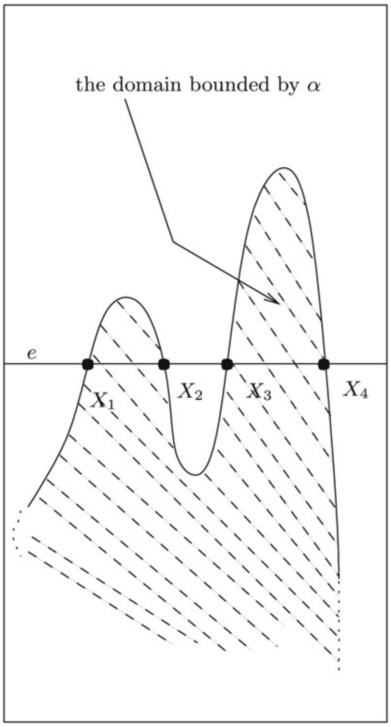

Fig. 2.

Proof. α is a compact boundaryless one-dimensional manifold, which is the boundary of the two-dimensional Riemannian manifold. We denote by A the domain bounded by α. We orient α so as the domain A is to the right of α. Let e be an edge in $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ which intersects α at $l > 2$ points. Assume that e is horizontal. Let $X_1, \ldots, X_l$ be the intersection points of α with e. We assume that the points are indexed consecutively from the left to the right. See Fig. 2.

We orient e in such a way that it enters A at the point X_1. α is contractible and hence, if we look at α and e as in the completion of $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$, we have $e \circ \alpha = 0$. Therefore, the number of points in the set $\alpha \cap e$ is even. It is easy to see that the line segments $X_{2j}X_{2j+1}$ for $j = 1, \ldots, \frac{l}{2} - 1$, do not intersect the interior of A.

Denote by $\alpha X_2, X_3$ the subarc of alpha that starts at X_2 and ends at X_3 (according to the orientation of α). We replace $\alpha X_2, X_3$ by the line segment X_2X_3 to obtain another embedded curve that we denote by α_1. Observe that $L(\alpha) \geq L(\alpha_1)$. Moreover, $A(\alpha) \leq A(\alpha_1)$, because A lies inside the domain bounded by α_1.

We now perform a small perturbation to α_1 at a small neighborhood of e so that the following statements hold:

- α_1 is transverse to e.
- There are no other intersection points of $\alpha_1 \cap e$ beside X_1 and $X_4, \ldots, X_n$.
- $L(\alpha) \geq L(\alpha_1)$ and $A(\alpha) \leq A(\alpha_1)$.

We apply successively the above procedure for the pairs $(X_4, X_5), \ldots, (X_{2n-2}, X_{2n-1})$. We obtain an embedded curve $\alpha_{\frac{l}{2}}$ which satisfies: $L(\alpha_{\frac{l}{2}}) \leq L(\alpha)$, $A(\alpha_{\frac{l}{2}}) \geq A(\alpha)$, and $\alpha_{\frac{l}{2}}$ intersects e at exactly two points: X_1 and X_n.

We apply the above procedure to every edge e' which intersects α. We obtain an embedded curve β which satisfies: $L(\beta) \leq L(\alpha)$, $A(\beta) \geq A(\alpha)$. Moreover, β intersects any edge e' at exactly two points or none. This completes the proof. □

From now on we consider only the curves that intersect any edge of $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ at exactly two points or none.

Definition 4.2. A θ-*curve* is a simple curve in $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ whose both endpoints lie on the same edge of a fundamental square. For a θ-curve β, $L(\beta)$ will denote its length, $A(\beta)$ will denote the area enclosed by the closed curve obtained

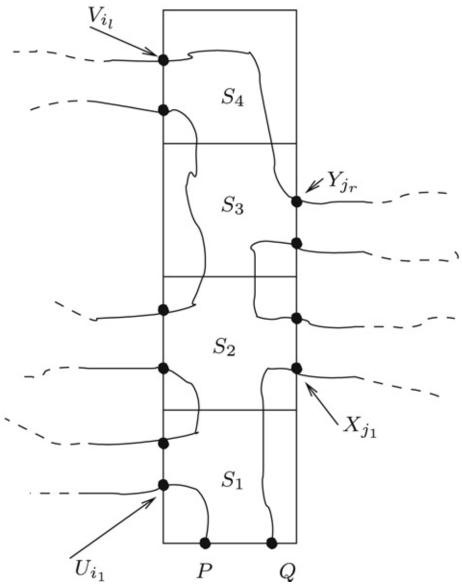

Fig. 3. Case 2 with $l = 3$ and $r = 2$.

from β by joining its two endpoints by a straight line segment. $d(\beta)$ will denote the distance between the two endpoints of β.

Theorem 1.2 is an easy consequence of the following lemma:

Lemma 4.3. *Let β be a θ-curve , then*

$$L(\beta) \geq \epsilon A(\beta) + \frac{\epsilon}{2} d(\beta). \quad (4)$$

We first show how Theorem 1.2 follows from Lemma 4.3. Without loss of generality α intersects (at exactly two points) an edge of a fundamental square. Let Q and P be these two intersection points. P and Q divide α into two curves each of which is a θ-curve . Let β_1 and β_2 be those two curves. By Lemma 4.3, $L(\beta_i) \geq \epsilon A(\beta_i) + \frac{\epsilon}{2}|PQ|$, for $i = 1, 2$. Therefore,

$$L(\alpha) = L(\beta_1) + L(\beta_2) \geq \epsilon(A(\beta_1) + A(\beta_2)) + 2\frac{\epsilon}{2}(|PQ|) \geq \epsilon A(\alpha).$$

Proof of Lemma 4.3. We prove the lemma by induction on the number n of fundamental squares that intersect the relative interior of β. The *relative interior of* β relates to the interior of the simple closed curve obtained from β and the straight line segment joining its two end points.

The case $n = 1$ follows directly from Theorem 3.6, after some suitable reductions: Denote by S the single fundamental square which contains β. Let P and Q be the endpoints of β and assume without loss of generality that P is to the left of Q and that the interval PQ lies on the bottom edge of S (see Fig. 3). We may assume that β lies entirely to the right of the perpendicular line to the segment PQ which touches P. Indeed, otherwise consider the line y that is perpendicular to PQ and is the tangent to β so that β lies entirely to the right of y. Let T be a point at which y touches β. Let P' be the intersection point of y with the bottom edge of S. Now modify β by replacing the subarc of β between P and T by the straight line segment on y between P' and T. We thus obtain a new θ-curve β' such that $L(\beta') \leq L(\beta)$, $A(\beta') \geq A(\beta)$, and $d(\beta') \geq d(\beta)$ (and therefore $\frac{\epsilon}{2}d(\beta') \geq \frac{\epsilon}{2}d(\beta)$). Observe that now β' lies entirely to the right of the perpendicular line through P'.

In a similar manner we can assume that β lies entirely to the left of the perpendicular line to the segment PQ which touches Q. We can now use Theorem 3.6 and conclude the case $n = 1$.

In order to complete the proof of Lemma 4.3 we have to consider the case $n > 1$, and complete the induction step. To this end, we consider the fundamental square S which contains the endpoints P and Q of β. Without loss of generality we assume that both P and Q lie on the bottom edge of S so that P is to the left of Q. Let us denote the four vertices of S in the clockwise order starting from the lower left vertex, by A, B, C, and D.

Denote $S_1 = S$ and for each integer $m > 1$ let S_m be the fundamental square adjacent to S_{m-1} in its top edge. Let $i_1 < i_2 < \cdots < i_l$ be all the indices such that β intersects the left edge of S_{i_k} (at exactly two points). For every $0 \le k \le l$, let us denote the two points of intersection of β and the left edge of S_{i_k} by U_{i_k} and V_{i_k} so that V_{i_k} is above the U_{i_k}.

Similarly, let $j_1 < j_2 < \cdots < j_r$ be all the indices such that β intersects the right edge of S_{j_k} (at exactly two points). For every $0 \le k \le r$, let us denote the two points of intersection of β and the right edge of S_{j_k} by X_{j_k} and Y_{j_k}, so that Y_{j_k} is above X_{j_k}. See Fig. 3.

We distinguish among three cases:

Case 1. $r = l = 0$. This case follows directly from Theorem 3.6 after similar adjustments to those made in the case $n = 1$.

Case 2. $r > 0$ and $l > 0$. In this case for every $1 \le k \le l-1$ we may replace the subarc of β between V_{i_k} and $U_{i_{k+1}}$ by a straight line segment (with a small perturbation in order that $\beta \subset \widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$), since this will decrease $L(\beta)$ and will increase $A(\beta)$. We may also replace the subarc of β between P and U_{i_1} by a straight line segment and thus assume that P almost coincides with A. Indeed, this will result in decreasing $L(\beta)$, increasing $A(\beta)$ and increasing $d(\beta)$, and it will be enough to prove that even after this adjustment $L(\beta) - \epsilon A(\beta) \ge \frac{\epsilon}{2} d(\beta)$ still holds.

Similarly, for every $1 \le k \le r-1$ we may replace the subarc of β between Y_{j_k} and $X_{j_{k+1}}$ by a straight line segment (with a small perturbation in order that $\beta \subset \widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$) since this will decrease $L(\beta)$ and will increase $A(\beta)$. We may also replace the subarc of β between Q and X_{j_1} by a straight line segment and thus assume that Q almost coincides with D.

Note that we may assume that the subarc of β between V_{i_l} and Y_{j_r} is concave. Consider the line m in $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ that is the perpendicular bisector of AD, and denote its (unique) intersection point with β by M. We now reflect the subarc of β between P and M with respect to the reflection line m. We thus obtain a θ-curve that we denote by β_l. For every $1 \le k \le l$, denote by β_{i_k} the θ-curve which is the subarc of β between U_{i_k} and V_{i_k}. Denote by V'_{i_l} the reflection of V_{i_l} with respect to m. Let β_M denote the subarc of β_l between V_{i_l} and V'_{i_l} (see Fig. 3). We have,

$$L(\beta_l) = 2\sum_{k=1}^{l} L(\beta_{i_k}) + L(\beta_M) + 2|PU_{i_1}| + 2\sum_{k=1}^{l-1} |V_{i_k}U_{i_{k+1}}| \tag{5}$$

$$A(\beta_l) = 2\sum_{k=1}^{l} A(\beta_{i_k}) + A(\beta_M) + |PU_{i_1}| + \sum_{k=1}^{l-1} |V_{i_k}U_{i_{k+1}}| + \sum_{k=1}^{l} |U_{i_k}V_{i_k}|. \tag{6}$$

By the induction hypothesis, for every $1 \le k \le l$, $L(\beta_{i_k}) - \epsilon A(\beta_{i_k}) \ge \frac{\epsilon}{2}|U_{i_k}V_{i_k}|$. By Theorem 3.6, $L(\beta_M) - \epsilon A(\beta_M) \ge \frac{\epsilon}{2}$.

Therefore,

$$\begin{aligned}
L(\beta_l) - \epsilon A(\beta_l) &\ge \frac{\epsilon}{2} + 2\sum_{k=1}^{l}\frac{\epsilon}{2}|U_{i_k}V_{i_k}| \\
&\quad + 2|PU_{i_1}| + 2\sum_{k=1}^{l-1}|V_{i_k}U_{i_{k+1}}| - \epsilon\left(|PU_{i_1}| + \sum_{k=1}^{l-1}|V_{i_k}U_{i_{k+1}}| + \sum_{k=1}^{l}|U_{i_k}V_{i_k}|\right) \\
&= \frac{\epsilon}{2} + (2-\epsilon)\left(|PU_{i_1}| + \sum_{k=1}^{l-1}|V_{i_k}U_{i_{k+1}}|\right) + \sum_{k=1}^{l}(\epsilon|U_{i_k}V_{i_k}| - \epsilon|U_{i_k}V_{i_k}|) \\
&= \frac{\epsilon}{2} + (2-\epsilon)\left(|PU_{i_1}| + \sum_{k=1}^{l-1}|V_{i_k}U_{i_{k+1}}|\right) \\
&\ge \frac{\epsilon}{2}.
\end{aligned}$$

In a similar manner we reflect the subarc of β between Q and M with respect to the reflection line m and obtain a θ-curve that we denote by β_r. The same arguments yield that

$$L(\beta_r) - \epsilon A(\beta_r) \geq \frac{\epsilon}{2}.$$

Observe that $2L(\beta) = L(\beta_l) + L(\beta_r)$ and that $2A(\beta) = A(\beta_l) + A(\beta_r)$. Combining this with the isoperimetric inequalities for β_l and β_r we obtain the desired result, namely,

$$L(\beta) - \epsilon A(\beta) \geq \frac{\epsilon}{2}.$$

Case 3. $l > 0$ and $r = 0$. As in case 2, for every $1 \leq k \leq l-1$ we may replace the subarc of β between V_{i_k} and $U_{i_{k+1}}$ by a straight line segment (with a small perturbation in order that $\beta \subset \widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$) since this will decrease $L(\beta)$ and will increase $A(\beta)$. We may also replace the subarc of β between P and U_{i_1} by a straight line segment and thus assume that P almost coincides with A. Indeed, this will result in decreasing $L(\beta)$, increasing $A(\beta)$ and increasing $d(\beta)$, and it will be enough to prove that even after this adjustment $L(\beta) - \epsilon A(\beta) \geq \frac{\epsilon}{2} d(\beta)$ still holds (observe that $\frac{\epsilon}{2} t$ is monotone increasing in t). Moreover, we may assume that the subarc of β between V_{i_l} and Q is concave, and that Q is the rightmost point of β in $S_1 \cup \ldots \cup S_l$. Indeed, otherwise we consider the line y perpendicular to AD that is a tangent of β and touches β at a point T. Then replace the subarc of β between Q and T by the straight line segment on y between T and Q', where Q' is the intersection point of y with AD. In this way we increase $A(\beta)$, decrease $L(\beta)$, and increase $\frac{\epsilon}{2} d(\beta)$.

We consider the line m in $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ that is the perpendicular bisector of AD. We distinguish between two cases.

Case 3(a): m intersects β.

In this case, as in Case 2, we denote the (unique) intersection point of m with β by M. We reflect the subarc of β between P and M with respect to the reflection line m. We thus obtain a θ-curve that we denote by β_l. The same arguments as in Case 2 yield the following isoperimetric inequality:

$$L(\beta_l) - \epsilon A(\beta_l) \geq \frac{\epsilon}{2}.$$

In a similar manner we reflect the subarc of β between Q and M with respect to the reflection line m and obtain a θ-curve that we denote by β_r. β_r satisfies the conditions in Theorem 3.6. Hence,

$$L(\beta_r) - \epsilon A(\beta_r) \geq 2\left(|PQ| - \frac{1}{2}\right)\frac{\epsilon}{2} = (2|PQ| - 1)\frac{\epsilon}{2}.$$

Observe that $2L(\beta) = L(\beta_l) + L(\beta_r)$ and that $2A(\beta) = A(\beta_l) + A(\beta_r)$. Combining this with the isoperimetric inequalities for β_l and β_r we obtain the desired result, namely,

$$L(\beta) - \epsilon A(\beta) \geq |PQ|\frac{\epsilon}{2}.$$

Case 3(b): m does not intersect β.

In this case we consider the line l_S that is perpendicular to AD at A. We think of the subarc of β between V_{i_l} and Q as a curve in $\mathbb{R}^2$ and reflect it with respect to the line l_S. We thus obtain a curve that we denote by β_r. Observe that β_r satisfies the conditions of Theorem 3.6. For every $1 \leq k \leq l$ denote by β_{i_k} the θ-curve that is the subarc of β between U_{i_k} and V_{i_k}.

We have:

$$L(\beta) \geq \sum_{k=1}^{l} L(\beta_{i_k}) + \frac{1}{2} L(\beta_r) \tag{7}$$

$$A(\beta) \geq \sum_{k=1}^{l} A(\beta_{i_k}) + \frac{1}{2} A(\beta_r). \tag{8}$$

By the induction hypothesis, for every $1 \leq k \leq l$ we have

$$L(\beta_{i_k}) \geq \epsilon A(\beta_{i_k}) + \frac{\epsilon}{2} d(\beta_{i_k}) \geq \epsilon A(\beta_{i_k}).$$

By Theorem 3.6,

$$L(\beta_r) \geq \epsilon A(\beta_r) + \frac{\epsilon}{2} d(\beta_r) = \epsilon A(\beta_r) + \frac{\epsilon}{2} 2d(\beta).$$

Therefore,

$$\begin{aligned} L(\beta) &\geq \sum_{k=1}^{l} L(\beta_{i_k}) + \frac{1}{2} L(\beta_r) \\ &\geq \sum_{k=1}^{l} \epsilon A(\beta_{i_k}) + \frac{1}{2} \epsilon A(\beta_r) + \frac{1}{2}\frac{\epsilon}{2} 2d(\beta) \\ &= \epsilon \left(\sum_{k=1}^{l} A(\beta_{i_k}) + \frac{1}{2} A(\beta_r) \right) + \frac{\epsilon}{2} d(\beta) \\ &= \epsilon A(\beta) + \frac{\epsilon}{2} d(\beta). \end{aligned}$$

This completes the proof of Lemma 4.3. □

Acknowledgments

We would like to thank Leonid Polterovich for introducing the problem to us and for many important comments. We also thank Misha Bialy for pointing out important references. Noga Alon's research was supported in part by the Hermann Minkowski Minerva Center for Geometry at Tel Aviv University.

References

[1] E. Bender, Area-perimeter relations for two dimensional lattices, Amer. Math. Monthly 69 (1962) 742–744.
[2] M. Bonk, A. Eremenko, Uniformly hyperbolic surfaces, Indiana Univ. Math. J. 49 (2000) 61–80.
[3] Yu.D. Burago, V.A. Zalgaller, Geometric inequalities, in: Grundlehren der Mathematischen Wissenschaften, vol. 285, Springer, 1988.
[4] H.L. Keng, Introduction to Number Theory, Springer, 1982.
[5] L. Polterovich, J.C. Sikorav, A linear isoperimetric inequality for the punctured plane. Manuscript, can be downloaded at: http://front.math.ucdavis.edu/math.GR/0106216.
[6] L. Polterovich, J.C. Sikorav, Personal communication.
[7] U. Schnell, S. Segura Gomis, Two problems concerning the area-perimeter ratio of lattice-point-free regions in the plane, Beiträge Algebra Geom. 37 (1) (1996) 1–8.
[8] P.W. Shor, Ch.J. Van Wyk, Detecting and decomposing self-overlapping curves, Comput. Geom. 2 (1992) 31–50.