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THE DEGREE OF THE EIGENVALUES OF GENERALIZED MOORE GEOMETRIES

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Using elementary methods it is proved that the eigenvalues of generalized Moore geometries of type $\mathrm{GM}_m(s, t, c)$ are of degree at most 3 with respect to the field of rational numbers, if $st > 1$.

1. Introduction

A generalized Moore geometry $\mathrm{GM}_m(s, t, c)$ is defined as a finite incidence structure of points and lines, such that each line contains $s+1$ points and each point lies on $t+1$ lines and with a corresponding point graph of diameter m. Moreover, any two distinct vertices in the point graph with distance $<m$ are connected by a unique path of length $\leqslant m$, whereas any two distinct vertices at distance m are connected by precisely c distinct shortest paths. By definition the vertex set of the point graph is the set of points of the incidence structure and two vertices are connected if and only if they are incident with a same line. For more details about special types of these structures we refer to [11, Section 1].

Generalized Moore geometries only seem to exist for small values of the diameter m, except in the trivial case when $s = t = 1$. Many non-existence theorems have been derived (cf. [4–14]). Since the point graph of a generalized Moore geometry is distance-regular these non-existence theorems are mostly proved in the context of the theory of distance-regular graphs (cf. [3]). The process of embedding $\mathrm{GM}_m(s, t, c)$ into this theory enhances the investigation of the so-called intersection matrix $L_m(s, t, c)$, defined as the $(m+1)\times(m+1)$-matrix

$$L_m(s,t,c)=\begin{bmatrix} 0 & 1 & & & & \\ s(t+1) & s-1 & 1 & & & \\ & st & s-1 & \ddots & & \\ & & st & \ddots & 1 & \\ & & & \ddots & s-1 & c \\ & & & & st & s(t+1)-c \end{bmatrix} \tag{1}$$

This tridiagonal matrix corresponds to the point graph of $GM_m(s, t, c)$ in the sense that all the eigenvalues of the adjacency matrix of that graph are also eigenvalues of $L_m(s, t, c)$ with multiplicity 1 (cf. [3]). Furthermore the theory of distance-regular graphs provides us with an expression for the multiplicity of such an eigenvalue, with which it occurs as eigenvalue of the adjacency matrix. This turns out to be a powerful criterion if one investigates whether a certain set of parameters s, t, and c can correspond to a really existing incidence structure, or stated equivalently, whether the matrix $L_m(s, t, c)$ is feasible. Obviously, these multiplicities are positive integers. If a certain matrix $L_m(s, t, c)$ has an eigenvalue for which this "multiplicity" is not in $\mathbb{Z}^+$ it cannot be the intersection matrix of an existing generalized Moore geometry. One expresses this by saying that $L_m(s, t, c)$ is not feasible in such a case. On the other hand, if $L_m(s, t, c)$ is feasible one still cannot be sure that it is realizable, although only a few examples are known of feasible matrices $L_m(s, t, c)$, which are proved to be non-realizable.

A major technique in proving the infeasibility of a matrix $L_m(s, t, c)$ consists of deriving an upper bound for the degree of its eigenvalues, considered as algebraic numbers with respect to the rational field $\mathbb{Q}$, and then showing in one way or another that this leads to contradictions for certain values of m.

Generally speaking, one can say that the lower this upper bound, the easier it becomes to eliminate the corresponding incidence structure. For example, in [1] Bannai and Ito showed that the eigenvalues of $GM_m(1, t, c)$, $t > 1$, are at most quadratic with respect to $\mathbb{Q}$ and in [2] they improved this result by proving that the eigenvalues are elements of $\mathbb{Q}$ (and hence of $\mathbb{Z}$, since these eigenvalues are algebraic integers). Another example is provided by $GM_m(s, t, s+1)$, the eigenvalues of which are at most quadratic. Moreover an irreducible defining polynomial has been derived for those eigenvalues which are precisely of degree 2 (cf. [10, 11]). The specific form of this polynomial was exploited in [10–14] as a tool for proving the infeasibility of $L_m(s, t, s+1)$ for $m > 5$ and for $m = 4$, and $st > 1$.

We remark that most of the calculations in the cited references are lengthy and cumbersome. Especially the derivations of upper bounds for the degrees of the eigenvalues are of a rather technical nature and appear quite complicated.

In this paper we prove that the degree of the eigenvalues of $L_m(s, t, c)$ is at most 3, for arbitrary values of the parameters s, t and c, with $st > 1$, if one requires feasibility. The used method is elementary and technicalities like e.g. the use of Chebyshev-polynomials (cf. [10, 11]) are avoided. At the same time a polynomial equation of degree 3 is derived which has to be satisfied by any eigenvalue of $L_m(s, t, c)$ in case of feasibility. This equation can possibly be used as a tool to prove the non-existence of other generalized Moore geometries.

2. Eigenvalues and eigenvectors of $L_m(s, t, c)$

Since the point graph Γ of $GM_m(s, t, c)$ is distance-regular (see Section 1) we can apply the theory of this type of graph as has been presented e.g. as in [3].

Here we summarize those results from [3] which we shall need in this paper, in an adapted notation.

The adjacency matrix A *of* Γ has $m+1$ distinct real eigenvalues

$$\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_m, \tag{2}$$

with the properties

$$\lambda_0 = k, \tag{3}$$

$$|\lambda_i| \leq k, \quad 0 \leq i \leq m, \tag{4}$$

where $k := s(t+1)$ is the valency of Γ. The eigenvalues λ_i, $0 \leq i \leq m$, are equal to the $m+1$ roots of the characteristic equation

$$|L_m(s, t, c) - \lambda I| = 0 \tag{5}$$

of the matrix $L_m(s, t, c)$ defined in (1). Therefore, the polynomial in λ in the left hand side of eq. (5) is the minimal polynomial of A.

Let λ be any of the eigenvalues (2) and let $y = (y_0, y_1, \dots, y_m)$ be the corresponding standard left eigenvector of $L_m(s, t, c)$, i.e.,

$$yL_m(s, t, c) = \lambda y, \quad y_0 = 1. \tag{6}$$

Substituting (1) and using the variables

$$\tau := \sqrt{st}, \tag{7}$$

and

$$\mu := \lambda + 1 - s, \tag{8}$$

which appear to be convenient in later calculations (cf. also [1]), one can easily verify that (6) is equivalent to

$$y_0 = 1, \qquad y_1 = (\tau^2 + s)^{-1}(\mu - 1 + s), \tag{9}$$

$$\tau^2 y_{i+1} - \mu y_i + y_{i-1} = 0, \quad 1 \leq i \leq m-1, \tag{10}$$

$$(\mu - \tau^2 - 1 + c)y_m - cy_{m-1} = 0. \tag{11}$$

We may regard μ in these relations as an indeterminate and interpret (9) and (10) as the definition relations of a sequence of polynomials in μ. Equation (11) then provides us with an equation between the last two polynomials of that sequence. The degree of eq. (11) as an equation in μ is equal to $m+1$ and its roots $\mu_0, \mu_1, \dots, \mu_m$ will—not quite correctly—be called the eigenvalues of $L_m(s, t, c)$ instead of $\lambda_0, \lambda_1, \dots, \lambda_m$ to which they are related by (8).

Together with the standard left eigenvector y one usually introduces the standard right eigenvector $z = (z_0, z_1, \dots, c^{-1}z_m)^{\mathrm{T}}$ defined according to

$$L_m(s, t, c)z = \lambda z, \quad z_0 = 1. \tag{12}$$

One can verify that the following relationship holds between the standard left and right eigenvectors belonging to a particular eigenvalue λ,

$$z_i = k_i y_i, \quad 0 \leqslant i < m, \tag{13}$$

$$z_m = c k_m y_m, \tag{14}$$

where the "i-valencies" are defined as

$$k_0 := 1, \qquad k_m := c^{-1} s^m t^{m-1}(t+1), \tag{15}$$

$$k_i := s^i t^{i-1}(t+1), \quad 0 < i < m. \tag{16}$$

We conclude this summary by giving the expression for the multiplicity $M(\lambda)$ of λ, considered as eigenvalue of the adjacency matrix A of Γ:

$$M(\lambda) = \frac{N}{\sum_{i=0}^{m} k_i y_i^2}, \tag{17}$$

where N denotes the number of vertices of Γ. It is this formula which has turned out to be a powerful tool to test whether a given matrix $L_m(s, t, c)$ can correspond to a distance-regular graph Γ of which it is the intersection matrix. (See also Section 1.) As a matter of fact one has in that case

$$M(\lambda) \in \mathbb{Z}^+. \tag{18}$$

3. Consequences of the characteristic equation

First we prove a useful lemma concerning the polynomials y_i, $0 \leqslant i \leqslant m$, defined in (9) and (10).

Lemma 3.1. *For $0 \leqslant i \leqslant m-2$ one has*

$$y_{i+2} y_i - y_{i+1}^2 = [\tau^{2i+2}(\tau^2 + s)^2]^{-1} k(\mu), \tag{19}$$

with

$$k(\mu) = (\mu - \tau^2 - 1)(s\mu + \tau^2 + s^2). \tag{20}$$

Proof. By mathematical induction one can prove quite easily the matrix equality

$$\frac{1}{\tau^{2i}} \begin{bmatrix} \mu & -1 \\ \tau^2 & 0 \end{bmatrix}^i \begin{bmatrix} y_2 & y_1 \\ y_1 & y_0 \end{bmatrix} = \begin{bmatrix} y_{i+2} & y_{i+1} \\ y_{i+1} & y_i \end{bmatrix}, \quad 0 \leqslant i \leqslant m-2, \tag{21}$$

applying the recurrence relation (10). Taking determinants of both sides of eq. (21) yields

$$y_{i+2} y_i - y_{i+1}^2 = \tau^{-2i}(y_2 y_0 - y_1^2), \quad 0 \leqslant i \leqslant m-2. \tag{22}$$

Substituting the explicit expressions for y_2, y_1 and y_0 then completes the proof of the lemma. □

In order to evaluate the expression for $M(\lambda)$ from eq. (17) we introduce polynomials x_i, $0 \leqslant i \leqslant m$, in the following way

$$x_0 := \tau^{-2}(\tau^2 + s), \tag{23a}$$

$$x_i := y_i z_i, \quad 0 \leqslant i \leqslant m. \tag{23b}$$

Lemma 3.2. *The polynomials* x_i, *defined by* (23a) *and* (23b), *satisfy the recurrence relation*

$$\tau^2 x_{i+1} + (2\tau^2 - \mu^2)x_i + \tau^2 x_{i-1} = -2(\tau^2 + s)^{-1}k(\mu), \quad \textit{for } 0 < i < m. \tag{24}$$

Proof. First we remark that from (13)–(16) follows

$$z_i = s^i t^{i-1}(t+1)y_i = \tau^{2i-2}(\tau^2 + s)y_i, \quad 0 < i \leqslant m. \tag{25}$$

Now we multiply (10) for some fixed value of $i(1 < i < m)$ by the expression $z_{i+1} + \mu z_i + \tau^2 z_{i-1}$ and obtain by (25)

$$\tau^2 x_{i+1} - \mu^2 x_i + \tau^2 x_{i-1} + 2\tau^{2i}(\tau^2 + s)y_{i+1}y_{i-1} = 0. \tag{26}$$

Applying Lemma 3.1 and using (25) once more then provides us with relation (24) for all i with $1 < i < m$. Due to the special choice (23a) for x_0 the relation also holds for $i = 1$. □

We use the variables x_m and x_{m-1}, defined in (23b) in the next two lemmas in order to present some equations, which have to be satisfied by the eigenvalues of $L_m(s, t, c)$.

Lemma 3.3. *The eigenvalues* μ_i, $0 \leqslant i \leqslant m$, *of* $L_m(s, t, c)$ *satisfy the equation*

$$(\mu - \tau^2 - 1 + c)^2 x_m - \tau^2 c^2 x_{m-1} = 0. \tag{27}$$

Proof. Multiply eq. (11) by the expression

$$(\mu - \tau^2 - 1 + c)y_m + c y_{m-1}, \tag{28}$$

and use (25). □

Lemma 3.4. *The eigenvalues* μ_i, $0 < i \leqslant m$, *of* $L_m(s, t, c)$ *satisfy the equation*

$$[(c-1)\mu + \tau^2 + (c-1)^2]x_m - c^2(\tau^2 + s)^{-1}(s\mu + \tau^2 + s^2) = 0. \tag{29}$$

Proof. Multiply eq. (11) by the expression

$$[(c-1)\mu + \tau^2 + 1 - c]y_m - c y_{m-1}. \tag{30}$$

This leads to

$$[-(\mu - \tau^2 - 1 + c)^2 + c\mu(\mu - \tau^2 - 1 + c)]y_m^2 - c^2 \mu y_m y_{m-1} + c^2 y_{m-1}^2 = 0. \tag{31}$$

If we substitute $\mu y_{m-1} = \tau^2 y_m + y_{m-2}$ (cf. eq. (10)) in the second term of (31) we obtain

$$(\mu - \tau^2 - 1)[(c-1)\mu + \tau^2 + (c-1)^2]y_m^2 - c^2(y_m y_{m-2} - y_{m-1}^2) = 0. \qquad (32)$$

Now we apply Lemma 3.1, yielding for the eigenvalues μ_i, $0 < i \leq m$, the relation

$$[(c-1)\mu + \tau^2 + (c-1)^2]y_m^2 - c^2[\tau^{2m-2}(\tau^2+s)^2]^{-1}(s\mu + \tau^2 + s^2) = 0,$$

which is equivalent to (29) by (14), (15) and (23b). □

We remark that eqs. (27) and (29), considered as equations in μ, are of degree $2m+2$ and $2m+1$ respectively and that the eigenvalues $\mu_1, \mu_2, \ldots, \mu_m$ which are dealt with in Lemma 3.4 are sometimes referred to as eigenvalues of the reduced characteristic equation.

4. The rationality condition

The multiplicity of an eigenvalue λ of the adjacency matrix of $\mathrm{GM}_m(s, t, c)$ will be denoted in this section by $M(\mu)$, where μ is related to λ according to (8). From the expression (17) we derive, using eqs. (13)–(16), that

$$\frac{N}{M(\mu)} = 1 + \sum_{i=1}^{m-1} x_i + \frac{1}{c} x_m, \qquad (33)$$

where $x_1, x_2, \ldots, x_m$ are defined by (23b). In order to evaluate the right hand side of eq. (33) we take the sum of both sides of equality (24) for all values of i between 0 and m, giving

$$\begin{aligned}
&\tau^2 \sum_{i=1}^{m-1} x_{i+1} + (2\tau^2 - \mu^2) \sum_{i=1}^{m-1} x_i + \tau^2 \sum_{i=1}^{m-1} x_{i-1} \\
&= (4\tau^2 - \mu^2) \sum_{i=1}^{m-1} x_i + \tau^2(x_0 - x_1 + x_m - x_{m-1}) \\
&= -2(m-1)(\tau^2+s)^{-1}k(\mu). \qquad (34)
\end{aligned}$$

Substitution of this relation in (33) and applying Lemma 3.3 yields

$$\begin{aligned}
(4\tau^2 - \mu^2)\frac{N}{M(\mu)} &= 4\tau^2 - \mu^2 + \tau^2(x_1 - x_0 - x_m + x_{m-1}) \\
&\quad - 2(m-1)(\tau^2+s)^{-1}k(\mu) + c^{-1}(4\tau^2 - \mu^2)x_m \\
&= 4\tau^2 - \mu^2 + \tau^2(x_1 - x_0) - 2(m-1)(\tau^2+s)^{-1}k(\mu) \\
&\quad - c^{-2}[(c-1)\mu^2 - 2(c-1-\tau^2)\mu \\
&\quad + (\tau^2-1)\{(c-1)^2 - \tau^2\} - 4\tau^2]x_m. \qquad (35)
\end{aligned}$$

Now, in case of feasibility, the factor

$$q := \frac{N}{M(\mu)} \tag{36}$$

is a rational number. Hence, we can use (35) as a necessary condition for eigenvalues to be an eigenvalue of the graph of a generalized Moore geometry. An eigenvalue must be such that the value of q defined by (35) and (36) is rational. Furthermore, since $k(\mu)$ and $x_1 - x_0$ are polynomials from $\mathbb{Q}[\mu]$ of the second degree and because of eq. (29) for x_m, condition (35) is equivalent to an algebraic equation over $\mathbb{Q}$ of the third degree. Hence, we state the following theorem.

Theorem 4.1. *If $L_m(s, t, c)$ is feasible, and $st > 1$, its eigenvalues are algebraic numbers of degree at most 3 with respect to $\mathbb{Q}$. Moreover, the eigenvalues μ_i, $1 \leqslant i \leqslant m$, satisfy the equation*

$$\left[\frac{(\tau^2+s)(4\tau^2-\mu^2)q}{\mu-\tau^2-1} + 2m(s\mu+\tau^2+s^2)\right][(c-1)\mu+\tau^2+(c-1)^2]$$
$$+ (\tau^2 - s(c-1))[(c-1+s)\mu + 2\tau^2 + 2s(c-1)] = 0. \tag{37}$$

Proof. We substitute $\tau^2(\tau^2+s)(x_1-x_0) = \tau^2(\mu-1+s)^2 - (\tau^2+s)^2$ and the expressions for $k(\mu)$ and x_m as follow from (20) and (29) respectively, in condition (35). After a straightforward calculation we get

$$\begin{aligned}
&[(\tau^2+s)(4\tau^2-\mu^2)q + 2m(\mu-\tau^2-1)(s\mu+\tau^2+s^2)][(c-1)\mu+\tau^2+(c-1)^2] \\
&= [s\mu^2 + 2s(s-1)\mu + (\tau^2-1)(\tau^2-s^2) - 4s^2][(c-1)\mu+\tau^2+(c-1)^2] \\
&\quad - [s\mu+\tau^2+s^2][(c-1)\mu^2 + 2(\tau^2-c+1)\mu \\
&\quad - (\tau^2-1)(\tau^2-(c-1)^2) - 4\tau^2] \\
&= [s(c-1)-\tau^2][(c-1+s)\mu^2 + \{2\tau^2+2s(c-1)-(c-1+s)(\tau^2+1)\}\mu \\
&\quad - 2(\tau^2+1)(\tau^2+s(c-1))] \\
&= [s(c-1)-\tau^2][\mu-\tau^2-1][(c-1+s)\mu+2\tau^2+2s(c-1)].
\end{aligned} \tag{38}$$

For eigenvalues $\mu \neq \tau^2+1$ it then follows that they satisfy eq. (37). By a straightforward calculation it can easily be shown that the only case in which the μ-dependent factors in (37) cancel is the case $s = t = 1$. Hence, if $st > 1$, the eigenvalues of $L_m(s, t, c)$ are of degree at most 3 with respect to $\mathbb{Q}$. □

Corollary 4.2. *The eigenvalues of the following generalized Moore geometries are at most quadratic with respect to $\mathbb{Q}$, if $st > 1$:*

(i) $\mathrm{GM}_m(s, t, 1)$,
(ii) $\mathrm{GM}_m(s, t, t+1)$,
(iii) $\mathrm{GM}_m(s, t, s+1)$,
(iv) $\mathrm{GM}_m(t, t, c)$.

Proof. Substitution of the various values of c and s shows that, apart from a factor $(\mu - \tau^2 - 1)^{-1}$, the left hand side of eq. (37) equals a product of a linear and a quadratic polynomial over $\mathbb{Q}$ in all four cases. □

Remarks 4.3. The explicit forms of the quadratic equations which can be split off from eq. (37) in the cases (i)–(iv) are omitted here. In case (iii) the resulting equation is identical with [11, eq. 6)], which was derived using properties of Chebyshev-polynomials (cf. [11] and the references cited there.) In cases (i), (ii) and (iv) the resulting equations are related to equations used in [5–9]. In all cited references the specific forms of the quadratic equations were applied to prove the infeasibility of the corresponding intersection matrices for certain values of m. If $s = t = 1$ one has that $c = 1$ or $c = 2$ and the corresponding graphs are the circuit graphs C_n. As is well-known these graphs can have eigenvalues of degree larger than 3 with respect to $\mathbb{Q}$ (see e.g. [3, Chapter 3]).

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