Extension operators via semigroups

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A B S T R A C T

The Roper–Suffridge extension operator and its modifications are powerful tools to construct biholomorphic mappings with special geometric properties. The first purpose of this paper is to analyze common properties of different extension operators and to define an extension operator for biholomorphic mappings on the open unit ball of an arbitrary complex Banach space. The second purpose is to study extension operators for starlike, spirallike and convex in one direction mappings. In particular, we show that the extension of each spirallike mapping is A-spirallike for a variety of linear operators A. Our approach is based on a connection of special classes of biholomorphic mappings defined on the open unit ball of a complex Banach space with semigroups acting on this ball.

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1. Introduction

One of the main purposes of the classical geometric function theory is the study of various classes of univalent and multivalent mappings. Convex, starlike and spirallike functions on the open unit disk $\Delta \in \mathbb{C}$ have been the objects of intensive study for over a century. A reader can be referred to the book of Goodman [8]. The study of different classes of biholomorphic mappings in multidimensional settings began later. In fact, the first survey appeared in 1977 (see [21]). Recent developments in this area are reflected in [7,11,4,19]. However, numerous well-known tools for the construction of mappings with special geometric properties on the open unit disk $\Delta := \{z \in \mathbb{C}: |z| < 1\}$ have no generalization for the multidimensional case. For example, until recently only a few concrete examples of convex, starlike and spirallike mappings in the open unit ball in $\mathbb{C}^n$ were known.

In 1995, Roper and Suffridge [20] introduced an extension operator, which provides a variety of required examples. Given a univalent function $f \in \text{Hol}(\Delta, \mathbb{C})$ normalized by $f(0) = f'(0) - 1 = 0$, they considered the mapping $\Phi[f]: \mathbb{B}^n \mapsto \mathbb{C}^n$ defined as follows:

$$\Phi[f](z_1, x) = \left(f(z_1), \sqrt{f'(z_1)}x\right), \tag{1.1}$$

where $x = (z_2, \dots, z_n)$. The Roper–Suffridge extension operator has remarkable properties. In particular:

- if f is a normalized convex function on Δ, then $\Phi[f]$ is a normalized convex mapping on $\mathbb{B}^n$, see [20];
- if f is a normalized starlike function on Δ, then $\Phi[f]$ is a normalized starlike mapping on $\mathbb{B}^n$, see [10];
- if f is a normalized μ-spirallike function on Δ, then $\Phi[f]$ is a normalized μI-spirallike mapping on $\mathbb{B}^n$, see, for example, [12,14];
- if f is a normalized Bloch function on Δ, then $\Phi[f]$ is a normalized Bloch mapping on $\mathbb{B}^n$, see [10].

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Several authors have discussed this operator and its generalizations. In particular, the operator

$$\Phi_\alpha[f](z_1, x) = \left(f(z_1), \left(f'(z_1)\right)^\alpha x\right),\tag{1.2}$$

where $\alpha \in [0, \frac{1}{2}]$, was introduced in [12].

For a locally biholomorphic mapping f defined on the unit ball of $\mathbb{C}^n$, Pfaltzgraff and Suffridge constructed in [17] an extension operator as follows:

$$\widehat{\Phi}_n[f](z, x) = \left(f(z), \left(J_f(z)\right)^{\frac{1}{n+1}} x\right),\tag{1.3}$$

where $z \in \mathbb{C}^n$, $x \in \mathbb{C}$, $\|z\|^2 + |x|^2 < 1$, and $J_f(z)$ is the complex Jacobian of the mapping f at the point z. It was shown in [13] that this operator preserves starlikeness.

Another extension operator was introduced in [10] for locally biholomorphic functions $f \in \text{Hol}(\Delta, \mathbb{C})$ by

$$\widetilde{\Phi}_\beta[f](z_1, x) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^\beta x\right),\tag{1.4}$$

where $\beta \in [0, 1]$. These extension operators and their combinations (with multiplier $(f'(z))^{\alpha_j}(\frac{f(z)}{z})^{\beta_j}$ in j-th coordinate) in the space $\mathbb{C}^n$ equipped with different concrete norms have been considered in numerous papers. Detailed references can be found in [6].

Note that, as we updated, all extension operators were studied for functions f satisfying the standard normalization $f(0) = 0$ and $f'(0) = 1$ (or $J_f(0) = \text{id}$, respectively).

The first purpose of this paper is to analyze common properties of different extension operators and to define an extension operator for biholomorphic mappings on the open unit ball of an arbitrary complex Banach space.

The second purpose is to study extension operators for mappings starlike or spirallike with respect to an arbitrary interior or a boundary point (see definitions in Section 2). Although the case of spirallikeness with respect to an interior point can often be reduced to a standard one ($f(0) = 0$), extension operators for starlike and spirallike mappings with respect to a boundary point have not been considered at all. The following effect is new even for the case of $f(0) = 0$: we show that the extension of each spirallike function is A-spirallike for a variety of linear operators A.

Our approach is based on several simple but effective observations:

(1) All extension operators mentioned above have the form:

$$f(x) \mapsto \left(f(x), \Gamma(f, x) y\right)$$

with a certain linear operator Γ depending on a mapping f and a point x. So, we have to understand which properties of Γ enable us to use it to construct an extension operator. We will say that operators having such properties are appropriate.

(2) A biholomorphic mapping is A-spirallike if and only if its image is S-invariant, where $S = \{e^{-tA}\}_{t \geqslant 0}$ is the semigroup of linear transformations. Similar relations between biholomorphic mappings and special semigroups also exist for other classes of biholomorphic mappings. Therefore, we must study extension operators for one-parameter continuous semigroups.

(3) Extension operators for a semigroup of biholomorphic self-mappings of the open unit ball and for a corresponding class of biholomorphic mappings do not necessarily coincide.

2. Preliminary notions

In this section we present some notions of non-linear analysis and geometric function theory which will be useful subsequently. A reader may be referred to as the book [19].

Let X be a complex Banach space with the norm $\|\cdot\|$. Denote by $\text{Hol}(D, E)$ the set of all holomorphic mappings on a domain $D \subset X$ which map D into a set $E \subset X$ and a set $\text{Hol}(D) := \text{Hol}(D, D)$.

We start with the notion of a one-parameter continuous semigroup.

Definition 2.1. Let D be a domain in a complex Banach space X. A family $S = \{F_t\}_{t \geqslant 0} \subset \text{Hol}(D)$ of holomorphic self-mappings of D is said to be a one-parameter continuous semigroup (in short, semigroup) on D if

$$F_{t+s} = F_t \circ F_s, \quad t, s \geqslant 0,\tag{2.1}$$

and for all $x \in D$,

$$\lim_{t \to 0^+} F_t(x) = x.\tag{2.2}$$

For example, if D is the unit ball of X and $A \in L(X)$ is an accretive operator, then the family $\{e^{-tA}\}_{t \geqslant 0}$ forms a semigroup of proper contractions of D. Moreover, each uniformly continuous semigroup of bounded linear operators can be represented by this form.

Definition 2.2. A semigroup $S = \{F_t\}_{t \geqslant 0}$ on D is said to be generated if for each $x \in D$, there exists the strong limit

$$f(x) := \lim_{t \to 0^+} \frac{1}{t}\big(x - F_t(x)\big). \tag{2.3}$$

In this case the mapping $f : D \mapsto X$ is called the (infinitesimal) generator of the semigroup S.

It was established in [18] that *a semigroup S of holomorphic self-mappings of D is differentiable with respect to the parameter $t \geqslant 0$ (hence, generated by a holomorphic mapping) if and only if it is locally uniformly continuous on D.*
The following notion connects semigroups on biholomorphically equivalent domains.

Definition 2.3. Let $\{F_t\}_{t \geqslant 0}$ and $\{\Psi_t\}_{t \geqslant 0}$ be semigroups on domains D and Ω of a complex Banach space, respectively. We say that the semigroups are conjugate if there is a biholomorphic mapping $h : D \mapsto \Omega$ such that

$$h \circ F_t = \Psi_t \circ h.$$

The mapping h in this relation is called the intertwining map for the semigroups.

An important class of mappings which serve intertwining maps with semigroups of linear transformations is the class of spirallike mappings.

Definition 2.4. (See [4,19], cf. [21,7,11].) Let h be a biholomorphic mapping defined on a domain D of a Banach space X. The mapping h is said to be spirallike if there is a bounded linear operator A such that the function $\operatorname{Re}\lambda$ is bounded away from zero on the spectrum of A and such that for each point $w \in h(D)$ and each $t \geqslant 0$, the point $e^{-tA}w$ also belongs to $h(D)$. In this case h is called A-spirallike. If A can be chosen to be the identity mapping, that is, $e^{-t}w \in h(D)$ for all $w \in h(D)$ and all $t \geqslant 0$, then h is called starlike.

In other words, a biholomorphic mapping $h \in \operatorname{Hol}(D, X)$ is A-spirallike if and only if it intertwines some semigroup on D with the semigroup $\{e^{-At}\}_{t \geqslant 0}$.

Remark 1. For mappings defined on the direct product $Z = X \times Y$ of two Banach spaces X and Y, it is relevant to consider a block-matrix $A = \left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right)$ with operators $A_{11} \in L(X)$, $A_{12} \in L(Y, X)$, $A_{21} \in L(X, Y)$ and $A_{22} \in L(Y)$ satisfying certain conditions. In such situation, the notion of "$\left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right)$-spirallikeness" should be understood in the same sense of Definition 2.4.

It follows by this definition that if h is an A-spirallike mapping then $0 \in \overline{h(D)}$.

- If $0 \in h(D)$, then there is a unique point $\tau \in D$ such that $h(\tau) = 0$, and we say that h is spirallike (starlike) with respect to an interior point.
- Otherwise, if $0 \in \partial h(D)$, we say that h is spirallike (starlike) with respect to a boundary point.

In the one-dimensional case, the class of spirallike functions with respect to a boundary point was introduced in [1] (see also references therein). It turns out that for each function h of this class there is a point τ, $|\tau| = 1$, such that $\lim_{r \to 1^-} h(r\tau) = 0$. The same conclusion also holds in many multidimensional situations. In fact, the validity of such claim depends on the validity of an analog of Lindelöf's principle (see, for instance, [5]).
Another class of mappings closely connected with dynamical systems consists of mappings convex in one direction. These mappings intertwine some semigroups on a given domain D with semigroups of shifts. More precisely:

Definition 2.5. Let h be a biholomorphic mapping defined on a domain D of a Banach space X, and let $\tau \in X$, $\|\tau\| = 1$. The mapping h is called convex in the direction τ if for each point $w \in h(D)$ and each $t \geqslant 0$, the point $w + t\tau$ also belongs to $h(D)$.

In the one-dimensional case, functions convex in one direction have been studied by many authors starting from the classical work of M.S. Robertson (see, for examples, [8]). Recently, the interest in these functions and their geometric properties has received an impetus because of their connection with the semigroup theory (see [3] and references therein).
Note also that the semigroups $\{e^{-At}\}_{t \geqslant 0}$ and $\{\cdot + t\tau\}_{t \geqslant 0}$ which appear in Definitions 2.4 and 2.5 are particular cases of the general semigroup of affine mappings $\{e^{-At} \cdot + \lambda \int_0^t e^{-As}\tau\, ds\}_{t \geqslant 0}$, where $A \in L(X)$, $\lambda \geqslant 0$ and $\tau \in X$, $\|\tau\| = 1$.

3. Appropriate operator-valued mappings

Let X and Y be two complex Banach spaces endowed with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, and let $\mathbb{D}_1$ and $\mathbb{D}_2$ be the open unit balls in these spaces. On the space $Z = X \times Y$ we wish to define a norm depending on $\|\cdot\|_X$ and $\|\cdot\|_Y$ only. Such a norm may be defined as follows. Let $p : [0, 1] \mapsto [0, 1]$ be a continuous function which satisfies the conditions:

(a) $p(0) = 1$, $p(1) = 0$;
(b) p is a strongly decreasing function;
(c) p is convex up: $p(\frac{s_1 + s_2}{2}) \geqslant \frac{1}{2}(p(s_1) + p(s_2))$ for all $s_1, s_2 \in [0, 1]$.

Then the set

$$\mathbb{D} := \left\{ (x, y) \in \mathbb{D}_1 \times \mathbb{D}_2 \subset Z : \|y\|_Y < p(\|x\|_X) \right\}$$

is the open unit ball in Z with respect to some norm $\|\cdot\|$. Actually, this norm is the Minkowski functional of the set $\mathbb{D}$. Under our assumption, $\|(x, y)\|$ is the unique solution $\lambda \geqslant \|x\|_X$ of the equation $\|y\|_Y = \lambda p(\frac{\|x\|_X}{\lambda})$. Obviously, Z equipped with this norm $\|\cdot\|$ is a complex Banach space.

In our study of extension operators we need the notion of appropriate operator-valued mappings. We define this in several steps. First, we deal with self-mappings of $\mathbb{D}_1$.

Definition 3.1. Let $\widehat{\mathcal{K}}$ be a subset of $\mathrm{Hol}(\mathbb{D}_1)$ consisting of biholomorphic mappings and closed with respect to composition, and let $\widehat{\Gamma} : \widehat{\mathcal{K}} \times \mathbb{D}_1 \mapsto L(Y)$ be a mapping continuous on $\widehat{\mathcal{K}}$ and holomorphic on $\mathbb{D}_1$. We say that $\widehat{\Gamma}$ is appropriate if it satisfies the following properties:

 (i) the identity mapping id_X of the space X belongs to $\widehat{\mathcal{K}}$, and $\widehat{\Gamma}(\mathrm{id}_X, x) = \mathrm{id}_Y$, the identity mapping of the space Y;
 (ii) $\widehat{\Gamma}$ satisfies the chain rule in the sense that $\widehat{\Gamma}(f, g(x))\widehat{\Gamma}(g, x) = \widehat{\Gamma}(f \circ g, x)$ for all $f, g \in \widehat{\mathcal{K}}$ and $x \in \mathbb{D}_1$;
(iii) for each $f \in \widehat{\mathcal{K}}$ and $x \in \mathbb{D}_1$, the operator $\widehat{\Gamma}(f, x)$ is invertible;
(iv) $\|\widehat{\Gamma}(f, x)\|_{L(Y)} \leqslant \frac{p(\|f(x)\|_X)}{p(\|x\|_X)}$ for all $f \in \widehat{\mathcal{K}}$ and $x \in \mathbb{D}_1$.

In the following examples we set $p(s) = (1 - s^q)^{1/r}$, where $q, r \geqslant 1$. Thus, the unit ball in the space $Z = X \times Y$ is defined by

$$\mathbb{D} = \left\{ (x, y) : \|x\|_X^q + \|y\|_Y^r < 1 \right\}.$$

Example 1. Let $X = \mathbb{C}^n$ be the Euclidean n-dimensional complex space. We consider the scalar operator $\widehat{\Gamma}(f, x) := (J_f(x))^\alpha \mathrm{id}_Y$, $\alpha > 0$. To verify whether this operator is appropriate, first we choose a branch of the power $(J_f(x))^\alpha$ such that condition (i) of Definition 3.1 holds. Furthermore, we denote by $\widehat{\mathcal{K}}$ a set consisting of biholomorphic self-mappings of $\mathbb{D}_1$. In particular, we can choose $\widehat{\mathcal{K}} = \widehat{\mathcal{K}^\tau}$, the subset of $\mathrm{Hol}(\mathbb{D}_1)$ consisting of all biholomorphic self-mappings of $\mathbb{D}_1$ with a fixed point $\tau \in \overline{\mathbb{D}_1}$.

Conditions (ii) and (iii) obviously are satisfied. In addition,

$$\left\| \widehat{\Gamma}(f, x) \right\|_{L(Y)} = \left| J_f(x) \right|^\alpha \leqslant \left(\frac{1 - \|f(x)\|_X^2}{1 - \|x\|_X^2} \right)^{\frac{(n+1)\alpha}{2}}$$

(see [13, Lemma 1.1]). Therefore, condition (iv) will follow by the inequality

$$\left(\frac{1 - \|f(x)\|_X^2}{1 - \|x\|_X^2} \right)^{\frac{(n+1)\alpha}{2}} \leqslant \frac{(1 - \|f(x)\|_X^q)^{1/r}}{(1 - \|x\|_X^q)^{1/r}}, \tag{3.1}$$

which obviously holds for $\alpha = \frac{2}{r(n+1)}$ and $q = 2$. To proceed, we rewrite (3.1) as

$$\frac{(1 - \|f(x)\|_X^2)^{\frac{(n+1)\alpha}{2}}}{(1 - \|f(x)\|_X^q)^{1/r}} \leqslant \frac{(1 - \|x\|_X^2)^{\frac{(n+1)\alpha}{2}}}{(1 - \|x\|_X^q)^{1/r}}.$$

Now, if all mappings in $\widehat{\mathcal{K}}$ satisfy $f(0) = 0$, then $\|f(x)\|_X \leqslant \|x\|_X$. Taking into account that the function $\frac{(1-t^2)^{\frac{(n+1)\alpha}{2}}}{(1-t^q)^{1/r}}$ is increasing in $t \in (0, 1)$ for $q \leqslant 2$ and $\alpha \leqslant \frac{2}{r(n+1)}$, we conclude that in this situation inequality (3.1) (hence, condition (iv)) holds.

In the next example $X = \mathbb{C}$, the complex plane, and $\mathbb{D}_1 = \Delta$, the open unit disk in $\mathbb{C}$.

Example 2. Consider the scalar operator $\widehat{\Gamma}(f, x) = (\frac{f(x)}{x})^\beta \mathrm{id}_Y$, $\beta > 0$. Namely, we set $\widehat{\mathcal{K}}$ to be the set of all univalent self-mappings of Δ with $f(0) = 0$. Similar to the above example, we choose a branch of the power $(\frac{f(x)}{x})^\beta$ such that

conditions (i)–(iii) of Definition 3.1 hold. Condition (iv) follows from the Schwarz Lemma: the inequality $|f(x)| \leqslant |x|$ implies that

$$\big\| \widehat{\Gamma}(f,x) \big\|_{L(Y)} = \left| \frac{f(x)}{x} \right|^\beta \leqslant 1 \leqslant \left(\frac{1-|f(x)|^q}{1-|x|^q} \right)^{1/r}.$$

As above, it is easy to modify this example for functions of the set $\widehat{\mathcal{K}^\tau}$ for any $\tau \in \Delta$.

Remark 2. In the introduction we mentioned papers where combinations of extension operators (1.2) and (1.4) were studied. Obviously, such combinations are included in our scheme; namely, we can consider non-scalar operators based on Examples 1 and 2 above.

In the next example, X is a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|_X$, and $\tau \in \partial \mathbb{D}_1 \subset X$. Also, we set $p(s) = (1-s^2)^{1/r}$.

Example 3. Consider the scalar operator $\widehat{\Gamma}(f,x) = \left(\frac{1-\langle f(x),\tau \rangle}{1-\langle x,\tau \rangle} \right)^{2/r} \mathrm{id}_Y$, defined on the set $\widehat{\mathcal{K}^\tau}$ of all biholomorphic self-mappings of $\mathbb{D}_1$ with the boundary attractive fixed point $\tau \in \partial \mathbb{D}_1$. As above, condition (i) follows by the selection of an appropriate branch of the power, conditions (ii) and (iii) hold automatically. Furthermore, by a multidimensional analog of the boundary Wolff–Schwarz Lemma (see, for example, [19])

$$\frac{1-\|x\|_X^2}{|1-\langle x,\tau \rangle|^2} \leqslant \frac{1-\|f(x)\|_X^2}{|1-\langle f(x),\tau \rangle|^2}.$$

Therefore,

$$\big\| \widehat{\Gamma}(f,x) \big\|_{L(Y)} = \left| \frac{1-\langle f(x),\tau \rangle}{1-\langle x,\tau \rangle} \right|^{2/r} \leqslant \frac{(1-\|f(x)\|_X^2)^{1/r}}{(1-\|x\|_X^2)^{1/r}},$$

i.e., condition (iv) is satisfied.

For each appropriate mapping $\widehat{\Gamma}$, one corresponds the extension operator $\widehat{\Phi} : \widehat{\mathcal{K}} \mapsto \mathrm{Hol}(\mathbb{D})$ defined by

$$\widehat{\Phi}[f](x,y) = \widehat{\Phi}_{\widehat{\Gamma}}[f](x,y) = \big(f(x), \widehat{\Gamma}(f,x)y \big). \tag{3.2}$$

In Section 4 below, we will study its modification as an extension operator for one-parameter semigroups.

Lemma 3.1. *Let $\widehat{\Gamma} : \widehat{\mathcal{K}} \times \mathbb{D}_1 \mapsto L(Y)$ be appropriate. Let $f, g \in \widehat{\mathcal{K}}$. Then:*

(a) $\widehat{\Phi}[f] \in \mathrm{Hol}(\mathbb{D})$;
(b) $\widehat{\Phi}[f \circ g] = \widehat{\Phi}[f] \circ \widehat{\Phi}[g]$.

For the original Roper–Suffridge operator (1.1), assertion (a) of this lemma can be found in [2].

Proof. Assertion (a) means that for each point $(x,y) \in \mathbb{D}$ the inequality

$$\big\| \widehat{\Gamma}(f,x)y \big\|_Y < p\big(\|f(x)\|_X \big)$$

holds. Indeed, since $(x,y) \in \mathbb{D}$, we have $\|y\|_Y < p(\|x\|_X)$. Therefore,

$$\big\| \widehat{\Gamma}(f,x)y \big\|_Y \leqslant \big\| \widehat{\Gamma}(f,x) \big\|_{L(Y)} \|y\|_Y \leqslant \frac{p(\|f(x)\|_X)}{p(\|x\|_X)} \|y\|_Y < p\big(\|f(x)\|_X \big).$$

To prove assertion (b), we just calculate

$$\begin{aligned} \big(\widehat{\Phi}[f] \circ \widehat{\Phi}[g] \big)(x,y) = \widehat{\Phi}[f]\big(\widehat{\Phi}[g](x,y) \big) &= \widehat{\Phi}[f]\big(g(x), \widehat{\Gamma}(g,x)y \big) \\ &= \big(f(g(x)), \widehat{\Gamma}(f,g(x))\widehat{\Gamma}(g,x)y \big) = \big((f \circ g)(x), \widehat{\Gamma}(f \circ g, x)y \big) \\ &= \big(\widehat{\Phi}[f \circ g] \big)(x,y). \quad \square \end{aligned}$$

Now we expand the notion of appropriate operators to biholomorphic mappings $\mathbb{D}_1 \mapsto X$.

Definition 3.2. Let a set $\widehat{\mathcal{K}} \subset \mathrm{Hol}(\mathbb{D}_1)$ and an appropriate mapping $\widehat{\Gamma}$ be given. Suppose that there are (a) a non-empty set $\mathcal{K} = \mathcal{K}_{\mathbb{D}_1} \subset \mathrm{Hol}(\mathbb{D}_1, X)$ consisting of biholomorphic mappings and (b) a mapping $\Gamma = \Gamma_{\mathbb{D}_1} : \mathcal{K} \times \mathbb{D}_1 \mapsto L(Y)$ continuous on $\mathcal{K}$ and holomorphic on $\mathbb{D}_1$ such that

(i) for all $h_1, h_2 \in \mathcal{K}$ with $h_1(\mathbb{D}_1) \subset h_2(\mathbb{D}_1)$, we have $h_2^{-1} \circ h_1 \in \widehat{\mathcal{K}}$;
(ii) $\Gamma(h, g(x))\widehat{\Gamma}(g, x) = \Gamma(h \circ g, x)$ for all $h \in \mathcal{K}$, $g \in \widehat{\mathcal{K}}$ and $x \in \mathbb{D}_1$;
(iii) for each $h \in \mathcal{K}$ and $x \in \mathbb{D}_1$, the operator $\Gamma(h, x)$ is invertible.

Then we say that $\Gamma = \Gamma_{\mathbb{D}_1}$ is appropriate.

Remark 3. For appropriate mappings considered in Examples 1 and 2 and defined on the set of mappings normalized by $f(0) = 0$, one can choose Γ to be defined by the same formula as $\widehat{\Gamma}$, that is, respectively, $\Gamma(h, x) = (J_h(x))^\alpha \mathrm{id}_Y$ or $\Gamma(h, x) = (\frac{h(x)}{x})^\beta \mathrm{id}_Y$, where $h(0) = 0$. As previously mentioned, the operator $\widehat{\Gamma}$ from Example 1 can be defined on the set $\widehat{\mathcal{K}^\tau}$, $\tau \in \overline{\mathbb{D}_1}$. In this case, we can again use the same formula.

Concerning Example 3, it is possible to proceed as follows. We choose some mapping $A : X \mapsto L(Y)$ and a set of biholomorphic mappings $\mathcal{K}$ such that $A(h(x))$ is invertible for all $h \in \mathcal{K}$ and $x \in \mathbb{D}_1$. Then we set $\Gamma(h, x) = (1 - \langle x, \tau \rangle)^{-2/r} A(h(x))$. For instance, we can choose $\mathcal{K}$ to be a set of biholomorphic mappings $h \in \mathrm{Hol}(\mathbb{D}_1, X)$ with $\langle h(x), \tau \rangle \neq 0$ for all $x \in \mathbb{D}_1$ and to define $\Gamma(h, x) = (\frac{\langle h(x), \tau \rangle}{1 - \langle x, \tau \rangle})^{2/r} \mathrm{id}_Y$.

Similar to (3.2), we define the extension operator $\Phi : \mathcal{K} \mapsto \mathrm{Hol}(\mathbb{D}, Z)$ by

$$\Phi[h](x, y) = \Phi_\Gamma[h](x, y) = \big(h(x), \Gamma(h, x)y\big). \tag{3.3}$$

This operator will be the main subject in Section 5. In particular, we will study its action on starlike and spirallike mappings.

Lemma 3.2. *Let $\widehat{\Gamma} : \widehat{\mathcal{K}} \times \mathbb{D}_1 \mapsto L(Y)$ and $\Gamma : \mathcal{K} \times \mathbb{D}_1 \mapsto L(Y)$ be appropriate. Let $h \in \mathcal{K}$ and $g \in \widehat{\mathcal{K}}$. Then*

$$\Phi[h \circ g] = \Phi[h] \circ \widehat{\Phi}[g].$$

In addition, $\Phi[h]$ is biholomorphic, and for $(z, w) \in \Phi[h](\mathbb{D})$ we have

$$\big(\Phi[h]\big)^{-1}(z, w) = \big(h^{-1}(z), \big(\Gamma\big(h, h^{-1}(z)\big)\big)^{-1} w\big).$$

Proof. The first assertion follows by the calculation:

$$\begin{aligned} \big(\Phi[h] \circ \widehat{\Phi}[g]\big)(x, y) &= \Phi[h]\big(\widehat{\Phi}[g](x, y)\big) = \Phi[h]\big(g(x), \widehat{\Gamma}(g, x)y\big) \\ &= \big(h(g(x)), \Gamma\big(h, g(x)\big)\widehat{\Gamma}(g, x)y\big) = \big((h \circ g)(x), \Gamma(h \circ g, x)y\big) \\ &= \big(\Phi[h \circ g]\big)(x, y). \end{aligned}$$

The last assertion is obvious. $\quad\square$

Now we are ready to turn to appropriate operators on domains biholomorphically equivalent to the unit ball $\mathbb{D}_1$.

Definition 3.3. Let $\Gamma : \mathcal{K} \times \mathbb{D}_1 \mapsto L(Y)$ be appropriate. Given a domain $\Omega \in X$ biholomorphically equivalent to the ball $\mathbb{D}_1$, we define the set $\mathcal{K}_\Omega$ to consist of all biholomorphic mappings $f \in \mathrm{Hol}(\Omega, X)$ for which there is a biholomorphic mapping h of $\mathbb{D}_1$ onto Ω such that both h and $f \circ h$ belong to $\mathcal{K}$. For $f \in \mathcal{K}_\Omega$ and $x \in \Omega$ we define the appropriate mapping Γ_Ω by

$$\Gamma_\Omega(f, x) := \Gamma\big(f \circ h, h^{-1}(x)\big)\big(\Gamma\big(h, h^{-1}(x)\big)\big)^{-1}.$$

The next assertion can be checked directly.

Lemma 3.3. *The mapping Γ_Ω is well defined in the sense that it is independent of the choice of a biholomorphic mapping $h \in \mathcal{K}$ of $\mathbb{D}_1$ onto Ω. Moreover, Γ_Ω has the following properties:*

(i) $\Gamma_\Omega(\mathrm{id}_X, x) = \mathrm{id}_Y$ *for all $x \in \Omega$;*
(ii) $\Gamma_\Omega(f, g(x))\Gamma_\Omega(g, x) = \Gamma_\Omega(f \circ g, x)$ *for all $f \in \mathcal{K}_\Omega$, $g \in \mathcal{K}_\Omega \cap \mathrm{Hol}(\Omega)$ and $x \in \Omega$;*
(iii) *for each $f \in \mathcal{K}_\Omega$ and $x \in \Omega$, the operator $\Gamma_\Omega(f, x)$ is invertible. In particular, if $h \in \mathcal{K}$ is a biholomorphic mapping of $\mathbb{D}_1$ onto Ω, then $h^{-1} \in \mathcal{K}_\Omega$ and $\Gamma_\Omega(h^{-1}, h(x)) = (\Gamma(h, x))^{-1}$.*

Proof. Let h_1, h_2 be biholomorphic mappings of $\mathbb{D}_1$ onto Ω such that $h_1, h_2, f \circ h_1, f \circ h_2 \in \mathcal{K}$. We have to show that

$$\Gamma\big(f \circ h_1, h_1^{-1}(x)\big)\big(\Gamma\big(h_1, h_1^{-1}(x)\big)\big)^{-1} = \Gamma\big(f \circ h_2, h_2^{-1}(x)\big)\big(\Gamma\big(h_2, h_2^{-1}(x)\big)\big)^{-1}$$

for all $x \in \Omega$. Denote $w = h_1^{-1}(x)$ and let $\phi := h_2^{-1} \circ h_1$ be an automorphism of $\mathbb{D}_1$ which belongs to $\mathcal{K}$ by Definition 3.2. Then the equality above can be rewritten as

$$\Gamma(f \circ h_1, w)\big(\Gamma(h_1, w)\big)^{-1} = \Gamma\big(f \circ h_2, \phi(w)\big)\big(\Gamma\big(h_2, \phi(w)\big)\big)^{-1}.$$

This relation holds by Definition 3.2 since $h_2 \circ \phi = h_1$.

Properties (i) and (iii) hold by Definition 3.3. To check property (ii), let consider the expression

$$\Gamma_\Omega\big(f, g(x)\big)\Gamma_\Omega(g, x)\Gamma\big(h, h^{-1}(x)\big) = \Gamma\big(f \circ h, h^{-1}\big(g(x)\big)\big)\big(\Gamma\big(h, h^{-1}\big(g(x)\big)\big)\big)^{-1}\Gamma\big(g \circ h, h^{-1}(x)\big)$$

$$= \Gamma\big(f \circ h, \psi(w)\big)\big(\Gamma\big(h, \psi(w)\big)\big)^{-1}\Gamma(h \circ \psi, w),$$

where we denote $\psi = h^{-1} \circ g \circ h$ and $w = h^{-1}(x)$. Since $(g \circ h)(\mathbb{D}_1) \subset h(\mathbb{D}_1)$, we conclude by Definition 3.2 (i) that $\psi \in \widehat{\mathcal{K}}$. Now, using condition (ii) of Definition 3.2, we obtain

$$\Gamma_\Omega\big(f, g(x)\big)\Gamma_\Omega(g, x)\Gamma\big(h, h^{-1}(x)\big) = \Gamma\big(f \circ h, \psi(w)\big)\widehat{\Gamma}(\psi, w) = \Gamma(f \circ h \circ \psi, w) = \Gamma\big(f \circ g \circ h, h^{-1}(x)\big),$$

so (ii) follows. □

In what follows, all operator-valued mappings $\widehat{\Gamma}$, Γ and Γ_Ω are assumed to be appropriate.

4. Extension operators for semigroups

In this section we study extension operators for one-parameter continuous semigroups. It turns out that for a given appropriate mapping, each semigroup on the unit ball of X admits a family of extensions.

Theorem 4.1. *Let $\widehat{\Gamma} : \widehat{\mathcal{K}} \times \mathbb{D}_1 \mapsto L(Y)$ be appropriate, i.e., conditions (i)–(iv) of Definition 3.1 are satisfied. Let $S = \{F_t\}_{t \geqslant 0} \subset \mathrm{Hol}(\mathbb{D}_1)$ be a semigroup on the ball $\mathbb{D}_1$ such that $S \subset \widehat{\mathcal{K}}$. Let $\Sigma = \{G_t\}_{t \geqslant 0}$ be a semigroup on the ball $\mathbb{D}_2$ such that each its element G_s, $s \geqslant 0$, satisfies $\|G_s(y)\|_Y \leqslant \|y\|_Y$ for all $y \in \mathbb{D}_2$ and commutes with operators $\widehat{\Gamma}(F_t, x)$ for all $t \geqslant 0$ and $x \in \mathbb{D}_1$:*

$$\widehat{\Gamma}(F_t, x) \circ G_s = G_s \circ \widehat{\Gamma}(F_t, x). \tag{4.1}$$

Then the family $\widetilde{S} = \{\widetilde{F}_t\}_{t \geqslant 0}$ defined by

$$\widetilde{F}_t(x, y) = \big(F_t(x), \widehat{\Gamma}(F_t, x)G_t(y)\big), \tag{4.2}$$

forms a semigroup on $\mathbb{D}$.

Remark 4. In the case when Σ is a uniformly continuous semigroup of proper contractions (hence, $G_t = e^{-Bt}$ for some accretive operator B, see [19]), the commutativity condition (4.1) can be replaced by the following one: *all operators $\widehat{\Gamma}(F_t, x)$, $t \geqslant 0$, $x \in \mathbb{D}_1$, commute with B*. In particular, the last condition always holds if $\widehat{\Gamma}(F_t, x)$ is a scalar operator for each $t \geqslant 0$ and $x \in \mathbb{D}_1$.

Proof. Since G_t is a contraction, it follows by Lemma 3.1 (a) that $\widetilde{F}_t$ is a self-mapping of $\mathbb{D}$ for each $t \geqslant 0$. The continuity of $\widehat{\Gamma}$ and condition (i) of Definition 3.1 imply that

$$\lim_{t \to 0^+} \widetilde{F}_t(x, y) = \lim_{t \to 0^+} \big(F_t(x), \widehat{\Gamma}(F_t, x)G_t(y)\big) = \big(x, \widehat{\Gamma}(\mathrm{id}_X, x)G_0(y)\big) = (x, y).$$

Similarly to the proof of Lemma 3.1 (b), we have for all $t, s > 0$:

$$\begin{aligned}(\widetilde{F}_t \circ \widetilde{F}_s)(x, y) = \widetilde{F}_t\big(\widetilde{F}_s(x, y)\big) &= \widetilde{F}_t\big(F_s(x), \widehat{\Gamma}(F_s, x)G_s(y)\big) \\ &= \big(F_t\big(F_s(x)\big), \widehat{\Gamma}\big(F_t, F_s(x)\big) \circ G_t \circ \widehat{\Gamma}(F_s, x) \circ G_s(y)\big) \\ &= \big(F_t\big(F_s(x)\big), \widehat{\Gamma}\big(F_t, F_s(x)\big)\widehat{\Gamma}(F_s, x) \circ G_t \circ G_s(y)\big) \\ &= \big((F_t \circ F_s)(x), \widehat{\Gamma}(F_t \circ F_s, x)G_{t+s}(y)\big) = \widetilde{F_{t+s}}(x, y).\end{aligned}$$

This calculation completes the proof. □

Corollary 4.1. *Let an appropriate mapping $\widehat{\Gamma}$ and semigroups $S \subset \widehat{\mathcal{K}}$ and $\Sigma \subset \mathrm{Hol}(\mathbb{D}_2)$ be as above. Denote by $\mathcal{M} \subset \mathbb{D}_1$ the stationary point set of S. Then the stationary point set $\widetilde{\mathcal{M}}$ of the extended semigroup $\widetilde{S}$ satisfies the following inclusion:*

$$\big\{(x, 0) \in \mathbb{D} : x \in \mathcal{M}\big\} \subset \widetilde{\mathcal{M}} \subset \big\{(x, y) \in \mathbb{D} : x \in \mathcal{M}\big\}.$$

To find the semigroup generator, we require the Frechét differentiability of $\widehat{\Gamma}$ in $f \in \widehat{\mathcal{K}}$, namely,

- *at each point $f \in \widehat{\mathcal{K}}$ the Frechét derivative (denoted by $\partial\widehat{\Gamma}(f, x)$) exists as a linear operator defined on* span$(\widehat{\mathcal{K}})$.

Just differentiating (4.2) at $t = 0^+$, we obtain the following assertion.

Corollary 4.2. *Let an appropriate mapping $\widehat{\Gamma}$ and semigroups $S \subset \widehat{\mathcal{K}}$ and $\Sigma \subset \mathrm{Hol}(\mathbb{D}_2)$ be as above, and let condition (•) be satisfied. If S is generated by a mapping $f \in \mathrm{Hol}(\mathbb{D}_1, X)$, and Σ is generated by a mapping $g \in \mathrm{Hol}(\mathbb{D}_2, Y)$, then the extended semigroup $\widetilde{S}$ defined by (4.2) is generated as well. Its generator $\widetilde{f}$ is defined by*

$$\widetilde{f}(x, y) = \big(f(x), \partial\widehat{\Gamma}(\mathrm{id}_X, x)[f]y + g(y)\big).$$

We proceed with the extension of conjugate semigroups.

Theorem 4.2. *Let $\{F_t\}_{t \geqslant 0} \subset \widehat{\mathcal{K}}$ and $\{\Psi_t\}_{t \geqslant 0} \subset \mathcal{K}_\Omega$ be conjugate semigroups acting on the unit ball $\mathbb{D}_1$ and a domain $\Omega \subset X$, respectively. Let $h \in \mathrm{Hol}(\mathbb{D}_1, \Omega) \cap \mathcal{K}$ be their intertwining map. Then the mapping $\widetilde{h} = \Phi[h]$ defined by (3.3) is the intertwining map for the semigroup $\widetilde{S} = \{\widetilde{F}_t\}_{t \geqslant 0}$ defined by (4.2) and the semigroup $\{\widetilde{\Psi}_t\}_{t \geqslant 0}$ acting on $\Phi[h](\mathbb{D})$ and defined by*

$$\widetilde{\Psi}_t(z, w) = \big(\Psi_t(z), \Gamma_\Omega(\Psi_t, z)\widetilde{G}_t(z, w)\big), \tag{4.3}$$

where

$$\widetilde{G}_t(z, w) = \Gamma\big(h, h^{-1}(z)\big)G_t\big(\Gamma_\Omega\big(h^{-1}, z\big)w\big).$$

Note that if all mappings G_t, $t \geqslant 0$, commute with $\Gamma(h, x)$ (for example, in the case described in Remark 4), then $\widetilde{G}_t(z, w) = G_t(w)$.

Proof. It has already be proven in Theorem 4.1 that the family $\widetilde{S} = \{\widetilde{F}_t\}_{t \geqslant 0}$ forms a semigroup on $\mathbb{D}$. Therefore, the family $\{\Phi[h] \circ \widetilde{F}_t \circ (\Phi[h])^{-1}\}_{t \geqslant 0}$ forms a semigroup on $\Phi[h](\mathbb{D})$ which is conjugate to $\widetilde{S}$ with the intertwining mapping $\Phi[h]$. Let us find its exact form. By Lemmas 3.2 and 3.3,

$$\big(\Phi[h]\big)^{-1}(z, w) = \big(h^{-1}(z), \big(\Gamma\big(h, h^{-1}(z)\big)\big)^{-1}w\big) = \big(h^{-1}(z), \Gamma_\Omega\big(h^{-1}, z\big)w\big).$$

Now, we substitute

$$\begin{aligned}
\widetilde{F}_t \circ \big(\Phi[h]\big)^{-1}(z, w) &= \widetilde{F}_t\big(h^{-1}(z), \Gamma_\Omega\big(h^{-1}, z\big)w\big) \\
&= \big(F_t\big(h^{-1}(z)\big), \widehat{\Gamma}\big(F_t, h^{-1}(z)\big)G_t\big(\Gamma_\Omega\big(h^{-1}, z\big)w\big)\big).
\end{aligned}$$

By Definition 3.2, $\widehat{\Gamma}(F_t, h^{-1}(z)) = (\Gamma(h, F_t \circ h^{-1}(z)))^{-1}\Gamma(h \circ F_t, h^{-1}(z))$. In addition, since h is the intertwining map for $\{F_t\}_{t \geqslant 0}$ and $\{\Psi_t\}_{t \geqslant 0}$, we conclude that $F_t \circ h^{-1} = h^{-1} \circ \Psi_t$. Therefore,

$$\begin{aligned}
\big(\Phi[h] \circ \widetilde{F}_t \circ \big(\Phi[h]\big)^{-1}\big)(z, w) &= \Phi[h]\big(h^{-1}\big(\Psi_t(z)\big), \big(\Gamma\big(h, h^{-1} \circ \Psi_t(z)\big)\big)^{-1}\Gamma\big(\Psi_t \circ h, h^{-1}(z)\big)G_t\big(\Gamma_\Omega\big(h^{-1}, z\big)w\big)\big) \\
&= \big(\Psi_t(z), \Gamma\big(\Psi_t \circ h, h^{-1}(z)\big)G_t\big(\Gamma_\Omega\big(h^{-1}, z\big)w\big)\big).
\end{aligned}$$

Finally, by Definition 3.3

$$\Gamma\big(\Psi_t \circ h, h^{-1}(z)\big) = \Gamma_\Omega(\Psi_t, z)\Gamma\big(h, h^{-1}(z)\big).$$

Thus,

$$\big(\Phi[h] \circ \widetilde{F}_t \circ \big(\Phi[h]\big)^{-1}\big)(z, w) = \big(\Psi_t(z), \Gamma_\Omega(\Psi_t, z)\Gamma\big(h, h^{-1}(z)\big)G_t\big(\Gamma_\Omega\big(h^{-1}, z\big)w\big)\big),$$

and the assertion follows. $\quad\square$

5. Starlikeness, spirallikeness and convexity in one direction

The main results of this section are Theorems 5.1 and 5.2 below. In these theorems, given a biholomorphic mapping h, we examine geometric properties of its extension $\Phi[h]$ defined by formula (3.3): $\Phi[h](x, y) = (h(x), \Gamma(h, x)y)$.

Theorem 5.1. *Let $h \in \mathrm{Hol}(\mathbb{D}_1, X)$ be an A-spirallike mapping. Suppose that $e^{-At} \circ h \in \mathcal{K}$ for all $t \geqslant 0$ and there is $C \in L(Y)$ such that*

$$\Gamma\big(e^{-At} \circ h, x\big) = e^{-Ct}\Gamma(h, x) \quad \text{for all } t \geqslant 0 \text{ and } x \in \mathbb{D}_1. \tag{5.1}$$

Then the mapping $\Phi[h]$ is $\begin{pmatrix} A & 0 \\ 0 & B+C \end{pmatrix}$-spirallike for any accretive operator $B \in L(Y)$ which commutes with C and with $\Gamma(h, x)$ for all $x \in \mathbb{D}_1$ and such that the function $\mathrm{Re}\,\lambda$ is bounded away from zero on the spectrum of $B + C$.
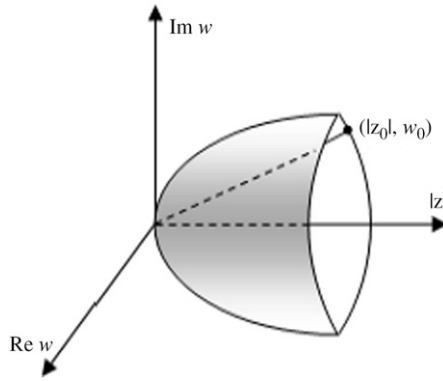
Fig. 1. The 'spiral' segment and the manifold inside $\Phi[h](\mathbb{D})$.

By definition, for each point of the image of a spirallike mapping there is a spiral curve which is contained in the image. Our theorem asserts that the image of the extension of a spirallike mapping contains not only a one-dimensional spiral curve but at least some manifold (of real codimension $(2n-1)$ when $X = \mathbb{C}^n$). We illustrate this effect in Examples 4 and 5 below.

Theorem 5.2. *Let a biholomorphic mapping $h \in \mathrm{Hol}(\mathbb{D}_1, X)$ be convex in the direction τ, where $\tau \in \partial\mathbb{D}_1$. Suppose that $h + t\tau \in \mathcal{K}$ for all $t \geqslant 0$ and there is $C \in L(Y)$ such that*

$$\Gamma(h + t\tau, x) = e^{-Ct}\Gamma(h, x) \quad \text{for all } t \geqslant 0 \text{ and } x \in \mathbb{D}_1. \tag{5.2}$$

Then for each point $(z, w) \in \Phi[h](\mathbb{D})$, the set $\Phi[h](\mathbb{D})$ contains the curve

$$\left\{ \left(z + t\tau, e^{-(B+C)t} w \right),\ t \geqslant 0 \right\},$$

for any accretive operator $B \in L(Y)$ which commutes with C and with $\Gamma(h, x)$ for all $x \in \mathbb{D}_1$ and such that the function $\mathrm{Re}\,\lambda$ is non-negative on the spectrum of $B + C$. In particular, if $\Gamma(h + t\tau, x) = \Gamma(h, x)$ for all $t \geqslant 0$ and $x \in \mathbb{D}_1$, then the mapping $\Phi[h]$ is convex in the direction $(\tau, 0)$.

Example 4. Let $X = \mathbb{C}^n$ with an arbitrary norm. Similar to the examples in Section 3, we define the unit ball in the space $Z = X \times Y = \mathbb{C}^n \times Y$ by

$$\mathbb{D} = \left\{ (x, y)\colon \|x\|_X^2 + \|y\|_Y^r < 1 \right\}, \quad r \geqslant 1.$$

Consider the appropriate mapping $\Gamma(h, x) = (J_h(x))^{\frac{2}{r(n+1)}} \mathrm{id}_Y$ (cf. Example 1 and Remark 3 above) and the corresponding extension operator

$$\Phi[h](x, y) = \left(h(x), \left(J_h(x) \right)^{\frac{2}{r(n+1)}} y \right).$$

(1) Let A be a diagonal matrix, $A = \mathrm{diag}(\mu_1, \dots, \mu_n)$ with $\mathrm{Re}\,\mu_j > 0$. Take any A-spirallike mapping h on the unit ball $\mathbb{D}_1 \subset X$ with respect to either an interior or a boundary point. Since $J_{e^{-At}h}(x) = e^{-\mathrm{tr}\,A \cdot t} J_h(x)$, where $\mathrm{tr}\,A = \mu_1 + \cdots + \mu_n$ is the trace of the matrix A, we get that the operator C in formula (5.1) is given by $C = \frac{2\,\mathrm{tr}\,A}{r(n+1)} \mathrm{id}_Y$. According to Theorem 5.1, the extended mapping $\Phi[h]$ is $\left(\begin{smallmatrix} A & 0 \\ 0 & B + \frac{2\,\mathrm{tr}\,A}{r(n+1)} \mathrm{id}_Y \end{smallmatrix} \right)$-spirallike for any accretive operator $B \in L(Y)$. To understand this effect, consider the simplest case $Y = \mathbb{C}$. In this situation all linear operators are just multiplication by scalars. We have that for any point $(z_0, w_0) \in \Phi[h](\mathbb{D})$, the image $\Phi[h](\mathbb{D})$ contains the set

$$\left\{ (z, w)\colon z = e^{-At} z_0,\ w = e^{-(\lambda + \frac{\mathrm{tr}\,A}{r})t} w_0,\ t \geqslant 0,\ \mathrm{Re}\,\lambda > 0 \right\},$$

or, equivalently,

$$\left\{ (z, w)\colon z = e^{-At} z_0,\ |w| < e^{-\frac{t\,\mathrm{Re}(\mathrm{tr}\,A)}{r}} |w_0|,\ t \geqslant 0 \right\}.$$

Schematically, this set is presented in Fig. 1.

(2) Let now $\tau \in \partial\mathbb{D}_1$. Take a mapping h convex in the direction τ. Since $J_{h+t\tau}(x) = J_h(x)$, we conclude that the operator C in formula (5.2) is zero. According to Theorem 5.2, for each point $(z_0, w_0) \in \Phi[h](\mathbb{D})$, the image $\Phi[h](\mathbb{D})$ contains the set

$$\left\{ (z_0 + t\tau, e^{-Bt} w_0),\ t \geqslant 0,\ B \in L(Y) \text{ is attractive} \right\}.$$
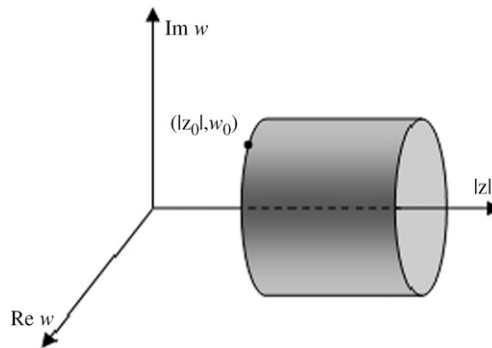
Fig. 2. The cylindric set inside $\Phi[h](\mathbb{D})$.

Once again, we restrict our consideration to the case $Y = \mathbb{C}$. Then for any point $(z_0, w_0) \in \Phi[h](\mathbb{D})$, the image $\Phi[h](\mathbb{D})$ contains the set

$$\{(z, w): z = z_0 + t\tau, \ |w| \leqslant |w_0|, \ t \geqslant 0\}$$

(see Fig. 2).

Example 5. Let X be a complex Hilbert space. As above, we define the unit ball in the space $Z = X \times Y$ by

$$\mathbb{D} = \{(x, y): \|x\|_X^2 + \|y\|_Y^r < 1\}, \quad r \geqslant 1.$$

Let $A \in L(X)$ be a bounded linear operator such that the function $\operatorname{Re}\lambda$ is bounded away from zero on the spectrum of A. Suppose that a vector $\tau \in X$, $\|\tau\|_X = 1$, is an eigenvector of the adjoint operator A^*, and $\bar{\lambda}$ is the corresponding eigenvalue.

Consider the appropriate mapping $\Gamma(h, x) = (\frac{\langle h(x), \tau \rangle}{1 - \langle x, \tau \rangle})^{2/r} \mathrm{id}_Y$ (cf. Example 3 and Remark 3 above) and the corresponding extension operator

$$\Phi[h](x, y) = \left(h(x), \left(\frac{\langle h(x), \tau \rangle}{1 - \langle x, \tau \rangle} \right)^{2/r} y \right).$$

Take an A-spirallike mapping $h \in \operatorname{Hol}(\mathbb{D}_1, X)$ with respect to a boundary point with $\lim_{r \to 1^-} h(r\tau) = 0$ and such that $\langle h(x), \tau \rangle \neq 0$ for all $x \in \mathbb{D}_1$. Clearly,

$$\langle e^{-At} h(x), \tau \rangle = \langle h(x), e^{-A^*t} \tau \rangle = e^{-\lambda t} \langle h(x), \tau \rangle.$$

Therefore, equality (5.1) holds for the operator $C = \frac{2\lambda}{r} \mathrm{id}_Y$. So, Theorem 5.1 asserts that the extended mapping $\Phi[h]$ is $\begin{pmatrix} A & 0 \\ 0 & B + \frac{2\lambda}{r}\mathrm{id}_Y \end{pmatrix}$-spirallike for any accretive operator $B \in L(Y)$.

In the particular case when X and Y are one-dimensional, we conclude that for any λ, $\operatorname{Re}\lambda > 0$, the extension of each λ-spirallike function with respect to a boundary point is $\begin{pmatrix} \lambda & 0 \\ 0 & \mu + \frac{2\lambda}{r} \end{pmatrix}$-spirallike for any number μ with non-negative real part. We see that if $r < 2$ then the extended mapping may be not λ-spirallike.

As previously mentioned in Section 2, the images of spirallike mappings and mappings convex in one direction are invariant under action of a linear semigroup of proper contractions and a semigroup of shifts, respectively. More generally, we can consider a semigroup of affine mappings. Thus, both Theorems 5.1 and 5.2 can be considered as consequences of the following general assertion, where we denote by $\Sigma = \Sigma(A, \lambda, \tau) = \{\Psi_t\}_{t \geqslant 0}$ the semigroup of affine mappings defined by

$$\Psi_t(z) = e^{-At} z + \lambda \int_0^t e^{-As} \tau \, ds,$$

where $A \in L(X)$, $\lambda \geqslant 0$ and $\tau \in X$, $\|\tau\|_X = 1$.

Theorem 5.3. Let $\Sigma = \Sigma(A, \lambda, \tau)$ be a semigroup of affine mappings. Let $h \in \operatorname{Hol}(\mathbb{D}_1, X)$ be biholomorphic, and $h(\mathbb{D}_1)$ be Σ-invariant. Suppose that $\Psi_t \circ h \in \mathcal{K}$ for all $t \geqslant 0$ and there is an operator $C \in L(Y)$ such that

$$\Gamma(\Psi_t \circ h, x) = e^{-Ct} \Gamma(h, x) \tag{5.3}$$

for all $t \geqslant 0$ and $x \in \mathbb{D}_1$. Let $\{G_s\}_{s \geqslant 0} \subset \operatorname{Hol}(\mathbb{D}_2)$ be a semigroup such that each its element G_s, $s \geqslant 0$, satisfies $\|G_s(y)\|_Y \leqslant \|y\|_Y$ for all $y \in \mathbb{D}_2$ and commutes with $\Gamma(h, x)$ for all $x \in \mathbb{D}_1$.

Then for each point $(z, w) \in \Phi[h](\mathbb{D})$, the image $\Phi[h](\mathbb{D})$ contains the curve

$$\left\{\left(\Psi_t(z), e^{-Ct} G_t(w)\right), \ t \geqslant 0\right\}.$$

Proof. Since $h(\mathbb{D}_1)$ is Σ-invariant, the family $S = \{F_t\}_{t \geqslant 0}$ with $F_t(x) = h^{-1} \circ \Psi_t \circ h(x)$ forms a semigroup on $\mathbb{D}_1$. By our assumption and condition (i) of Definition 3.2 we conclude that $S \subset \widehat{\mathcal{K}}$. Obviously, h is the intertwining map for the semigroups S and Σ (acting on the domain $\Omega = h(\mathbb{D}_1)$).

By Theorem 4.2, the image of the mapping $\Phi[h]$ contains together with each point $(z, w) \in \Phi[h](\mathbb{D})$ the whole semigroup trajectory $\{\widetilde{\Psi}_t(z, w), \ t \geqslant 0\}$, where $\widetilde{\Psi}_t$ is defined by (4.3). It follows by (5.3) that

$$\begin{aligned}
\Gamma_\Omega(\Psi_t, z) &= \Gamma\left(\Psi_t \circ h, h^{-1}(z)\right)\left(\Gamma\left(h, h^{-1}(z)\right)\right)^{-1} \\
&= e^{-Ct} \Gamma\left(h, h^{-1}(z)\right)\left(\Gamma\left(h, h^{-1}(z)\right)\right)^{-1} = e^{-Ct}.
\end{aligned}$$

In addition, $\widetilde{G}_t(z, w) = G_t(w)$. Therefore,

$$\widetilde{\Psi}_t(z, w) = \left(\Psi_t(z), e^{-Ct} G_t(w)\right),$$

and the assertion is proved. $\square$

Proof of Theorem 5.1. Let $h \in \mathrm{Hol}(\mathbb{D}_1, X)$ be an A-spirallike mapping which satisfies (5.1). Let $B \in L(Y)$ be an accretive operator which commutes with C. Then the semigroup $\{e^{-Bs}\}_{s \geqslant 0}$ consists of proper contractions with respect to the norm $\|\cdot\|_Y$. Since

$$\begin{aligned}
\widehat{\Gamma}(F_t, x) &= \left(\Gamma\left(h, F_t(x)\right)\right)^{-1} \Gamma(h \circ F_t, x) \\
&= \left(\Gamma\left(h, F_t(x)\right)\right)^{-1} \Gamma\left(e^{-At} \circ h, x\right) \\
&= \left(\Gamma\left(h, F_t(x)\right)\right)^{-1} e^{-Ct} \Gamma(h, x),
\end{aligned}$$

we conclude that if B commutes with C and with $\Gamma(h, x)$ for all $x \in \mathbb{D}_1$, then B commutes with all operators $\widehat{\Gamma}(F_t, x)$, $t \geqslant 0$, $x \in \mathbb{D}_1$. Thus, we can apply Theorem 5.3 with $\Psi_t = e^{-At}$ and $G_s = e^{-Bs}$ (see Remark 4). According to this theorem, for each point $(z, w) \in \Phi[h](\mathbb{D})$ the image $\Phi[h](\mathbb{D})$ contains the curve

$$\left\{\left(e^{-At}z, e^{-Ct} G_t(w)\right), \ t \geqslant 0\right\} = \left\{\left(e^{-At}z, e^{-(B+C)t}w\right), \ t \geqslant 0\right\}.$$

So, by Definition 2.4 (see also Remark 1), the mapping $\Phi[h]$ is $\left(\begin{smallmatrix} A & 0 \\ 0 & B+C \end{smallmatrix}\right)$-spirallike. The proof is complete. $\square$

Proof of Theorem 5.2. Let $h \in \mathrm{Hol}(\mathbb{D}_1, X)$ be a mapping convex in the direction τ which satisfies (5.2). Let $B \in L(Y)$ be an accretive operator which commutes with C. Then the semigroup $\{e^{-Bs}\}_{s \geqslant 0}$ consists of proper contractions with respect to the norm $\|\cdot\|_Y$. As in the proof of Theorem 5.1, we conclude that if B commutes with C and with $\Gamma(h, x)$ for all $x \in \mathbb{D}_1$, then B commutes with all operators $\widehat{\Gamma}(F_t, x)$, $t \geqslant 0$, $x \in \mathbb{D}_1$. Once again, we can apply Theorem 5.3 with $\Psi_t(z) = z + t\tau$ and $G_s(w) = e^{-Bs}w$. This theorem implies that for each point $(z, w) \in \Phi[h](\mathbb{D}_1)$ the image $\Phi[h](\mathbb{D}_1)$ contains the curve

$$\left\{\left(z + t\tau, e^{-Ct} G_t(w)\right), \ t \geqslant 0\right\} = \left\{\left(z + t\tau, e^{-(B+C)t}w\right), \ t \geqslant 0\right\}.$$

The proof is complete. $\square$

6. Concluding remarks

6.1. Bloch type mappings

Proposition 6.1. *Let Γ be an appropriate operator. Suppose that a mapping $h \in \mathcal{K}$ satisfies the following conditions*:

  (i) $\sup_{x \in \mathbb{D}_1} \|h'(x)\|_{L(X)}(1 - \|x\|_X^2) < \infty$;
 (ii) $\sup_{x \in \mathbb{D}_1} \|\Gamma(h, x)\|_{L(Y)}(1 - \|x\|_X^2) < \infty$;
(iii) $\sup_{x \in \mathbb{D}_1} \|\frac{\partial}{\partial x}\Gamma(h, x)\|_{L(X, L(Y))} p(\|x\|_X)(1 - \|x\|_X^2) < \infty$.

Then $\sup_{(x, y) \in \mathbb{D}} \|\Phi[h]'(x, y)\|_{L(Z)}(1 - \|(x, y)\|^2) < \infty$.

Proof. Differentiating $\Phi[h]$ we get

$$\Phi[h]'(x, y)\big[(z, w)\big] = \left(h'(x)z, \frac{\partial}{\partial x}\Gamma(h, x)[z]y + \Gamma(h, x)w\right).$$

The direct estimation leads us to

$$
\begin{aligned}
\big\| \Phi[h]'(x, y) \big\|_{L(Z)} &= \sup_{(z,w)\in\mathbb{D}} \big\| \Phi[h]'(x, y)\big[(z, w)\big] \big\| \\
&\leqslant \sup_{(z,w)\in\mathbb{D}} \left(\big\| h'(x)z \big\|_X + \left\| \frac{\partial}{\partial x}\Gamma(h, x)[z] \right\|_{L(Y)} p\big(\|x\|_X\big) + \big\| \Gamma(h, x) \big\|_{L(Y)} \|w\|_Y \right) \\
&\leqslant \big\| h'(x) \big\|_{L(X)} + \left\| \frac{\partial}{\partial x}\Gamma(h, x) \right\|_{L(X,L(Y))} p\big(\|x\|_X\big) + \big\| \Gamma(h, x) \big\|_{L(Y)}.
\end{aligned}
$$

Therefore,

$$
\sup_{(x,y)\in\mathbb{D}} \big\| \Phi[h]'(x, y) \big\|_{L(Z)}\big(1 - \big\|(x, y)\big\|^2\big) \leqslant \big\| h'(x) \big\|_{L(X)}\big(1 - \|x\|_X^2\big) + \big\| \Gamma(h, x) \big\|_{L(Y)}\big(1 - \|x\|_X^2\big)
$$

$$
+ \left\| \frac{\partial}{\partial x}\Gamma(h, x) \right\|_{L(X,L(Y))} p\big(\|x\|_X\big)\big(1 - \|x\|_X^2\big),
$$

and the assertion follows. $\quad\square$

6.2. Open questions

(a) It seems to be possible to repeat a similar construction for non-linear operators Γ. At the same time, we know of no concrete example of an extension operator of the form (3.3) with non-linear Γ. The question could be to find such examples.

(b) As a rule, the convexity property is more delicate. For instance, quoting [11], we note that it seems to be difficult to perturb either the extension operator or the domain without losing the convexity-preserving property. The original Roper–Suffridge operator (1.1) preserves the convexity of the image of the p-ball only if $p = 2$, i.e., of the Euclidean ball. On the other hand, if f is convex, then the extended mapping defined by formula (1.2) is convex if and only if $\beta = \frac{1}{2}$ (see, [9]). So, it is natural to examine which conditions on Γ allow the extension operator (3.3) to preserve the convexity.

(c) Our scheme does not cover the extension operators introduced by Muir [15,16]. We ask: how to expand it to include his operators.

Acknowledgments

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