On the Inversion of the Sign of One Basis in an Oriented Matroid

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In this paper we study the operation of inversion of the sign of one basis in an oriented matroid. This operation generalizes for oriented matroids the "switching" of a triangle introduced by G. Ringel for arrangements of pseudolines.

1. INTRODUCTION

The problem of generating all possible orientations of a given orientable matroid is of great interest in oriented matroid theory. There are two important classes of matroids for which an answer is known: binary orientable matroids and uniform matroids of rank 3.

By [1, Proposition 6.21, all orientations of an orientable binary matroid can be obtained from any given one by reversing signs on subsets of elements.

The case of uniform matroids of rank 3 is a consequence of a result of G. Ringel [12] (see also [13]). Given any two simple arrangements $\mathcal{A}$ and $\mathcal{A}'$ of $n$ pseudolines in the projective plane $PR(2)$ there is a sequence of simple arrangements $\mathcal{A}_0, \mathcal{A}_1, ..., \mathcal{A}_k$ such that $\mathcal{A}_0$ and $\mathcal{A}$ (resp. $\mathcal{A}_k$ and $\mathcal{A}'$) determine isomorphic cell complexes and, $\mathcal{A}_i$ is obtained from $\mathcal{A}_{i-1}$ by switching a triangle. By the theorem of representability of J. Folkman and J. Lawrence [5] an arrangement of pseudolines is equivalent to a rank 3 oriented matroid.

Every oriented matroid can be characterized by a list of signs of its ordered bases, i.e., an orientation of its bases [7] (for an axiomatic treatment of ordered bases see [10]). Given a uniform oriented matroid of rank 3, switching a triangle in the corresponding arrangement of pseudolines amounts to inverting the sign of the basis constituted by the 3 pseudolines of the triangle (see [3]).
The generalization of the result of G. Ringel to higher dimensions has been conjectured by R. Cordovil and M. Las Vergnas [4]; any two orientations of an uniform matroid are related by a sequence of inversions of one basis sign.

In the present paper we make a first step in the direction of this conjecture by studying invertible bases of an oriented matroid. We show in Proposition 2.2 that the existence of an invertible basis imposes a strong condition on the underlying matroid (this condition is trivially satisfied by uniform matroids). Our main result is Theorem 3.1 which characterizes invertible bases in terms of acyclic reorientations.

As a consequence of this theorem the existence of an invertible basis in an oriented matroid is related to the following conjecture (M. Las Vergnas [8]): Every oriented matroid of rank \( r \) has an acyclic reorientation with exactly \( r \) extremal points. This conjecture is a generalization of a result of P. Camion [2] in the real case.

2. Notation and Preliminary Results

We will be using the notation of [1, 7, 8] with minor changes.

Let \( M \) be an oriented matroid over a finite set \( E \). We denote by \( \mathcal{C} \), resp. \( \mathcal{C}^{-} \), the set of signed circuits, resp. signed cocircuits of \( M \), and by \( \varepsilon \), resp. \( \varepsilon^{-} \) one of the two orientations of bases of \( M \) ([7, 10]).

An orientation \( \varepsilon \) of the bases of \( M \) is a mapping from the set of permutations of bases of \( M \) to \( \{-1, 1\} \). Since \( \varepsilon \) is alternating, for simplicity, in what follows, we always suppose that \( E \) is totally ordered. Given a basis \( B \) of \( M \) we will denote by \( \varepsilon(B) \), the value of \( \varepsilon \) on the permutation of \( B \) increasing with respect to the ordering of \( E \).

Let \( \varepsilon \) be an orientation of the bases of a matroid \( M \). We denote by \( \varepsilon_{B} \) the mapping from the set \( \mathcal{B} \) of ordered bases of \( M \) to \( \{-1, 1\} \) defined by

\[
\varepsilon_{B}(B) = -\varepsilon(B)
\]

\[
\varepsilon_{B}(B') = \varepsilon(B') \quad \text{if} \quad B' \neq B.
\]

If \( \varepsilon_{B} \) is an orientation of the bases of \( M \) we say that \( \varepsilon_{B} \) is the reorientation of \( M \) obtained from \( \varepsilon \) by inversion of the sign of \( B \) and we say that \( B \) is an invertible basis of \( M \).

If \( B \) is an invertible basis of \( M \) we denote by \( M_{B} \) the oriented matroid whose orientation is defined by \( \varepsilon_{B} \). We will denote by \( \mathcal{C}_{B} \), resp. \( \mathcal{C}^{\perp}_{B} \), the sets of signed circuits, resp. signed cocircuits, of \( M_{B} \) and by \( \varepsilon^{\perp}_{B} \) one of the two orientations of the cobasis of \( M_{B} \).
PROPOSITION 2.1. Let $M$ be an oriented matroid, without loops over a set $E$. Suppose that $B$ is an invertible basis of $M$ and that for some $b \in B$ the hyperplane $H = \overline{B} - b$ satisfies $(E - B) \cap H \neq \emptyset$. Then $E - H = \{b\}$; i.e., $b$ is an isthmus of $M$.

Proof. Let $B$ be an invertible basis of $M$ and suppose that for some $b \in B$ the hyperplane $H = \overline{B} - b$ satisfies $(E - B) \cap H \neq \emptyset$. Suppose there is an element $y \neq b$ such that $y \in E - H$.

Let $Y$, resp. $Y'$, be a signed cocircuit of $M$, resp. $M_b$ with support $E - H$. By hypothesis $B_0 = B - b$ is a basis of $H$ and, since $(E - B) \cap H \neq \emptyset$ and $M$ has no loops, there is a basis $B_1$ of $H$, $B_1 \neq B_0$.

By definition of $e_B$ we obtain the following equalities relating the signs of $b$ and $y$ in $Y$ and $Y'$:

\begin{align*}
(a) \quad &\text{using } B_0, \quad sg_{Y'}(b) \cdot sg_{Y}(y) = e_B(b, B_0) e_B(y, B_0) = -e(b, B_0) e(y, B_0) = -sg_{Y}(b) \cdot sg_{Y}(y) \\
(b) \quad &\text{using } B_1, \quad sg_{Y'}(b) sg_{Y}(y) = e_B(b, B_1) e_B(y, B_1) = e(b, B_1) e(y, B_1) - sg_{Y}(b) sg_{Y}(y) \text{ contradicting (a).}
\end{align*}

PROPOSITION 2.2. Let $M$ be an oriented matroid without loops. Suppose $B$ is an invertible basis of $M$ and let $B'$ be the set of isthmuses of $M$. Then for every $x \in E - B$, $(B - B') \cup \{x\}$ is a circuit of $M$.

This proposition is an immediate consequence of Proposition 2.1 and of the definition of isthmus.

Remark 2.3. Proposition 2.2 shows that the existence of an invertible basis in an oriented matroid imposes a strong condition on the underlying matroid. We give next examples of particular situations:

(2.3A) Family of oriented matroids which have no invertible basis.
Consider $E = \{e_0, e_1, \ldots, e_n\} \subseteq \mathbb{R}^{r-1}$, where $e_0 = (0, \ldots, 0)$, $e_1 = (1, 0, \ldots, 0)$, ..., $e_{r-1} = (0, \ldots, 0, 1)$ and for $n > r - 1$, $n = k(r - 1) + p$ with $0 < p \leq r - 1$, $e_n = (k + 1)p$. For any $n > 2r - 1$, $\text{Aff}(E)$, the matroid of affine dependencies of $E$ over $\mathbb{R}$, has no basis verifying the condition of Proposition 2.2 and hence has no invertible bases.

(2.3B) Family of oriented matroids which have invertible bases but admit two different orientations which cannot be related by successive inversions of the sign of one basis.
Let $E \subseteq \mathbb{R}^{r-1}$ be one of the subsets of the preceding example. Choose $p, p_1, p_2 \in \mathbb{R}^{r-1}$ such that $p, p_1 \in \text{poscon}(E)$ with $p$ and $p_1$ in general position in $E \cup \{p, p_1\}$, $p_2 \in \mathbb{R}^{r-1} - \{\text{poscon}(E) \cup \text{negcon}(E)\}$, with $p$ and $p_2$ in general position in $E \cup \{p, p_2\}$.

Let $M' = \text{Aff}(E \cup \{p, p_1\})$ and $M = \text{Aff}(E \cup \{p, p_2\})$. It is clear that $M$
and \( M' \) have the same underlying matroid and it is not difficult to verify that they do have invertible bases. To show that \( M' \) cannot be obtained from \( M \) by successive inversions of the sign of one basis consider for \( i = 1, \ldots, r - 1 \) the hyperplane \( H_i = \text{aff}(\{e_0, e_1, \ldots, e_i, \ldots, e_r\}) \). In \( M \) for all \( i = 1, \ldots, r - 1 \), \( H_i \) does not separate \( p \) from \( p_1 \). In \( M' \) there is a pair \( i_0, j_0 \in \{1, \ldots, r - 1\} \) such that \( H_{i_0} \) separates \( p \) and \( p_2 \) and \( H_{j_0} \) does not separate \( p \) and \( p_2 \), implying that to obtain \( M' \) from \( M \) it would be necessary to change, for some \( i = 1, \ldots, r - 1 \) a basis of the form \( B_{i_0} + p_i \), where \( B_{H_i} \) is a basis of \( H_j, j = 1, 2 \). Those bases do not fulfill the condition of Proposition 2.2.

(2.3C) When the underlying matroid is a uniform matroid, every basis satisfies the condition of Proposition 2.2, and in this case given two representable orientations \( \mathcal{O} \) and \( \mathcal{O}' \) of the uniform matroid \( \bigcup_{n_r} \), it is always possible to find a sequence of basis \( B_1, \ldots, B_k \) such that, if \( \mathcal{O}_0 = \mathcal{O} \) and for \( i = 1, \ldots, k \), \( \mathcal{O}_i = \mathcal{O}_{i-1} \), \( \mathcal{O}_i \) is a representable orientation of \( \bigcup_{n_r} \) and \( \mathcal{O}_k = \mathcal{O}' \).

Indeed, suppose \( E = \{e_1, \ldots, e_n\} \) and \( E' = \{e'_1, \ldots, e'_n\} \) are two sets of points in general position in \( \mathbb{R}^{r-1} \). Consider \( \mathcal{O} = \text{Aff}(E) \) and \( \mathcal{O}' = \text{Aff}(E') \). Let \( \mathcal{H}_i = \{H_i : H \text{ hyperplane of } \mathcal{O}, e_i \notin H\} \), \( \mathcal{H}'_i = \{H'_i : H'_i \text{ hyperplane of } \mathcal{O}', e'_i \notin H'_i\} \), and \( L_1 \) the straight line \( L_1 = e_1 + \lambda(e'_1 - e_1) \). If necessary replacing \( e'_1 \) by a point \( p'_1 \) in the same cell of \( \mathbb{R}^{r-1} - \text{aff}(H'_1) \), we can suppose \( L_1 \) intersects the hyperplanes of \( \mathcal{H}_i \) one by one. Let \( H_1, \ldots, H_{k_1} \) be the sequence of hyperplanes of \( \mathcal{H}_i \) intersecting \( L_1 \) between \( e_1 \) and \( e'_1 \), corresponding to the sequence \( 0 < \lambda_1 < \cdots < \lambda_{k_1} < 1 \) in the parametrisation of \( L_1 \).

For \( j = 1, \ldots, k_1 \), let \( B_j = H_j \cup e_1 \) then \( \mathcal{O}_j = \mathcal{O}_j. \mathcal{O}_j \mathcal{O}_j \) is a representable orientation of \( \bigcup_{n_r} : \mathcal{O}_j = \text{Aff}(E, e_1 + x_j), \) where \( x_j = e_1 + x_j(e'_1 - e_1), \lambda_j < x_j < \lambda_{j+1} \), and \( \mathcal{O}_{k_1} = \text{Aff}(E - e_1 + e'_1) \). Iterating this procedure we obtain \( \mathcal{O}' \) from \( \mathcal{O} \) by successive inversions of one basis.

This simple argument shows that the conjecture of R. Cordovil and M. Las Vergnas is true if we consider representable orientations. From this point of view the result of Ringel [12] on arrangements of pseudolines states the veracity of this conjecture for rank 3 uniform matroids.

The next proposition gives a characterization of invertible bases of an oriented matroid in terms of oriented circuits and cocircuits.

It is an easy consequence of the orthogonality between the families of oriented circuits and cocircuits of an oriented matroid (see [1]) and the canonical correspondence between orientation of circuits and orientations of bases (see [5, 10]).

**Proposition 2.4.** Let \( M \) be a matroid without loops and isthmuses over a set \( E \). Then the following propositions are equivalent:

(i) \( B \) is an invertible basis of \( M \).
(ii) \( E - B \) is an invertible basis of \( M^\perp \).

(iii) For every \( x \in E - B \), \( B + x \) is a circuit of \( M \) and \( \mathcal{C}_B \) the signature of circuits of \( H \) defined by: \( X = (X^+, X^-) \in \mathcal{C}_B \) if either \( B \subseteq X \) and \( B X \in \mathcal{C} \) or \( B \not\subseteq X \) and \( X \in \mathcal{C} \), is an orientation of the circuits of \( H \).

(iv) For every \( b \in B \), \( (E - B) + b \) is a cocircuit of \( M(E) \) and \( \mathcal{C}_{E - B}^\perp \) the signature of cocircuits of \( M(E) \) defined by \( Y = (Y^+, Y^-) \in \mathcal{C}_{E - B}^\perp \) if either \( (E - B) \subseteq Y \) and \( Y \in \mathcal{C}^\perp \) or \( (E - B) \not\subseteq Y \) and \( Y \in \mathcal{C}^\perp \), is an orientation of the cocircuits of \( M \).

Results obtained for matroids without loops and isthmuses can be easily extended to general matroids.

3. The Main Results

Let \( M \) be an oriented matroid of rank \( r \) on a set \( E \). By the theorem of representability of J. Folkman and J. Lawrence [5] we can associate with \( M \) an arrangement of pseudohyperplanes in \( \mathbb{P} \mathbb{R}(r - 1) \), \( \mathcal{H} = \{ H_e \}_{e \in E} \), such that there is a one-to-one correspondence between the maximal cells of the cell complex \( \mathcal{C}(M) \) determined by the arrangement \( \mathcal{H} \) in \( \mathbb{P} \mathbb{R}(r - 1) \) and the set of acyclic reorientations of \( M \). More precisely, to each maximal cell of \( \mathcal{C}(M) \) supported by the pseudohyperplanes \( \{ H_e \}_{e \in E} \), \( E' \subseteq E \) corresponds an acyclic reorientation of \( M \) whose extremal points are exactly the points of \( E' \).

Theorem 3.1 below characterizes the invertible bases of an oriented matroid \( M \) in terms of acyclic reorientations of \( M \). It says that \( B \) is an invertible basis of \( M \) if there is an \( r \)-simplex \( s \) in \( \mathcal{C}(M) \) supported by the pseudohyperplanes \( \{ H_e \}_{e \in B} \) such that every face of dimension \( k \), \( 0 \leq k \leq r \) is the intersection of exactly the \( r - k \) pseudohyperplanes of \( s \) containing it.

In particular, this theorem shows that to reverse the sign of an invertible basis of an oriented matroid corresponds, in Lawrence's representation, to slightly modifying any one of the pseudohyperplanes \( H_e, e \in B \), in such a way that the point \( V = \bigcap_{e \in B - e} H_e \), is in the new cell complex on the "other side" of \( H_e \) and the only cells which are affected are those in the "neighborhood" of \( V \).

The notion of local perturbation of an oriented matroid was introduced by K. Fukuda [6] and A. Mandel [11].

In Corollary 3.6 we relate inversions of the sign of one basis, which are local perturbations that do not change the underlying matroid with point perturbations, which topologically are also slight modifications of one hyperplane in the neighborhood of a point, but which do change the underlying matroid.
We remark that once Theorem 3.1 is proved, Lemma 3.2 gives a practical way of finding the invertible bases of an oriented matroid.

**Theorem 3.1.** Let $M$ be a simple matroid of rank $r$ on a set $E$. Then $B$ is an invertible basis of $M$ if and only if:

(i) For every $x \in E - B$, $B + x$ is a circuit of $M$ and

(ii) there is a subset $A \subseteq E$ such that $\lambda A M$ is acyclic with exactly $r$ extremal points, which are the points of $B$.

To prove this theorem we need some preliminary results. We remark that we have already proved (in Proposition 2.2) that if $B$ is an invertible basis of a simple matroid, then condition (i) of Theorem 3.1 is satisfied.

**Lemma 3.2.** Let $M$ be a simple matroid of rank $r$ on a set $E$. Let $B$ be an invertible basis of $M$. Then there is a partition of $B$ into two disjoint subsets $B^+$, $B^-$ such that every signed circuit $(X', X^-)$ of $M$ with support $B + x$, $x \in E - B$, satisfies one of the following conditions:

(a) $X^+ \cap B = B^+$ and $X^- \cap B = B^-$

(b) $X^+ \cap B = B^-$ and $X^- \cap B = B^+$.

**Proof.** To prove this lemma it is enough to show that under the hypothesis, given two signed circuits of $M$, $X = (X^+, X^-)$ and $Y = (Y^+, Y^-)$ with supports respectively $B + x$ and $B + y$ if $(X^+ \cap Y^-) \cup (X^- \cap Y^+) \neq \emptyset$ then $(X^+ \cap Y^+) \cup (X^- \cap Y^-) = \emptyset$.

We are going to prove this by contradiction:

Let $X = (X^+, X^-)$ and $Y = (Y^+, Y^-)$ be two signed circuits of $M$ with supports respectively $B + x$, $B + y$, $x \neq y$ and suppose that $(X^+ \cap Y^-) \cup (X^- \cap Y^+) \neq \emptyset$ and $(X^+ \cap Y^+) \cup (X^- \cap Y^-) \neq \emptyset$.

Let $b \in (X^+ \cap Y^-) \cup (X^- \cap Y^+)$ and $b' \in (X^+ \cap Y^+) \cup (X^- \cap Y^-)$. By definition of orientation of circuits there is a circuit $Z \in \mathcal{C}$ such that $b' \in Z$, $Z^+ \subseteq (X^+ \cup Y^+)^- b$ and $Z^- \subseteq (X^- \cup Y^-)^- b$. By the minimality of $X$ and $Y$, $x, y \in Z$ and the following relations are verified:

$$sg_{\mathcal{C}}(b') = sg_{X}(b') = sg_{Y}(b')$$

$$sg_{\mathcal{C}}(x) = sg_{X}(x)$$

$$sg_{\mathcal{C}}(y) = sg_{Y}(y).$$

Since by hypothesis $B$ is an invertible basis of $M$, using Proposition 2.4(iii), $\mathcal{C}_B$ is an orientation of circuits of $M$ and $X' = \chi X$, $Y' = \chi Y$ are elements of $\mathcal{C}_B$ with $b \in (X''^+ \cap Y'^-) \cup (X''^- \cap Y'^+)$ and $b' \in (X'^+ \cap Y'^+) \cup (X'^- \cap Y'^-)$. Hence there is a signed circuit $Z' \in \mathcal{C}_B$.
such that \( b' \in Z' \), \( Z'^+ \subseteq (X'^+ \cup Y'^+) - b \), \( Z'^- \subseteq (X'^- \cup Y'^-) - b \), and \( Z' \) satisfies the following relations.

\[
\begin{align*}
\text{sg}_Z(b') &= \text{sg}_X(b') = \text{sg}_Y(b') = \text{sg}_Z(b') \\
\text{sg}_X(x) &= \text{sg}_Y(x) = -\text{sg}_Z(x) \\
\text{sg}_Z(y) &= \text{sg}_Y(x) = -\text{sg}_Z(y)
\end{align*}
\]

By definition of \( \mathcal{C}_B \), \( Z' \in \mathcal{C} \). Since \( B \uplus x \) is a circuit of \( M \), \( B_1 = B - b + x \) is a basis of \( M \); hence there is only one circuit of \( M \) with support contained in \( B_1 + y \), therefore \( Z' = \pm Z \), contradicting (\(*\)).

The next proposition will be used in the proof of Lemma 3.4 and recalls the fundamental results of [8].

**Proposition 3.3.** Let \( M \) be a simple oriented matroid of rank \( r \geq 2 \) on a set \( E \). Let \( B \) be a basis of \( M \) such that for every \( x \in E - B \) the signed set \( (B, \{x\}) \) is a signed circuit of \( M \). Then

(i) \( M \) is acyclic with exactly \( r \) extremal points which are the points of \( E \).

(ii) Every signed circuit \( X = (X^+, X^-) \) of \( M \) such that \( B \nsubseteq X \) satisfies \( X^+ \cap (E - B) \neq \emptyset \) and \( X^- \cap (E - B) \neq \emptyset \).

**Lemma 3.4.** Let \( M \) be a simple oriented matroid of rank \( r \) over a set \( E \). Let \( B \) be a basis of \( M \) such that for every \( x \in E - B \) the signed set \( (B, \{x\}) \) is a signed circuit of \( M \). Then \( B \) is an invertible basis of \( M \).

**Proof.** By hypothesis for every \( x \in E - B \), \( (B, \{x\}) \) is a signed circuit of \( M \). By Proposition 2.4(iii) we only need to prove that \( \mathcal{C}_B \) is an orientation of the circuits of \( M \).

By [9, Theorem 2.11], to prove that \( \mathcal{C}_B \) is an orientation of the circuits of \( M \) it is enough to verify that every modular pair of signed circuits of \( \mathcal{C}_B \) satisfies the elimination property. Thus we are going to prove that for every modular pair of signed circuits \( X, Y \) of \( \mathcal{C}_B \) and every \( x \in X^+ \cap Y^- \) there is a signed circuit \( Z \in \mathcal{C}_B \) such that

\[
Z^+ \subseteq (X^+ \cup Y^+) - x \quad \text{and} \quad Z^- \subseteq (X^- \cup Y^-) - x.
\]

We recall that \( X, Y \) is a modular pair of (signed) circuits of \( M \) if \( r(X \cup Y) + r(X \cap Y) = r(X) + r(Y) \), where \( r \) is the rank function of \( M \). This implies, in particular, that for every \( x \in X \cap Y \), \( (X \cap Y) - x \) contains exactly one circuit (two opposite signed circuits).

Let \( X, Y \) be a modular pair of signed circuits of \( \mathcal{C}_B \). If \( B \nsubseteq X \cup Y \), by definition of \( \mathcal{C}_B \), \( X \) and \( Y \) are signed circuits of \( \mathcal{C} \). If \( x \in X^+ \cap Y^- \) let \( Z \in \mathcal{C} \)
be a signed circuit of $\mathcal{C}$ such that $Z \subseteq (X \cup Y) - x$. Then $B \not\subseteq Z$ and $Z \in \mathcal{C}_B$ satisfies (*).

If $B \subseteq X \cup Y$, we consider separately the following three cases: $X, Y \in \mathcal{C}_B - \mathcal{C}$; $X \in \mathcal{C}_B - \mathcal{C}$ and $Y \in \mathcal{C}_B \cap \mathcal{C}$; $X, Y \in \mathcal{C}_B \cap \mathcal{C}$.

(1) If $X, Y \in \mathcal{C}_B - \mathcal{C}$ we can suppose that $X - (B + x, \emptyset)$ and $Y = (\emptyset, B + y)$, $x, y \in E - B$, $x \neq y$. Let $b \in B = X^+ \cap Y^-$. By elimination of $b$ between the circuits $(B, x)$ and $(y, B)$ of $\mathcal{C}$, there is a signed circuit $Z \in \mathcal{C}$ with $Z^+ \subseteq B + y - b$ and $Z^- \subseteq B + x - b$. Since $b \notin Z$, $Z \in \mathcal{C}_B$ and satisfies (*).

(2) If $X \in \mathcal{C}_B - \mathcal{C}$ and $Y \in \mathcal{C}_B \cap \mathcal{C}$. Being, by hypothesis, $X, Y$ a modular pair of circuits we can suppose that the support of $Y$ is of the form $Y = B_0 \cup \{x, y\}$ with $x, y \in E - B$, $x \neq y$ and $B_0 \not\subseteq B$, and that $X = (B + x, \emptyset)$.

Let $B_0^+$ and $B_0^-$ be the sets defined by $B_0^+ = B_0 \cap X^+$, $B_0^- = B_0 \cap Y^-$. By Proposition 3.3(ii) either $Y = (B_0^+ + x, B_0^- + y)$ or $Y = (B_0^+ + y, B_0^- + x)$.

We suppose $Y = (B_0^+ + y, B_0^- + x)$. The other case is analogous.

The elimination of $x \in X^+ \cap Y^-$ has no problems since $Z = (B + y, \emptyset)$ is a circuit of $\mathcal{C}_B$.

Let $b \in B_0^- = (X^+ \cap Y^-)$. $X(x) - (B, x)$, $X(y) - (y, B)$ are, by hypothesis, signed circuits of $M$ and $b \notin X(x)^+ \cap X(y)^-$. So there is a signed circuit $Z$ of $M$ such that $Z^+ \subseteq B + y - b$, $Z^- \subseteq B + x - b$.

Denoting by $B_1^+$ and $B_1^-$ the sets $B_1^+ = B \cap Z^+$, $B_1^- = B \cap Z^-$, $Z$ is the circuit $Z = (B_1^+ + y, B_1^- + x)$ which, since $b \notin Z$, is a signed circuit of $\mathcal{C}_B$.

To prove that $Z$ satisfies (*) we are going to show that $B_1^- \subseteq B_0^-$. $Y$ and $-Z$ are signed circuits of $\mathcal{C}$, eliminating $x$ between $Y$ and $-Z$ one of the circuits $(B, y)$ or $(y, B)$ satisfies (*) (for $X = -Z$). Since $b \in Y^- - Z$, it is the circuit $(y, B)$ which satisfies (*), implying that $B \subseteq B_0^- \cup B_1^+$ and $B - B_1^+ \subseteq B_0^-$. Since $B_1^- \cup B_1^+ \not\subseteq B$ and $B_1^- \cap B_1^+ = \emptyset$ we have $B_1^- \subseteq B_0^-$.

(3) Let us consider now a modular pair of circuits $X, Y$ with $X, Y \in \mathcal{C}_B \cap \mathcal{C}$ (i.e., $B \not\subseteq X$ and $B \not\subseteq Y$). For every $x \in X^+ \cap Y^-$ there is a signed circuit $Z \in \mathcal{C}$ such that $Z^+ \subseteq (X^+ \cap Y^+) - x$ and $Z^- \subseteq (X^- \cup Y^-) - x$. If $B \not\subseteq Z$ then $Z \in \mathcal{C}_B$ verifying (*). If $B \subseteq Z$ then $Z$ is of the form $(B, y)$ or $(y, B)$ for some $y \in E - B$. By hypothesis $X, Y$ are a modular pair of circuits. $B + y$ is the only circuit contained in $(X \cup Y) - x$ if and only if $X \cap (E - B) = \{x, y\} = Y \cap (E - B)$. By Proposition 3.3(ii) since $x \in X^+ \cap Y^-$ we must have $y \in X^- \cap Y^+$, thus $Z' = y, Z$ is the circuit of $\mathcal{C}_B$ satisfying (*).

Proof of Theorem 3.1. Let $B$ be an invertible basis of $M$. Let $B^1$, $B$ be the partition of $B$ satisfying the conditions of Lemma 3.2.

Let $\mathcal{X}$ be the set of signed circuits of $M$ defined by $\mathcal{X} = \{X \in \mathcal{C}: B \subseteq X$
and $X^+ \cap B = B^+$ and $X^- \cap B^- = B^-$. Let $A_1$ be the subset of $E$ defined by $A_1 = \{ x \in E - B : x \in X^+ \text{ for some } X \in \mathcal{X} \}$ and let $A = A_1 \cup B^-$. Let $\bar{M}$ contains the signed circuits $(B, x)$ for every $x \in E - B$. By Proposition 3.3(i) $\bar{M}$ is acyclic with exactly $r$ extremal points which are the points of $B$.

It follows from the definition that $B$ is an invertible basis of $M$ if and only if for every $A \subseteq E$, $B$ is an invertible basis of $\bar{M}$. The converse part of the theorem is then an immediate consequence of Proposition 3.3 and Lemma 3.4.

Given an oriented matroid $M = (E, \varepsilon)$ and a basis $B$ of $M$, let $\varepsilon_B : \mathcal{B} \to \{0, 1, -1\}$ be defined as

$$
\varepsilon_B^0 (B') = \begin{cases} 
\varepsilon (B') & \text{if } B' \neq B \\
0 & \text{if } B' = B.
\end{cases}
$$

Corollary 3.6, below, states that if $B$ is an invertible basis of $M$ then $\varepsilon_B^0$ defines an orientation of bases of a new matroid $M_B^0$. This operation of passing from $M$ to $M_B^0$ can be seen as a particular case of inversion of a point perturbation.

Theorem 3.5 is a version of point perturbation theorem of K. Fukuda and A. Mandel:

**Theorem 3.5** (Point Perturbation Theorem, K. Fukuda [6], A. Mandel [11]). Let $M$ be an oriented matroid over a set $E$, $\mathcal{C}^\perp$ the family of its signed cocircuits.

Suppose $H$ is an hyperplane of $M$ and $h$ is an element of $H$ such that:

(a) $h$ is in general position in $H$

(b) The closure of $H - h$ in $M$ is $H$.

Consider $\mathcal{L} = \{ L \subseteq H : L \text{ is an hyperline of } M \text{ and } h \notin L \}$. For every $L \in \mathcal{L}$ let $(H_L^+, H_L^-)$ be the signed cocircuit of the restriction of $M$ to $H$, such that $h \in H_L^-$. Then for any of the two signed cocircuits $X$ of $M$ with support $E - H$, $\mathcal{C}^\perp = (\mathcal{C}^\perp - \{ \pm X \}) \cup \{ \pm \tilde{X} \} \cup \{ \pm X_L \}_{L \in \mathcal{L}}$ is the set of cocircuits of an oriented matroid over $E$, where $\tilde{X} = (X^+ + h, X^-)$ and $X_L = (X^+ + H_L^+, X^- + H_L^-, h)$, $L \in \mathcal{L}$.

**Corollary 3.6.** Let $M$ be an oriented matroid, $B$ an invertible basis of $M$. Then $\varepsilon_B^0$ is an orientation of bases of a matroid $M_B^0$ with set of bases $\mathcal{B} - \{B\}$, $\mathcal{B}$ being the set of bases of $M$. Moreover $M$ and $M_B$ are obtained from $M_B^0$ by point perturbation of $b \in B$ with respect to the hyperplane $B$.

**Proof.** If $B = \{ b_1, ..., b_r \}$ is an invertible basis of $M$ then $E - B$ is an invertible basis of $M^\perp$ and by Lemma 3.2 there is a partition of $E - B$ in
two disjoint subsets \((E - B)^+, (E - B)^-\) such that for every \(i = 1, ..., r\) one of the signed cocircuits \(X_i\) with support \(E - B + b_i\) satisfies \(X_i^+ - b_i = (E - B)^+\) and \(X_i^- - b_i = (E - B)^-\).

It is an easy consequence of this fact to prove that \(C^0_B = (C^\perp - \{ \pm X_i \}) \cup \{ \pm ((E - B)^+, (E - B)^-)\}\) is then an orientation of cocircuits of a matroid \(M^0_B\), whose orientation of basis is \(e^0_B\) and also that \(M\) and \(M^0_B\) are obtained by point perturbation of \(M^0_B\).

Note added in proof. J. P. Roudneff and B. Sturmfels give [14] an alternative proof of Theorem 3.1 for the uniform case using bases axioms. Applying Lemma 3.2 this proof is easily adaptable to the general case.

REFERENCES

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