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Eigen series solutions to terminal-state tracking optimal control problems and exact controllability problems constrained by linear parabolic PDEs

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Abstract

Terminal-state tracking optimal control problems for linear parabolic equations are studied in this paper. The control objectives are to track a desired terminal state and the control is of the distributed type. Explicit solution formulae for the optimal control problems are derived in the form of eigen series. Pointwise-in-time L^2 norm estimates for the optimal solutions are obtained and approximate controllability results are established. Exact controllability is shown when the target state vanishes on the boundary of the spatial domain. One-dimensional computational results are presented which illustrate the terminal-state tracking properties for the solutions expressed by the series formulae.

Keywords: Terminal-state tracking; Optimal control; Parabolic equations; Eigen series; Approximate and exact controllability

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1. Introduction

In this paper we study terminal-state tracking optimal control problems for a linear second order parabolic partial differential equation (PDE) defined over the time interval $[0, T] \subset [0, \infty)$ and on a bounded, C^2 (or convex) spatial domain $\Omega \subset \mathbb{R}^d$, $d = 1$, 2 or 3. Let a target function $W \in L^2(\Omega)$ and an initial condition $w \in L^2(\Omega)$ be given and let $f \in L^2((0, T) \times \Omega)$ denote the distributed control. The optimal control problems we study are to minimize the terminal-state tracking functional

$$\mathcal{J}(u, f) = \frac{T}{2} \int_\Omega \bigl|u(T, \mathbf{x}) - W(\mathbf{x})\bigr|^2 \, d\mathbf{x} + \frac{\gamma}{2} \int_0^T \int_\Omega \bigl|f(t, \mathbf{x})\bigr|^2 \, d\mathbf{x}\, dt \tag{1.1}$$

or

$$\mathcal{K}(u, f) = \frac{T}{2} \int_\Omega \bigl|u(T, \mathbf{x}) - W(\mathbf{x})\bigr|^2 \, d\mathbf{x} + \frac{\gamma}{2} \int_0^T \int_\Omega \bigl|f(t, \mathbf{x}) - F(t, \mathbf{x})\bigr|^2 \, d\mathbf{x}\, dt \tag{1.2}$$

(where γ is a positive constant and F is a given reference function) subject to the parabolic PDE

$$u_t - \operatorname{div}\bigl[A(\mathbf{x})\nabla u\bigr] = f, \quad (t, \mathbf{x}) \in (0, T) \times \Omega, \tag{1.3}$$

with the homogeneous boundary condition

$$u = 0, \quad (t, \mathbf{x}) \in (0, T) \times \partial\Omega, \tag{1.4}$$

and the initial condition

$$u(0, \mathbf{x}) = w(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{1.5}$$

In (1.3), $A(\mathbf{x})$ is a symmetric matrix-valued, $C^1(\overline{\Omega})$ function that is uniformly positive definite.

Similar optimal control problems have been studied in the literature from different aspects or in different settings. For instance, in [15] the existence and regularity of an optimal solution was studied; in [2] the connection between optimal solutions and controllability was examined, and in [22] eigen series solutions were studied wherein the control f was assumed to belong to a bounded set in $L^2((0, T) \times \Omega)$ (due to the boundedness constraint the tracking functional of [22] did not contain the term involving f). Both optimal control problems and controllability problems are studied in this paper. Our main achievements concerning optimal control problems include: the introduction of an F in (1.2) that results in an optimal solution that approaches the target more effectively (even for $t \ll T$ and moderate parameter γ); the derivation and justification of explicit eigen series solution formulae for optimal solutions; pointwise-in-time estimates for optimal solutions and the approximately controllability properties for the optimal solutions. A distinctive feature of this work is that the desired terminal-state W and the admissible state u are allowed to have nonmatching boundary conditions, though the reference function F need be suitably chosen in the formulation of cost functional (1.2) (the details about the choice of F will be revealed in Section 2).

Terminal-state tracking problems are optimal control problems in their own right. They are also closely related to approximate and exact controllability problems which were studied in, among others, [1–5,7–14,17–20]. As mentioned in the foregoing the boundary value for the target state W may be nonzero so that the parabolic problem (1.3)–(1.5) in general is not exactly controllable when the solution for (1.3)–(1.5) is defined in the standard weak sense (see [6]). Contributions of this paper on controllability consist of the proof of approximate controllability when the target state has an inhomogeneous boundary value and the derivation of explicit series solution formulae for the exact controllability problem when the target state vanishes on the boundary.

In Section 2 we formulate the optimal control problems and controllability problems in an appropriate mathematical framework. In Section 3 we review and establish certain results concerning eigen functions expansions for both spatial and temporal–spatial functions. In Section 4 we derive explicit eigen series solution formulae for the optimal control problems. In Section 5 we derive pointwise-in-time estimates for the optimal solutions and show that as the parameter $\gamma \to 0$, the optimal solutions at the terminal time T approach the target state W. In Section 6 we justify eigen series solution formulae for the exact controllability problem by assuming homogeneous boundary values for the target state. In Section 7 we present some one-dimensional computational results that illustrate the terminal-state tracking properties for the solutions expressed by the series formulae of Section 4.

2. Formulation of optimal control and controllability problems

Throughout we freely make use of standard Sobolev space notations $H^m(\Omega)$ and $H_0^1(\Omega)$. We denote the norm for Sobolev space $H^m(\Omega)$ by $\|\cdot\|_m$. Note that $H^0(\Omega) = L^2(\Omega)$ so that $\|\cdot\|_0$ is the $L^2(\Omega)$ norm. We will need the temporal–spatial function space

$$H^{2,1}\big((0,T)\times\Omega\big) = \big\{v \in L^2\big(0,T; H^2(\Omega)\big)\colon\ v_t \in L^2\big(0,T; L^2(\Omega)\big)\big\}.$$

A temporal–spatial function $v(t,\mathbf{x})$ often will be simply written as $v(t)$.

Functional (1.1) can be written as

$$\mathcal{J}(u,f) = \frac{T}{2}\big\|u(T) - W\big\|_0^2 + \frac{\gamma}{2}\int_0^T \big\|f(t)\big\|_0^2\,dt. \tag{2.1}$$

Regarding functional (1.2) the idea for constructing the reference function F is that we first choose a reference function $U(t,\mathbf{x})$ satisfying $U(T,\mathbf{x}) = W$ (i.e., U is a given path that reaches W at time T) and then set

$$F = U_t - \operatorname{div}\big[A(\mathbf{x})\nabla U\big] \quad \text{in } [0,T]\times\Omega.$$

However, W (and thus U) in general does not vanish on the boundary. The series method to be studied in this paper will involve eigen series expressions for reference functions F and U. The validity of these expressions require U to vanish on the boundary. To resolve this difficulty we choose the reference function $F = F^{(\gamma)}$ (which is dependent of γ) as follows. We first choose a one-parameter set of functions $\{W^{(\gamma)}\colon \gamma > 0\} \subset H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\|W^{(\gamma)} - W\|_0 \to 0 \quad \text{as } \gamma \to 0. \tag{2.2}$$

(If $W \in H^2(\Omega) \cap H_0^1(\Omega)$, then we may simply choose $W^{(\gamma)} = W$ that is independent of γ. In general, W has an inhomogeneous boundary condition and $W^{(\gamma)}$ approximates W in the $L^2(\Omega)$ sense.) Next, for each given $\gamma > 0$, we choose a function $V^{(\gamma)}(t, \mathbf{x})$ that satisfies

$$\begin{aligned} &V^{(\gamma)} \in L^2\big(0, T; H^2(\Omega) \cap H_0^1(\Omega)\big), \qquad V_t^{(\gamma)} \in L^2\big(0, T; L^2(\Omega)\big), \\ &V^{(\gamma)}(T) = W^{(\gamma)} \quad \text{in } \Omega; \end{aligned} \tag{2.3}$$

in other words, $V^{(\gamma)}$ is an arbitrarily chosen smooth path that reaches $W^{(\gamma)}$ at time T. By virtue of (2.2)–(2.3) we have

$$\|V^{(\gamma)}(T) - W\|_0 \to 0 \quad \text{as } \gamma \to 0. \tag{2.4}$$

We also assume

$$\|V^{(\gamma)}(0)\|_0 \leqslant C \quad \text{where } C > 0 \text{ is a constant independent of } \gamma. \tag{2.5}$$

The choices of a $V^{(\gamma)}$ that satisfies (2.3)–(2.5) are certainly nonvacuous, e.g., the steady-state function $V^{(\gamma)}(t, \cdot) = W^{(\gamma)}$ is a particular and convenient choice. Here we allow for more general choices of such a path $V^{(\gamma)}(t, \cdot)$ than the steady-state one. The reference function F is now defined by

$$F = F^{(\gamma)} \equiv V_t^{(\gamma)} - \operatorname{div}\big[A(\mathbf{x})\nabla V^{(\gamma)}\big] \quad \text{in } (0, T) \times \Omega. \tag{2.6}$$

Functional (1.2) may be written

$$\begin{aligned} \mathcal{K}(u, f) &= \frac{T}{2}\|u(T) - W\|_0^2 + \frac{\gamma}{2}\int_0^T \|f(t) - F(t)\|_0^2\, dt \\ &= \frac{T}{2}\|u(T) - W\|_0^2 + \frac{\gamma}{2}\int_0^T \left\| f(t) - \frac{d}{dt}V^{(\gamma)}(t) - \operatorname{div}\big[A(\mathbf{x})\nabla V^{(\gamma)}(t)\big]\right\|_0^2 dt. \end{aligned} \tag{2.7}$$

The solution to the constraint equations (1.3)–(1.5) is understood in the following weak sense.

Definition 2.1. Let $f \in L^2((0, T); L^2(\Omega))$ and $w \in L^2(\Omega)$ be given. u is said to be a solution of (1.3)–(1.5) if $u \in L^2((0, T); H_0^1(\Omega))$, $u_t \in L^2((0, T); H^{-1}(\Omega))$, and u satisfies

$$\begin{aligned} &\langle u_t(t), \phi\rangle + \int_\Omega \big[A(\mathbf{x})\nabla u(t)\big] \cdot \nabla\phi\, d\mathbf{x} = \langle f(t), \phi\rangle \quad \forall \phi \in H_0^1(\Omega), \text{ a.e. } t \in (0, T), \\ &u(0) = w \quad \text{in } \Omega, \end{aligned} \tag{2.8}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Remark. A weak solution in the sense of Definition 2.1 belongs to $C([0,T];L^2(\Omega))$, see [6].

An admissible element for the optimal control problem is a pair (u, f) that satisfies the initial boundary value problem (2.8). The precise definition is given as follow.

Definition 2.2. Let $w \in L^2(\Omega)$ be given. A pair (u, f) is said to be an *admissible element* if $u \in L^2((0,T); H_0^1(\Omega))$, $u_t \in L^2((0,T); H^{-1}(\Omega))$, $f \in L^2((0,T); L^2(\Omega))$, and (u, f) satisfies Eq. (2.8). The set of all admissible elements is denoted by $\mathcal{V}_{\rm ad}((0,T), w)$ or simply $\mathcal{V}_{\rm ad}$.

The optimal control problems we study can be concisely stated as:

(OP1) seek a pair $(\hat{u}, \hat{f}) \in \mathcal{V}_{\rm ad}$ such that $\mathcal{J}(\hat{u}, \hat{f}) = \inf_{(u,f)\in\mathcal{V}_{\rm ad}} \mathcal{J}(u, f)$ where the functional $\mathcal{J}$ is defined by (2.1);

and

(OP2) seek a pair $(\hat{u}, \hat{f}) \in \mathcal{V}_{\rm ad}$ such that $\mathcal{K}(\hat{u}, \hat{f}) = \inf_{(u,f)\in\mathcal{V}_{\rm ad}} \mathcal{K}(u, f)$ where the functional $\mathcal{K}$ is defined by (2.7).

The existence and uniqueness of optimal solutions for (OP1) and (OP2) follow from classical optimal control theories (see, e.g., [15]):

Theorem 2.3. *Assume that $w \in L^2(\Omega)$ and $W \in L^2(\Omega)$. Then there exists a unique solution $(\hat{u}, \hat{f}) \in \mathcal{V}_{\rm ad}$ to* (OP1) *and to* (OP2). *If, in addition, $w \in H_0^1(\Omega)$, then $\hat{u} \in H^{2,1}((0,T)\times\Omega)$.*

The approximate and exact controllability problems are formulated as follows:

(AP-CON) seek a one-parameter set $\{(u_\epsilon, f_\epsilon)\colon \epsilon > 0\} \subset \mathcal{V}_{\rm ad}$ such that

$$\lim_{\epsilon\to 0} \left\| u_\epsilon(T) - W \right\|_0 = 0$$

and

(EX-CON) seek a pair $(u, f) \in \mathcal{V}_{\rm ad}$ such that

$$u(T) = W \quad \text{in } \Omega.$$

Of course, exact controllability, whenever it holds, implies approximate controllability. In particular, if w and W belong to $H_0^1(\Omega)$, then the exact controllability holds.

Theorem 2.4. *Assume that $w \in H_0^1(\Omega)$. Then* (EX-CON) *has a solution if and only if $W \in H_0^1(\Omega)$.*

Proof. If (EX-CON) has a solution (u, f), then regularity for parabolic PDEs [6, Theorem 5, p. 360; Theorem 4, p. 288] implies $u \in H^{2,1}(Q)$ and $u \in C([0,T]; H^1(\Omega))$ so that $W = u(T) \in H^1(\Omega)$. Since $u = 0$ on $(0,T) \times \partial\Omega$, we have that

$$\|W\|_{1/2,\partial\Omega} = \lim_{t\to T-} \big\|u(T) - u(t)\big\|_{1/2,\partial\Omega} \leqslant C \lim_{t\to T-} \big\|u(T) - u(t)\big\|_1 = 0,$$

where $\|\cdot\|_{1/2,\partial\Omega}$ denotes the norm for the Sobolev space $H^{1/2}(\partial\Omega)$. Thus, $W \in H_0^1(\Omega)$.

Conversely, assume that $W \in H_0^1(\Omega)$. Let $\tilde{u}$ be a function satisfying

$$\tilde{u} \in H^{2,1}\big((0,T) \times \Omega\big), \qquad \tilde{u} = 0 \quad \text{on } (0,T) \times \partial\Omega, \qquad \tilde{u}|_{t=0} = w \in H_0^1(\Omega).$$

The existence of such a $\tilde{u}$ is guaranteed by the trace theorem [16, vol. II, Theorem 2.3, p. 18] or by the existence and regularity results (see [6]) for the parabolic problem

$$u_t - \Delta u = 0 \quad \text{in } (0,T) \times \Omega, \qquad u = 0 \quad \text{in } (0,T) \times \partial\Omega, \qquad u|_{t=0} = w.$$

Likewise, there exists a $\tilde{\tilde{u}}$ satisfying

$$\tilde{\tilde{u}} \in H^{2,1}\big((0,T) \times \Omega\big), \qquad \tilde{\tilde{u}} = 0 \quad \text{on } (0,T) \times \partial\Omega, \qquad \tilde{\tilde{u}}|_{t=T} = W \in H_0^1(\Omega).$$

We choose a function $\theta = \theta(t) \in C^\infty[0,T]$ such that

$$\theta(t) = 1 \quad \forall t \in [0, T/3] \quad \text{and} \quad \theta(t) = 0 \quad \forall t \in [2T/3, T],$$

and set $u = \theta(t)\tilde{u} + [1 - \theta(t)]\tilde{\tilde{u}}$ in $(0,T) \times \Omega$. Clearly,

$$\begin{aligned} &u \in H^{2,1}\big((0,T) \times \Omega\big), \qquad u = 0 \quad \text{on } (0,T) \times \partial\Omega,\\ &u|_{t=0} = w, \qquad u|_{t=T} = W. \end{aligned}$$

By defining $f \equiv u_t - \mathrm{div}[A(\mathbf{x})\nabla u] \in L^2((0,T) \times \Omega)$ we see that (u, f) solves the exact controllability problem (EX-CON). $\square$

Remark. The exact controllability result of [2, Theorem 3.7] was stated imprecisely. The proof of that theorem, in fact, required the target state to have the homogeneous boundary condition.

3. Results concerning eigen function expansions

The main objective of this paper is to find explicit solution formulae, expressed in terms of eigen-function expansions, for optimal control problems (OP1) and (OP2) and for controllability problem (EX-CON). In this section we will review some properties for the eigen pairs and eigen function expansions.

We recall the following lemma (see [6, Theorem 1, p. 335]).

Lemma 3.1. *The set Λ of all eigen values for the elliptic operator $-\mathrm{div}(A(\mathbf{x})\nabla)$ may be written $\Lambda = \{\lambda_i\}_{i=1}^\infty \subset \mathbb{R}$ where*

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots \quad \textit{and} \quad \lambda_i \to \infty \quad \textit{as } i \to \infty.$$

Furthermore, there exists a set of corresponding eigen functions $\{e_i\}_{i=1}^{\infty} \subset H^2(\Omega) \cap H_0^1(\Omega)$ *which forms an orthonormal basis of* $L^2(\Omega)$ *(with respect to the* $L^2(\Omega)$ *inner product).*

In the sequel we let $\{(\lambda_i, e_i)\}_{i=1}^{\infty}$ denote a set of eigen pairs as stated in Lemma 3.1.

Lemma 3.2. *The set* $\{e_i/\sqrt{\lambda_i}\}_{i=1}^{\infty}$ *forms an orthonormal basis of* $H_0^1(\Omega)$ *with respect to the inner product*

$$(u, v) \mapsto B[u, v] \equiv \int_{\Omega} A(\mathbf{x})\nabla u \cdot \nabla v \, d\mathbf{x} \quad \forall u, v \in H_0^1(\Omega). \tag{3.1}$$

The set $\{e_i/\lambda_i\}_{i=1}^{\infty}$ *forms an orthonormal basis of* $H^2(\Omega) \cap H_0^1(\Omega)$ *with respect to the inner product*

$$(u, v) \mapsto \tilde{B}[u, v] \equiv \int_{\Omega} \operatorname{div}\bigl[A(\mathbf{x})\nabla u\bigr] \operatorname{div}\bigl[A(\mathbf{x})\nabla v\bigr] \, d\mathbf{x} \quad \forall u, v \in H^2(\Omega) \cap H_0^1(\Omega). \tag{3.2}$$

Proof. The first statement of this lemma is proved in [6, Theorem 1, p. 335; step 3, p. 337]. The proof for the second statement is a verbatim repetition of [6, Theorem 1, p. 335; step 3, p. 337] with the inner product $B[\cdot,\cdot]$ replaced by $\tilde{B}[\cdot,\cdot]$ (defined in (3.2)). □

Based on Lemmas 3.1 and 3.2 we may establish the following characterizations of $H_0^1(\Omega)$.

Lemma 3.3. *Assume* $y \in L^2(\Omega)$ *and* $y = \sum_{i=1}^{\infty} y_i e_i$ *in* $L^2(\Omega)$. *Then the following statements are equivalent:*

(i) $y \in H_0^1(\Omega)$;
(ii) $y = \sum_{i=1}^{\infty} y_i e_i$ *in* $H_0^1(\Omega)$;
(iii) $\sum_{i=1}^{\infty} \lambda_i |y_i|^2 < \infty$.

Proof. We first prove (i) implies (ii). But this follows from [6, Theorem 1, p. 335; steps 2 and 3, p. 337].

We next prove (ii) implies (iii). Assume $y = \sum_{i=1}^{\infty} y_i e_i$ in $H_0^1(\Omega)$. By Lemma 3.2 we may write $y = \sum_{i=1}^{\infty} \bar{y}_i e_i/\sqrt{\lambda_i}$ in $H_0^1(\Omega)$ and $\sum_{i=1}^{\infty} |\bar{y}_i|^2 = B[y, y] < \infty$. Comparing $y = \sum_{i=1}^{\infty} \bar{y}_i e_i/\sqrt{\lambda_i}$ and $y = \sum_{i=1}^{\infty} y_i e_i$ in $L^2(\Omega)$ we obtain $\bar{y}_i = \sqrt{\lambda_i}\, y_i$ so that $\sum_{i=1}^{\infty} \lambda_i |y_i|^2 = \sum_{i=1}^{\infty} |\bar{y}_i|^2 < \infty$.

Finally, we prove (iii) implies (i). Assume that $\sum_{i=1}^{\infty} \lambda_i |y_i|^2 < \infty$. We note that the definition of the eigen pairs implies

$$B[e_i, v] = \lambda_i \int_{\Omega} e_i v \, d\mathbf{x} \quad \forall v \in H_0^1(\Omega)$$

so that $B[e_i, e_j] = 0$ if $j \neq i$ and $B[e_i, e_i] = \lambda_i$. Thus,

$$B\left[\sum_{i=n}^{n+p} y_i e_i, \sum_{j=n}^{n+p} y_j e_j\right] = \sum_{i=n}^{n+p} \lambda_i |y_i|^2$$

so that $\{\sum_{i=1}^{n} y_i e_i\}_{n=1}^{\infty} \subset H_0^1(\Omega)$ is a Cauchy sequence with respect to the $H_0^1(\Omega)$ norm induced by the $B[\cdot,\cdot]$ inner product. Hence $\sum_{i=1}^{\infty} y_i e_i = \bar{y}$ in $H_0^1(\Omega)$ for some $\bar{y} \in H_0^1(\Omega)$. But $y = \sum_{i=1}^{\infty} y_i e_i$ in $L^2(\Omega)$ and we conclude $y = \bar{y} \in H_0^1(\Omega)$. $\square$

Similar arguments yield the following characterizations of $H^2(\Omega) \cap H_0^1(\Omega)$.

Lemma 3.4. *Assume $y \in L^2(\Omega)$ and $y = \sum_{i=1}^{\infty} y_i e_i$ in $L^2(\Omega)$. Then the following statements are equivalent*:

(i) $y \in H^2(\Omega) \cap H_0^1(\Omega)$;
(ii) $y = \sum_{i=1}^{\infty} y_i e_i$ *in* $H^2(\Omega) \cap H_0^1(\Omega)$;
(iii) $\sum_{i=1}^{\infty} |\lambda_i|^2 |y_i|^2 < \infty$.

The main results of this section are the two theorems below concerning term-by-term differentiations of eigen series for functions in $H^{2,1}((0,T) \times \Omega) \cap C([0,T]; H_0^1(\Omega))$. We first quote a lemma (see [21, Lemma 1.1, p. 169] and [6, Theorem 2, p. 286]):

Lemma 3.5. *Assume $u \in L^2(0,T; L^2(\Omega))$ and $u_t \in L^2(0,T; L^2(\Omega))$. Then*

$$-\int_0^T \phi'(t) \int_\Omega u(t) v \, d\mathbf{x}\, dt = \int_0^T \phi(t) \int_\Omega u_t(t) v \, d\mathbf{x}\, dt \quad \forall \phi \in C_0^\infty(0,T),\ \forall v \in L^2(\Omega).$$

Theorem 3.6. *Assume that $u \in H^{2,1}((0,T) \times \Omega)$, $u = 0$ on $(0,T) \times \partial\Omega$ and*

$$u(t) = \sum_{i=1}^{\infty} u_i(t) e_i \quad \text{in } L^2(\Omega),\ \text{a.e. } t \in (0,T).$$

Then

$$\sum_{i=1}^{\infty} \int_0^T \left(\left|u_i'(t)\right|^2 + |\lambda_i|^2 \left|u_i(t)\right|^2\right) dt = \|u_t\|^2_{L^2(0,T;L^2(\Omega))} + \int_0^T \tilde{B}[u,u]\, dt < \infty, \tag{3.3}$$

$$\sum_{i=1}^{\infty} |\lambda_i| \left|u_i(0)\right|^2 dt < \infty, \tag{3.4}$$

$$u_t(t) = \sum_{i=1}^{\infty} u_i'(t) e_i \quad \text{in } L^2(\Omega),\ \text{a.e. } t \in (0,T) \tag{3.5}$$

and

$$-\operatorname{div}\big[A(\mathbf{x})\nabla u(t)\big] = \sum_{i=1}^{\infty} \lambda_i u_i(t) e_i \quad \textit{in } L^2(\Omega),\ \textit{a.e. } t. \tag{3.6}$$

Proof. We first note the continuous embedding $H^{2,1}((0,T)\times\Omega) \hookrightarrow C([0,T]; H^1(\Omega))$ and the boundary condition $u = 0$ on $(0,T)\times\partial\Omega$ imply that $u(t) \in H_0^1(\Omega)$ for every $t \in [0,T]$. By Lemma 3.3 we have

$$u(t) = \sum_{i=1}^{\infty} u_i(t) e_i \quad \text{in } H_0^1(\Omega), \forall t \in [0,T].$$

In particular, since $u(0) \in H_0^1(\Omega)$, Lemma 3.3 yields (3.4).

Using the $L^2(\Omega)$ orthonormality of $\{e_i\}$ we have

$$\|u\|^2_{L^2(0,T;L^2(\Omega))} = \int_0^T \big\|u(t)\big\|_0^2\,dt = \int_0^T \sum_{i=1}^{\infty} \big|u_i(t)\big|^2\,dt \geqslant \int_0^T \big|u_j(t)\big|^2\,dt \quad \forall j$$

so that each $u_j \in L^2(0,T)$. Since $u_t \in L^2(0,T; L^2(\Omega))$, we may write

$$u_t(t) = \sum_{i=1}^{\infty} v_i(t) e_i \quad \text{in } L^2(\Omega),\ \text{a.e. } t$$

and

$$\|u_t\|^2_{L^2(0,T;L^2(\Omega))} = \int_0^T \big\|u_t(t)\big\|_0^2\,dt = \int_0^T \sum_{i=1}^{\infty} \big|v_i(t)\big|^2\,dt \geqslant \int_0^T \big|v_j(t)\big|^2\,dt \quad \forall j \tag{3.7}$$

so that each $v_j \in L^2(0,T)$. Using Lemma 3.5 we have that

$$-\int_0^T \phi'(t) \int_\Omega u(t) e_j\,d\mathbf{x}\,dt = \int_0^T \phi(t) \int_\Omega u_t(t) e_j\,d\mathbf{x}\,dt$$
$$\forall \phi \in C_0^\infty(0,T),\ j = 1,2,\ldots.$$

Substituting series expressions for u and u_t into the last equation and using the $L^2(\Omega)$ orthonormality of $\{e_i\}$ we obtain

$$-\int_0^T \phi'(t) u_j(t)\,dt = \int_0^T \phi(t) v_j(t)\,dt \quad \forall \phi \in C_0^\infty(0,T),\ j = 1,2,\ldots,$$

so that $v_j = u_j'$ for $j = 1,2,\ldots$. This proves (3.5).

Since $u(t) \in H^2(\Omega) \cap H_0^1(\Omega)$ for almost every t, Lemma 3.4 implies that

$$u(t) = \sum_{i=1}^{\infty} u_i(t) e_i \quad \text{in } H^2(\Omega) \cap H_0^1(\Omega),\ \text{a.e. } t$$

so that

$$-\operatorname{div}\big[A(\mathbf{x})\nabla u(t)\big] = \sum_{i=1}^{\infty} -\operatorname{div}\big[A(\mathbf{x})\nabla u_i(t)e_i\big] = \sum_{i=1}^{\infty} \lambda_i u_i(t)e_i \quad \text{in } L^2(\Omega), \text{ a.e. } t,$$

i.e., (3.6) holds.

From (3.6) we obtain

$$\int_0^T \tilde{B}[u,u]\,dt = \int_0^T \big\|\operatorname{div}\big[A(\mathbf{x})\nabla u(t)\big]\big\|_0^2\,dt = \int_0^T \sum_{i=1}^{\infty} |\lambda_i|^2\big|u_i(t)\big|^2\,dt. \tag{3.8}$$

Adding up (3.7) and (3.8) and applying the Monotone Convergence theorem we arrive at (3.3). □

Theorem 3.7. *Assume that the set of functions $\{u_i(t)\}_{i=1}^{\infty} \subset H^1(0,T)$ satisfies*

$$\sum_{i=1}^{\infty} \int_0^T \big(\big|u_i'(t)\big|^2 + |\lambda_i|^2\big|u_i(t)\big|^2\big)\,dt < \infty \tag{3.9}$$

and

$$\sum_{i=1}^{\infty} |\lambda_i|\big|u_i(0)\big|^2\,dt < \infty. \tag{3.10}$$

Then the function u formally defined by $u(t) = \sum_{i=1}^{\infty} u_i(t)e_i$ satisfies

$$u \in H^{2,1}\big((0,T)\times\Omega\big), \qquad u = 0 \quad \textit{on } (0,T)\times\partial\Omega,$$

$$u_t(t) = \sum_{i=1}^{\infty} u_i'(t)e_i \quad \textit{in } L^2(\Omega),\ \textit{a.e. } t, \tag{3.11}$$

and

$$-\operatorname{div}\big[A(\mathbf{x})\nabla u(t)\big] = \sum_{i=1}^{\infty} \lambda_i u_i(t)e_i \quad \textit{in } L^2(\Omega),\ \textit{a.e. } t. \tag{3.12}$$

Proof. We note that

$$\sum_{i=1}^{\infty} \int_0^T \big|u_i(t)\big|^2\,dt \leqslant \frac{1}{|\lambda_1|^2} \sum_{i=1}^{\infty} \int_0^T |\lambda_i|^2\big|u_i(t)\big|^2\,dt < \infty$$

so that $u(t) = \sum_{i=1}^{\infty} u_i(t)e_i$ in $L^2(\Omega)$ for almost every $t \in (0,T)$.

By assumption (3.9) we are justified to define $f \in L^2((0,T);L^2(\Omega))$ as the series function

$$f = \sum_{i=1}^{\infty} f_i(t)e_i \equiv \sum_{i=1}^{\infty} \big[u_i'(t) + \lambda_i u_i(t)\big]e_i \quad \text{in } L^2(\Omega), \text{ a.e. } t \in (0,T).$$

It is well known that $H^1(0,T)$ is continuously embedded into $C[0,T]$ so that $u_i(0)$ is well defined for each i. Assumption (3.10) and Lemma 3.3 imply that $u|_{t=0} \in H_0^1(\Omega)$ where $u|_{t=0} = \sum_{i=1}^{\infty} u_i(0)e_i$.

Let $\tilde{u}$ be the solution for the parabolic problem

$$\begin{aligned} &\tilde{u}_t - \operatorname{div}\big[A(\mathbf{x})\nabla\tilde{u}\big] = f \quad \text{in } (0,T)\times\Omega, \qquad \tilde{u} = 0 \quad \text{on } (0,T)\times\partial\Omega, \\ &\tilde{u}|_{t=0} = u|_{t=0} \end{aligned} \tag{3.13}$$

in the sense of Definition 2.1. Regularity for parabolic PDEs implies $\tilde{u} \in H^{2,1}((0,T)\times\Omega)$. We write $\tilde{u} = \sum_{i=1}^{\infty} \tilde{u}_i(t)e_i$ in $L^2(\Omega)$ for almost every $t \in (0,T)$. Employing Theorem 3.6 we have

$$\tilde{u}_t(t) = \sum_{i=1}^{\infty} \tilde{u}_i'(t)e_i \quad \text{in } L^2(\Omega), \text{ a.e. } t \tag{3.14}$$

and

$$-\operatorname{div}\big[A(\mathbf{x})\nabla\tilde{u}(t)\big] = \sum_{i=1}^{\infty} \lambda_i\tilde{u}_i(t)e_i \quad \text{in } L^2(\Omega), \text{ a.e. } t. \tag{3.15}$$

Thus, we may write (3.13) in the series form

$$\begin{cases} \sum_{i=1}^{\infty}\big[\tilde{u}_i'(t) + \lambda_i\tilde{u}_i(t)\big]e_i = \sum_{i=1}^{\infty} f_i(t)e_i \quad \text{in } L^2(\Omega), \text{ a.e. } t, \\ \sum_{i=1}^{\infty} \tilde{u}_i(0) = \sum_{i=1}^{\infty} u_i(0)e_i \quad \text{in } L^2(\Omega) \end{cases}$$

so that for each i,

$$\tilde{u}_i(t) + \lambda_i\tilde{u}_i(t) = f_i(t) \quad \text{in } (0,T), \qquad \tilde{u}_i(0) = u_i(0). \tag{3.16}$$

From the definition of f_i we see that each u_i satisfies the same equations as $\tilde{u}_i$. The uniqueness of the solution for the initial value problem (3.16) implies $u_i \equiv \tilde{u}_i$ in $(0,T)$ for each i so that $u(t) = \tilde{u}(t)$ in $L^2(\Omega)$ for every t. Hence, $u = \tilde{u} \in H^{2,1}((0,T)\times\Omega)$ and $u = \tilde{u} = 0$ on $(0,T)\times\partial\Omega)$. Also, Eqs. (3.14) and (3.15) yield (3.11) and (3.12). $\square$

4. Solutions of the optimal control problems

We express all functions involved as $L^2(\Omega)$-convergent series of $\{e_i\}$:

$$u(t,\mathbf{x}) = \sum_{i=1}^{\infty} u_i(t)e_i(\mathbf{x}), \qquad f(t,\mathbf{x}) = \sum_{i=1}^{\infty} f_i(t)e_i(\mathbf{x}), \qquad w(\mathbf{x}) = \sum_{i=1}^{\infty} w_i e_i(\mathbf{x}),$$

$$W(\mathbf{x}) = \sum_{i=1}^{\infty} W_i e_i(\mathbf{x}), \qquad V^{(\gamma)}(t,\mathbf{x}) = \sum_{i=1}^{\infty} V_i^{(\gamma)}(t)e_i(\mathbf{x}).$$

We work out below an explicit formula for the optimal solution of (OP1) expressed as a series of eigen functions $\{e_i\}$. (For the existence of optimal solutions, see Theorem 2.3.)

Theorem 4.1. *Assume $w \in H_0^1(\Omega)$ and $W \in L^2(\Omega)$. Let $(\hat{u}, \hat{f}) \in H^{2,1}((0,T) \times \Omega) \times L^2((0,T) \times \Omega)$ be the solution of* (OP1). *Then*

$$\hat{u}(t, \mathbf{x}) = \sum_{i=1}^{\infty} \hat{u}_i(t) e_i(\mathbf{x}), \tag{4.1}$$

where

$$\begin{aligned} \hat{u}_i(t) = w_i \left(e^{-\lambda_i t} - \frac{T e^{-\lambda_i T}(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i \gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})} \right) \\ + W_i \frac{T(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i \gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})}. \end{aligned} \tag{4.2}$$

Proof. Let (u, f) be an arbitrary admissible element, then $u \in H^{2,1}((0,T) \times \Omega) \cap C([0,T]; H_0^1(\Omega))$. We may write $u = \sum_{i=1}^{\infty} u_i(t) e_i$ and $f = \sum_{i=1}^{\infty} f_i(t) e_i$ in $L^2(\Omega)$ for almost every t. Moreover, Theorem 3.6 implies

$$u_t = \sum_{i=1}^{\infty} u_i'(t) e_i \quad \text{in } L^2(\Omega), \text{ a.e. } t$$

and

$$-\operatorname{div}\big[A(\mathbf{x}) \nabla u\big] = \sum_{i=1}^{\infty} \lambda_i u_i(t) e_i \quad \text{in } L^2(\Omega), \text{ a.e. } t.$$

Thus we may rewrite the constraint equations (2.8) as

$$\begin{cases} \int_\Omega \Big(\sum_{j=1}^{\infty} \big[u_j'(t) + \lambda_j u_j(t)\big] e_j \Big) e_i \, d\mathbf{x} = \int_\Omega \Big(\sum_{j=1}^{\infty} f_j(t) e_j \Big) e_i \, d\mathbf{x}, \quad i = 1, 2, \ldots, \\ \int_\Omega \Big(\sum_{j=1}^{\infty} u_j(0) e_j \Big) e_i = \int_\Omega \Big(\sum_{j=1}^{\infty} w_j e_j \Big) e_i \, d\mathbf{x}, \quad i = 1, 2, \ldots, \end{cases}$$

so that for each i,

$$u_i'(t) + \lambda_i u_i(t) = f_i(t) \quad \text{in } (0,T), \qquad u_i(0) = w_i. \tag{4.3}$$

The functional $\mathcal{J}$ also can be written in the series form

$$\mathcal{J}(u, f) = \frac{T}{2} \sum_{i=1}^{\infty} \big| u_i(T) - W_i \big|^2 + \frac{\gamma}{2} \sum_{i=1}^{\infty} \int_0^T \big| f_i(t) \big|^2 \, dt. \tag{4.4}$$

The optimal control problem (OP1) is recast into:

$(\widetilde{\text{OP1}})$ minimize functional (4.4) subject to the constraints (4.3) for all $i = 1, 2, \ldots$.

Since the constraint equations are fully uncoupled for each i, the optimal control problem $(\text{O}\tilde{\text{P}}1)$ is equivalent to:

$(\widetilde{\text{OP1}}_i)$ for each $i = 1, 2, \ldots$, minimize $\mathcal{J}_i(u_i, f_i)$ subject to the constraints (4.3), where the functional $\mathcal{J}_i(u_i, f_i)$ is defined by

$$\mathcal{J}_i(u_i, f_i) = \frac{T}{2}\left|u_i(T) - W_i\right|^2 + \frac{\gamma}{2}\int_0^T \left|f_i(t)\right|^2 dt.$$

The pair $(\hat{u}, \hat{f}) = (\sum_{i=1}^{\infty} \hat{u}_i(t)e_i(\mathbf{x}), \sum_{i=1}^{\infty} \hat{f}_i(t)e_i(\mathbf{x}))$ is the solution for (OP1) if and only if $(\hat{u}_i, \hat{f}_i)$ is the solution for $(\widetilde{\mathrm{OP}1}_i)$ for every i.

To solve the constrained minimization problem $(\tilde{\mathrm{OP}}1_i)$ we introduce a Lagrange multiplier ξ_i and form the Lagrangian

$$\begin{aligned}\mathcal{L}_i(u_i, f_i, \xi_i) &= \frac{T}{2}\left|u_i(T) - W_i\right|^2 - u_i(T)\xi_i(T) + w_i\xi_i(0) \\ &\quad + \int_0^T \left(\frac{\gamma}{2}\left|f_i(t)\right|^2 + u_i(t)\xi_i'(t) - \lambda_i u_i(t)\xi_i(t) + f_i(t)\xi_i(t)\right) dt.\end{aligned}$$

By taking variations of the Lagrangian with respect to ξ_i, u_i and f_i, respectively, we obtain an optimality system which consists of (4.3),

$$\xi_i'(t) - \lambda_i\xi_i(t) = 0 \quad \text{in } (0, T), \qquad \xi_i(T) = T\big(u_i(T) - W_i\big) \tag{4.5}$$

and

$$\xi_i(t) = -\gamma f_i(t). \tag{4.6}$$

We proceed to solve for $(\hat{u}_i, \hat{f}_i)$ from the optimality system formed by (4.3), (4.5) and (4.6).

By eliminating ξ_i from (4.5)–(4.6) we have

$$f_i'(t) - \lambda_i f_i(t) = 0 \quad \text{in } (0, T), \qquad f_i(T) = -\frac{T}{\gamma}\big(u_i(T) - W_i\big). \tag{4.7}$$

Combining (4.7) and (4.3) we arrive at a second order ordinary differential equation with initial and terminal conditions:

$$\begin{cases} u_i''(t) - \lambda_i^2 u_i(t) = 0 \quad \text{in } (0, T), \\ u_i(0) = w_i, \\ u_i'(T) + \lambda_i u_i(T) = -\frac{T}{\gamma}\big(u_i(T) - W_i\big). \end{cases} \tag{4.8}$$

The general solution to this differential equation is

$$u_i(t) = C_1 e^{-\lambda_i t} + C_2 e^{\lambda_i t}.$$

The initial and terminal conditions yield:

$$\begin{cases} C_1 + C_2 = w_i, \\ \frac{T}{\gamma}e^{-\lambda_i T}C_1 + \big(2\lambda_i e^{\lambda_i T} + \frac{T}{\gamma}e^{\lambda_i T}\big)C_2 = \frac{T}{\gamma}W_i. \end{cases}$$

Solving for C_1 and C_2 and then plugging them into the general solution we find the formula for the solution $\hat{u}_i$ to (4.8) and that formula is precisely (4.2). Hence, the solution to (OP1) is expressed by (4.1)–(4.2). $\square$

Similarly, we may derive an explicit formula for the optimal solution of (OP2).

Theorem 4.2. *Assume that* $w \in H_0^1(\Omega)$, $W \in L^2(\Omega)$, $W^{(\gamma)} \in H^2(\Omega) \cap H_0^1(\Omega)$, $V^{(\gamma)}$ *satisfies (2.3) and* F *is defined by* (2.6). *Let* $(\hat{u}, \hat{f}) \in H^{2,1}((0,T) \times \Omega) \times L^2((0,T) \times \Omega)$ *be the solution of* (OP2). *Then*

$$\hat{u}(t, \mathbf{x}) = \sum_{i=1}^{\infty} \hat{u}_i(t) e_i(\mathbf{x}), \tag{4.9}$$

where

$$\begin{aligned} \hat{u}_i(t) = {} & V_i^{(\gamma)}(t) + \big[w_i - V_i^{(\gamma)}(0)\big]\left(e^{-\lambda_i t} - \frac{T e^{-\lambda_i T}(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i \gamma e^{\lambda_i T} + T e^{\lambda_i T} - T e^{-\lambda_i T}}\right) \\ & + \big[W_i - V_i^{(\gamma)}(T)\big] \frac{T(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i \gamma e^{\lambda_i T} + T e^{\lambda_i T} - T e^{-\lambda_i T}}. \end{aligned} \tag{4.10}$$

Proof. As in the proof of Theorem 4.1 we may write the constraint equations as

$$u_i'(t) + \lambda_i u_i(t) = f_i(t) \quad \text{in } (0,T), \qquad u_i(0) = w_i \tag{4.11}$$

for $i = 1, 2, \ldots.$

To simplify the notation we drop the superscript $(\cdot)^{(\gamma)}$ and write V in place of $V^{(\gamma)}$. Since $V \in H^{2,1}((0,T) \times \Omega)$, we are justified by Theorem 3.6 to express (2.6) in the series form

$$\begin{aligned} & \sum_{i=1}^{\infty} F_i(t) e_i = F(t, \mathbf{x}) = V_t - \operatorname{div}\big[A(\mathbf{x}) \nabla V\big] = \sum_{i=1}^{\infty} \Big[V_i'(t) + \lambda_i V_i(t)\Big] e_i \\ & \quad \text{in } L^2(\Omega), \text{ a.e. } t, \end{aligned} \tag{4.12}$$

so that

$$F_i(t) = V_i'(t) + \lambda_i V_i(t) \quad \text{a.e. } t.$$

The functional $\mathcal{K}$ also can be written in the series form

$$\begin{aligned} \mathcal{K}(u, f) &= \frac{T}{2} \sum_{i=1}^{\infty} \big|u_i(T) - W_i\big|^2 + \frac{\gamma}{2} \sum_{i=1}^{\infty} \int_0^T \big|f_i(t) - F_i(t)\big|^2 \, dt \\ &= \frac{T}{2} \sum_{i=1}^{\infty} \big|u_i(T) - W_i\big|^2 + \frac{\gamma}{2} \sum_{i=1}^{\infty} \int_0^T \big|f_i(t) - V_i'(t) - \lambda_i V_i(t)\big|^2 \, dt. \end{aligned} \tag{4.13}$$

The optimal control problem (OP2) is recast into:

$(\widetilde{\mathrm{OP2}})$ minimize functional (4.13) subject to the constraints (4.11) for all $i = 1, 2, \ldots.$

Since the constraint equations are fully uncoupled for each i, the optimal control problem $(\widetilde{\mathrm{OP2}})$ is equivalent to:

$(\widetilde{\mathrm{OP2}}_i)$ for each $i = 1, 2, \ldots,$ minimize $\mathcal{K}_i(u_i, f_i)$ subject to the constraints (4.11), where the functional $\mathcal{K}_i(u_i, f_i)$ is defined by

$$\mathcal{K}_i(u_i, f_i) = \frac{T}{2}\left|u_i(T) - W_i\right|^2 + \frac{\gamma}{2}\int_0^T \left|f_i(t) - V_i'(t) - \lambda_i V_i(t)\right|^2 dt.$$

The pair $(\hat{u}, \hat{f}) = (\sum_{i=1}^\infty \hat{u}_i(t)e_i(\mathbf{x}), \sum_{i=1}^\infty \hat{f}_i(t)e_i(\mathbf{x}))$ is the solution for (OP2) if and only if $(\hat{u}_i, \hat{f}_i)$ is the solution for $(\widetilde{\mathrm{OP2}}_i)$ for every i.

To solve the constrained minimization problem $(\widetilde{\mathrm{OP2}}_i)$ we introduce a Lagrange multiplier ξ_i and form the Lagrangian

$$\begin{aligned}
&\mathcal{L}_i(u_i, f_i, \xi_i) \\
&\quad = \frac{T}{2}\left|u_i(T) - W_i\right|^2 - u_i(T)\xi_i(T) + w_i\xi_i(0) \\
&\qquad + \int_0^T \left(\frac{\gamma}{2}\left|f_i(t) - V_i'(t) - \lambda_i V_i(t)\right|^2 + u_i(t)\xi_i'(t) - \lambda_i u_i(t)\xi_i(t) + f_i(t)\xi_i(t)\right) dt.
\end{aligned}$$

By taking variations of the Lagrangian with respect to ξ_i, u_i and f_i, respectively, we obtain an optimality system which consists of (4.11),

$$\xi_i'(t) - \lambda_i\xi_i(t) = 0 \quad \text{in } (0, T), \qquad \xi_i(T) = T\big(u_i(T) - W_i\big) \tag{4.14}$$

and

$$\xi_i(t) = -\gamma\left[f_i(t) - V_i'(t) - \lambda_i V_i(t)\right] \quad \text{in } (0, T). \tag{4.15}$$

We proceed to solve for $(\hat{u}_i, \hat{f}_i)$ from the optimality system formed by (4.11), (4.14) and (4.15). By eliminating ξ_i from (4.14)–(4.15) we have

$$\begin{cases} f_i'(t) - \lambda_i f_i(t) = V_i''(t) - \lambda_i^2 V_i(t) \quad \text{in } (0, T), \\ f_i(T) = V_i'(T) + \lambda_i V_i(T) - \frac{T}{\gamma}\big(u_i(T) - W_i\big). \end{cases} \tag{4.16}$$

Combining (4.16) and (4.11) we arrive at a second order ordinary differential equation with initial and terminal conditions:

$$\begin{cases} u_i''(t) - \lambda_i^2 u_i(t) = V_i''(t) - \lambda_i^2 V_i(t) \quad \text{in } (0, T), \\ u_i(0) = w_i, \\ u_i'(T) + \lambda_i u_i(T) = V_i'(T) + \lambda_i V_i(T) - \frac{T}{\gamma}\big(u_i(T) - W_i\big). \end{cases} \tag{4.17}$$

Evidently, $V_i(t)$ is a particular solution of this differential equation so that the general solution is

$$u_i(t) = V_i(t) + C_1 e^{-\lambda_i t} + C_2 e^{\lambda_i t}.$$

The initial and terminal conditions yield:

$$\begin{cases} C_1 + C_2 = w_i - V_i(0), \\ \frac{T}{\gamma}e^{-\lambda_i T}C_1 + \big(2\lambda_i e^{\lambda_i T} + \frac{T}{\gamma}e^{\lambda_i T}\big)C_2 = \frac{T}{\gamma}\left[W_i - V_i(T)\right]. \end{cases}$$

Solving for C_1 and C_2 and then plugging them into the general solution we find formula (4.10) for the solution $\hat{u}_i$ to (4.17). Hence, the solution to (OP2) is expressed by (4.9). □

Remark. In order for series expressions (4.12) to be valid, $V(t,\mathbf{x}) = \sum_{i=1}^{\infty} V_i(t)e_i(\mathbf{x})$ must satisfy $\sum_{i=1}^{\infty} |\lambda_i|^2 |V_i(t)|^2 < \infty$ for almost every t. But then by Lemma 3.4, $V(t) = V(t,\cdot)$ must belong to $H^2(\Omega) \cap H_0^1(\Omega)$. This is precisely the reason for choosing $W^{(\gamma)} \in H^2(\Omega) \cap H_0^1(\Omega)$ that approximates W so as to define V and F.

Remark. As in the proof of Theorem 6.1 we may verify that the optimal solution $\hat{u}$ given by (4.1)–(4.2) or (4.9)–(4.10) indeed belongs to $H^{2,1}((0,T)\times\Omega)$ and satisfies $\hat{u} = 0$ on $(0,T)\times\partial\Omega$.

5. Dynamics of the optimal control solutions

In this section we will derive pointwise-in-time estimates for $\|\hat{u}(t) - W\|_0$ (in the case of (OP1)) or $\|\hat{u}(t) - V^{(\gamma)}(t)\|_0$ (in the case of (OP2)) where $\hat{u}$ is the optimal solution for (OP1) or (OP2). The derivation will be based on the explicit solution formulae that were expressed as series of eigen functions $\{e_i\}$. We recall that $\{e_i\}$ is orthonormal in $L^2(\Omega)$ so that for any function $\phi(\mathbf{x}) = \sum_{i=1}^{\infty} \phi_i e_i(\mathbf{x})$ in $L^2(\Omega)$ we have $\|\phi\|_0^2 = \sum_{i=1}^{\infty} |\phi_i|^2$.

Lemma 5.1. *Let $\lambda > 0$ be given. Then $2\lambda t \leqslant e^{\lambda t} - e^{-\lambda t} \leqslant e^{\lambda T} - e^{-\lambda T}$ for all $t \in [0,T]$.*

Proof. The right inequality follows from the fact that the function $g(t) \equiv e^{\lambda t} - e^{-\lambda t}$ is increasing on $[0,T]$ (as $g'(t) \geqslant 0$). The left inequality can be proved by the power series expression for exponential functions:

$$e^{\lambda t} - e^{-\lambda t} = \sum_{m=0}^{\infty} \frac{\lambda^m t^m}{m!} - \sum_{m=0}^{\infty} \frac{(-1)^m \lambda^m t^m}{m!} = 2\sum_{m=1}^{\infty} \frac{\lambda^{2m-1} t^{2m-1}}{(2m-1)!} \geqslant 2\lambda t.$$

This completes the proof. □

Theorem 5.2. *Assume $w \in H_0^1(\Omega)$ and $W \in L^2(\Omega)$. Let $(\hat{u}, \hat{f}) \in H^{2,1}((0,T)\times\Omega) \times L^2((0,T)\times\Omega)$ be the solution of* (OP1)*. Then*

$$\|\hat{u}(t) - W\|_0^2 \leqslant 6e^{-2\lambda_1 t}\|w\|_0^2 + 3\|W\|_0^2 \quad \forall t \in [0,T] \tag{5.1}$$

and for every integer $n \geqslant 1$,

$$\|\hat{u}(T) - W\|_0^2 \leqslant \frac{2\gamma^2\|w\|_0^2}{T^4} + \frac{8\gamma^2}{T^2} \sup_{1\leqslant i\leqslant n} \frac{|\lambda_i|^2}{(1-e^{-2\lambda_i T})^2} \sum_{i=1}^{n} |W_i|^2 + 2\sum_{i=n+1}^{\infty} |W_i|^2. \tag{5.2}$$

Furthermore, the optimal solution $\hat{u}$ as a function of the parameter γ satisfies the approximate controllability property $\lim_{\gamma\to 0} \|\hat{u}(T) - W\|_0 = 0$.

Proof. Let $t \in [0, T]$ be given. Using solution formulae (4.1)–(4.2) and adding/subtracting terms we have:

$$
\begin{aligned}
\|\hat{u}(t) - W\|_0^2 &= \sum_{i=1}^{\infty} |u_i(t) - W_i|^2 \\
&= \sum_{i=1}^{\infty} \bigg| w_i \left(e^{-\lambda_i t} - \frac{T e^{-\lambda_i T}(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i \gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})} \right) \\
&\qquad + W_i \frac{T(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i \gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})} - W_i \bigg|^2 \\
&= \sum_{i=1}^{\infty} \bigg\{ w_i (e^{-\lambda_i t} - e^{-\lambda_i T}) + \frac{2\lambda_i \gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T}) - T(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i \gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})} \\
&\qquad \times e^{-\lambda_i T} w_i - \frac{2\lambda_i \gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T}) - T(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i \gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})} W_i \bigg\}^2.
\end{aligned}
\tag{5.3}
$$

Applying the inequality $|\sum_{i=1}^{3} a_i|^2 \leqslant 3 \sum_{i=1}^{3} |a_i|^2$ to (5.3) and using the relation

$$
0 \leqslant \frac{2\lambda_i \gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T}) - T(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i \gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})} \leqslant 1 \quad \text{(see Lemma 5.1)}
$$

we obtain

$$
\|\hat{u}(t) - W\|_0^2 \leqslant 3\|w\|_0^2 \sup_{1 \leqslant i < \infty} |e^{-\lambda_i t} - e^{-\lambda_i T}|^2 + 3e^{-2\lambda_1 T} \|w\|_0^2 + 3\|W\|_0^2
$$

so that (5.1) holds.

Using formulae (4.1)–(4.2) with $t = T$ we have, for each integer $n \geqslant 1$,

$$
\begin{aligned}
&\|\hat{u}(T) - W\|_0^2 \\
&\quad = \sum_{i=1}^{\infty} |u_i(T) - W_i|^2 \\
&\quad = \sum_{i=1}^{\infty} \left\{ \frac{2\lambda_i \gamma}{2\lambda_i \gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})} w_i - \frac{2\lambda_i \gamma e^{\lambda_i T}}{2\lambda_i \gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})} W_i \right\}^2 \\
&\quad \leqslant 2 \sum_{i=1}^{\infty} \left| \frac{2\lambda_i \gamma}{2\lambda_i \gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})} \right|^2 |w_i|^2 \\
&\qquad + 2 \sum_{i=1}^{\infty} \left| \frac{2\lambda_i \gamma}{2\lambda_i \gamma + T(1 - e^{-2\lambda_i T})} \right|^2 |W_i|^2
\end{aligned}
$$

$$\leqslant \frac{8\gamma^2}{T^2} \sup_{1\leqslant i<\infty} \frac{|\lambda_i|^2}{(e^{\lambda_i T} - e^{-\lambda_i T})^2} \sum_{i=1}^{\infty} |w_i|^2 + \frac{8\gamma^2}{T^2} \sup_{1\leqslant i\leqslant n} \frac{|\lambda_i|^2}{(1 - e^{-2\lambda_i T})^2} \sum_{i=1}^{n} |W_i|^2$$
$$+ 2 \sum_{i=n+1}^{\infty} |W_i|^2. \tag{5.4}$$

Using Lemma 5.1 we have

$$\frac{|\lambda_i|^2}{(e^{\lambda_i T} - e^{-\lambda_i T})^2} \leqslant \frac{|\lambda_i|^2}{(2\lambda_i T)^2} = \frac{1}{4T^2} \quad \forall i. \tag{5.5}$$

Combining (5.4) and (5.5) we arrive at (5.2).

It remains to prove $\lim_{\gamma\to 0} \|\hat{u}(T) - W\|_0 = 0$. Let $\epsilon > 0$ be given. There exists an n such that

$$\sum_{i=n+1}^{\infty} |W_i|^2 < \frac{\epsilon^2}{6}.$$

Holding this n fixed, we may choose a γ_0 such that

$$\frac{8|\gamma_0|^2}{T^2} \sup_{1\leqslant i\leqslant n} \frac{|\lambda_i|^2}{(1 - e^{-2\lambda_i T})^2} \sum_{i=1}^{n} |W_i|^2 < \frac{\epsilon^2}{3} \quad \text{and} \quad \frac{2|\gamma_0|^2 \|w\|_0^2}{T^4} < \frac{\epsilon^2}{3}.$$

Thus, we obtain from (5.2) that $\|\hat{u}(T) - W\|_0 < \epsilon$ for each $\gamma \in [0, \gamma_0]$. □

We may similarly derive a pointwise-in-time $L^2(\Omega)$ estimate for the solution of (OP2).

Theorem 5.3. *Assume that $w \in H_0^1(\Omega)$, $W \in L^2(\Omega)$, $W^{(\gamma)} \in H^2(\Omega) \cap H_0^1(\Omega)$, $V^{(\gamma)}$ satisfies* (2.3) *and F is defined by* (2.6). *Let $(\hat{u}, \hat{f}) \in H^{2,1}((0,T)\times\Omega) \times L^2((0,T)\times\Omega)$ be the solution of* (OP2). *Then*

$$\left\|\hat{u}(t) - V^{(\gamma)}(t)\right\|_0^2 \leqslant 3e^{-2\lambda_1 t}\left\|w - V^{(\gamma)}(0)\right\|_0^2 + 3e^{-2\lambda_1 T}\left\|w - V^{(\gamma)}(0)\right\|_0^2$$
$$+ 3\left\|W - V^{(\gamma)}(T)\right\|_0^2 \quad \forall t \in [0,T], \tag{5.6}$$

and

$$\left\|\hat{u}(T) - V^{(\gamma)}(T)\right\|_0^2 \leqslant \frac{2\gamma^2}{T^4}\left\|w - V^{(\gamma)}(0)\right\|_0^2 + 2\left\|W - V^{(\gamma)}(T)\right\|_0^2. \tag{5.7}$$

If, in addition, hypotheses (2.2), (2.4) *and* (2.5) *hold, then the optimal solution $\hat{u}$ as a function of the parameter γ satisfies* $\lim_{\gamma\to 0} \|\hat{u}(T) - W\|_0 = 0$.

Proof. Using solution formulae (4.9)–(4.10) and writing V in place of $V^{(\gamma)}$ (for notation brevity) we obtain:

$$\left\|\hat{u}(t) - V(t)\right\|_0^2$$
$$= \sum_{i=1}^{\infty}\left|u_i(t) - V_i(t)\right|^2$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} \biggl\{ \bigl[w_i - V_i(0)\bigr] e^{-\lambda_i t} + \frac{[W_i - V_i(T)] T (e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i \gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})} \\
&\qquad + \frac{[V_i(0) - w_i] T e^{-\lambda_i T}}{2\lambda_i \gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})} \bigl(e^{\lambda_i t} - e^{-\lambda_i t}\bigr) \biggr\}^2 \\
&= \sum_{i=1}^{\infty} \biggl\{ \bigl[w_i - V_i(0)\bigr] \bigl(e^{-\lambda_i t} - e^{-\lambda_i T}\bigr) + \frac{[W_i - V_i(T)] T (e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i \gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})} \\
&\qquad + \frac{2\lambda_i \gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T}) - T(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i \gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})} \bigl[w_i - V_i(0)\bigr] e^{-\lambda_i T} \biggr\}^2
\end{aligned}
\tag{5.8}
$$

so that

$$
\begin{aligned}
\bigl\| \hat{u}(t) - V(t) \bigr\|_0^2 &\leqslant 3 \sum_{i=1}^{\infty} \bigl[w_i - V_i(0)\bigr]^2 \bigl| e^{-\lambda_i t} - e^{-\lambda_i T} \bigr|^2 + 3 \sum_{i=1}^{\infty} \bigl| W_i - V_i(T) \bigr|^2 \\
&\quad + 3 \sum_{i=1}^{\infty} \biggl| \frac{2\lambda_i \gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T}) - T(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i \gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})} \biggr|^2 \\
&\qquad \times \bigl[w_i - V_i(0)\bigr]^2 e^{-2\lambda_i T} \\
&\leqslant 3 e^{-2\lambda_1 t} \bigl\| w - V(0) \bigr\|_0^2 + 3 \bigl\| W - V(T) \bigr\|_0^2 + 3 e^{-2\lambda_1 T} \bigl\| w - V(0) \bigr\|_0^2 \\
&\forall t \in [0, T],
\end{aligned}
$$

i.e., (5.6) holds.

Setting $t = T$ in (5.8) and using (5.5) we have

$$
\begin{aligned}
&\bigl\| \hat{u}(T) - V(T) \bigr\|_0^2 \\
&\quad \leqslant 2 \sum_{i=1}^{\infty} \biggl| \frac{2\lambda_i \gamma}{2\lambda_i \gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})} \biggr|^2 \bigl[w_i - V_i(0)\bigr]^2 + 2 \sum_{i=1}^{\infty} \bigl| W_i - V_i(T) \bigr|^2 \\
&\quad \leqslant 8\gamma^2 \bigl\| w - V(0) \bigr\|_0^2 \sup_{1 \leqslant i < \infty} \frac{|\lambda_i|^2}{T^2 (e^{\lambda_i T} - e^{-\lambda_i T})^2} + 2 \bigl\| W - V(T) \bigr\|_0^2 \\
&\quad \leqslant \frac{2\gamma^2}{T^4} \bigl\| w - V(0) \bigr\|_0^2 + 2 \bigl\| W - V(T) \bigr\|_0^2.
\end{aligned}
$$

This proves (5.7).

The relation $\lim_{\gamma \to 0} \|\hat{u}(T) - W\|_0 = 0$ follows easily from the triangle inequality

$$
\bigl\| \hat{u}(T) - W \bigr\|_0 \leqslant \bigl\| \hat{u}(T) - V(T) \bigr\|_0 + \bigl\| V(T) - W \bigr\|_0,
$$

estimate (5.7) and assumption (2.2). □

The particular choice of $V^{(\gamma)}(t) \equiv W^{(\gamma)}$ satisfies (2.3)–(2.5). Thus Theorem 5.3 yields the following result.

Corollary 5.4. *Assume that* $w \in H_0^1(\Omega)$, $W \in L^2(\Omega)$, $W^{(\gamma)} \in H^2(\Omega) \cap H_0^1(\Omega)$ *satisfying* (2.2), $V^{(\gamma)}(t,\cdot) \equiv W^{(\gamma)}$ *and* F *is defined by*

$$F = F^{(\gamma)} \equiv -\operatorname{div}\big[A(\mathbf{x})\nabla W^{(\gamma)}\big] \quad \text{in } (0,T)\times\Omega.$$

Let $(\hat{u},\hat{f}) = (\sum_{i=1}^{\infty}\hat{u}_i(t)e_i(\mathbf{x}), \sum_{i=1}^{\infty}\hat{f}_i(t)e_i(\mathbf{x})) \in H^{2,1}((0,T)\times\Omega)\times L^2((0,T)\times\Omega)$ *be the solution of* (OP2) *given by*

$$\hat{u}_i(t) = W_i^{(\gamma)} + \big[w_i - W_i^{(\gamma)}\big]\left(e^{-\lambda_i t} - \frac{Te^{-\lambda_i T}(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i\gamma e^{\lambda_i T} + Te^{\lambda_i T} - Te^{-\lambda_i T}}\right) + \big[W_i - W_i^{(\gamma)}\big]\frac{T(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i\gamma e^{\lambda_i T} + Te^{\lambda_i T} - Te^{-\lambda_i T}}. \tag{5.9}$$

Then

$$\big\|\hat{u}(t) - W^{(\gamma)}\big\|_0^2 \leqslant 3e^{-2\lambda_1 t}\big\|w - W^{(\gamma)}\big\|_0^2 + 3e^{-2\lambda_1 T}\big\|w - W^{(\gamma)}\big\|_0^2 + 3\big\|W - W^{(\gamma)}\big\|_0^2 \quad \forall t \in [0,T],$$

and

$$\big\|\hat{u}(T) - W^{(\gamma)}\big\|_0^2 \leqslant \frac{2\gamma^2}{T^4}\big\|w - W^{(\gamma)}\big\|_0^2 + 2\big\|W - W^{(\gamma)}\big\|_0^2.$$

Moreover, the optimal solution $\hat{u}$ *as a function of* γ *satisfies* $\lim_{\gamma\to 0}\|\hat{u}(T) - W\|_0 = 0$.

When $W \in H^2(\Omega)\cap H_0^1(\Omega)$ we may simply choose $W^{(\gamma)} = W$ and $V^{(\gamma)} \equiv W$. Then Corollary 5.4 reduces to the following.

Corollary 5.5. *Assume that* $w \in H_0^1(\Omega)$, $W \in H^2(\Omega)\cap H_0^1(\Omega)$, $V^{(\gamma)} \equiv W$ *and* F *is defined by*

$$F \equiv -\operatorname{div}\big[A(\mathbf{x})\nabla W\big] \qquad \text{in } (0,T)\times\Omega.$$

Let $(\hat{u},\hat{f}) = (\sum_{i=1}^{\infty}\hat{u}_i(t)e_i(\mathbf{x}), \sum_{i=1}^{\infty}\hat{f}_i(t)e_i(\mathbf{x})) \in H^{2,1}((0,T)\times\Omega)\times L^2((0,T)\times\Omega)$ *be the solution of* (OP2) *given by*

$$\hat{u}_i(t) = W_i + [w_i - W_i]\left(e^{-\lambda_i t} - \frac{Te^{-\lambda_i T}(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i\gamma e^{\lambda_i T} + Te^{\lambda_i T} - Te^{-\lambda_i T}}\right). \tag{5.10}$$

Then

$$\big\|\hat{u}(t) - W\big\|_0^2 \leqslant 2e^{-2\lambda_1 t}\|w - W\|_0^2 + 2e^{-2\lambda_1 T}\|w - W\|_0^2 \quad \forall t \in [0,T]$$

and

$$\big\|\hat{u}(T) - W\big\|_0^2 \leqslant \frac{2\gamma^2}{T^4}\|w - W\|_0^2.$$

Moreover, the optimal solution $\hat{u}$ *as a function of* γ *satisfies* $\lim_{\gamma\to 0}\|\hat{u}(T) - W\|_0 = 0$.

6. Solutions to the exact controllability problem

Recall that the exact controllability problem (EX-CON) is solvable if $w \in H_0^1(\Omega)$ and $W \in H_0^1(\Omega)$. Formally setting $\gamma = 0$ in (4.2) and (4.10) we expect to obtain solution formulae for the exact controllability problem (EX-CON). But these formulae needs justification as infinite series functions are involved. We first examine the solution obtained by setting $\gamma = 0$ in (4.2).

Theorem 6.1. *Assume that $w \in H_0^1(\Omega)$ and $W \in H_0^1(\Omega)$. Then the functions*

$$u(t,\mathbf{x}) = \sum_{i=1}^{\infty} u_i(t)e_i(\mathbf{x}) \quad \text{and} \quad f(t,\mathbf{x}) = \sum_{i=1}^{\infty} f_i(t)e_i(\mathbf{x}),$$

where

$$u_i(t) = w_i e^{-\lambda_i t} - w_i \frac{e^{-\lambda_i T}(e^{\lambda_i t} - e^{-\lambda_i t})}{e^{\lambda_i T} - e^{-\lambda_i T}} + W_i \frac{e^{\lambda_i t} - e^{-\lambda_i t}}{e^{\lambda_i T} - e^{-\lambda_i T}} \tag{6.1}$$

and

$$f_i(t) = -2\lambda_i w_i \frac{e^{-\lambda_i T} e^{\lambda_i t}}{e^{\lambda_i T} - e^{-\lambda_i T}} + 2\lambda_i W_i \frac{e^{\lambda_i t}}{e^{\lambda_i T} - e^{-\lambda_i T}}, \tag{6.2}$$

form a solution pair to the exact controllability problem (EX-CON).

Proof. Since $u_i(0) = w_i$ and $u_i(T) = W_i$, we have that $u(0) = w$ and $u(T) = W$. To show that the pair (u, f) is a solution to (EX-CON) we need to show that $u_t - \operatorname{div}[A(\mathbf{x})\nabla u] = f$ in $(0, T \times \Omega)$ and $u = 0$ on $(0,T) \times \partial\Omega$ and we will do so by employing Theorem 3.7.

We proceed to verify the assumptions of Theorem 3.7.

Lemma 3.3 and the assumptions $w, W \in H_0^1(\Omega)$ imply

$$\sum_{i=1}^{\infty} |\lambda_i||w_i|^2 < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} |\lambda_i||W_i|^2 < \infty. \tag{6.3}$$

Since $u_i(0) = w_i$, we obviously have $\sum_{i=1}^{\infty} \lambda_i |u_i(0)|^2 = \sum_{i=1}^{\infty} \lambda_i |w_i|^2 < \infty$.

Let $C_T = 1 - e^{-2\lambda_1 T} \in (0,1)$. Then we have $2\lambda_i T \geqslant 2\lambda_1 T = -\ln(1 - C_T)$ so that $e^{2\lambda_i T} \geqslant 1/(1 - C_T)$, or equivalently,

$$e^{\lambda_i T} - e^{-\lambda_i T} \geqslant C_T e^{\lambda_i T} \quad \forall i.$$

From (6.1) and the last inequality we have

$$\begin{aligned}
&\int_0^T |\lambda_i|^2 \big|u_i(t)\big|^2\,dt \\
&\quad\leqslant 3|\lambda_i|^2|w_i|^2 \int_0^T e^{-2\lambda_i t}\,dt + \frac{3|\lambda_i|^2|w_i|^2}{(C_T)^2 e^{4\lambda_i T}} \int_0^T e^{2\lambda_i t}\,dt + \frac{3|\lambda_i|^2|W_i|^2}{(C_T)^2 e^{2\lambda_i T}} \int_0^T e^{2\lambda_i t}\,dt \\
&\quad\leqslant 3|\lambda_i|^2|w_i|^2 \frac{1}{2\lambda_i} + \frac{3|\lambda_i|^2|w_i|^2}{(C_T)^2 e^{4\lambda_i T}} \cdot \frac{e^{2\lambda_i T}}{2\lambda_i} + \frac{3|\lambda_i|^2|W_i|^2}{(C_T)^2 e^{2\lambda_i T}} \cdot \frac{e^{2\lambda_i T}}{2\lambda_i}
\end{aligned}$$

$$\leqslant |\lambda_i||w_i|^2\left(\frac{3}{2}+\frac{3}{2(C_T)^2e^{2\lambda_1 T}}\right)+|\lambda_i||W_i|^2\frac{3}{2(C_T)^2}. \tag{6.4}$$

Combining (6.4) and (6.3) we arrive at $\sum_{i=1}^{\infty}\int_0^T |\lambda_i|^2|u_i(t)|^2\,dt<\infty$.

Differentiation of (6.1) yields

$$u_i'(t)=-\lambda_i w_i e^{-\lambda_i t}-\lambda_i w_i\frac{e^{-\lambda_i T}(e^{\lambda_i t}+e^{-\lambda_i t})}{e^{\lambda_i T}-e^{-\lambda_i T}}+\lambda_i W_i\frac{e^{\lambda_i t}+e^{-\lambda_i t}}{e^{\lambda_i T}-e^{-\lambda_i T}}.$$

Note that $e^{\lambda_i t}+e^{-\lambda_i t}\leqslant 2e^{\lambda_i t}$ so that estimations similar to those of (6.4) lead us to

$$\int_0^T \left|u_i'(t)\right|^2 dt\leqslant|\lambda_i||w_i|^2\left(\frac{3}{2}+\frac{6}{(C_T)^2e^{2\lambda_i T}}\right)+|\lambda_i||W_i|^2\frac{6}{(C_T)^2}<\infty.$$

Thus we have verified all assumptions of Theorem 3.7. Using that theorem we conclude that $u\in H^{2,1}((0,T)\times\Omega)$, $u=0$ on $(0,T)\times\partial\Omega$, and

$$\begin{aligned}u_t(t)-\operatorname{div}\bigl[A(\mathbf{x})\nabla u(t)\bigr]&=\sum_{i=1}^{\infty}u_i'(t)e_i+\sum_{i=1}^{\infty}\lambda_i u_i(t)e_i\\&=\sum_{i=1}^{\infty}\left(-2\lambda_i w_i\frac{e^{-\lambda_i T}e^{\lambda_i t}}{e^{\lambda_i T}-e^{-\lambda_i T}}+2\lambda_i W_i\frac{e^{\lambda_i t}}{e^{\lambda_i T}-e^{-\lambda_i T}}\right)e_i\quad\text{in } L^2(\Omega),\ \text{a.e. } t.\end{aligned}$$

By a comparison of the last relation with (6.2) we deduce $f(t)=u_t(t)-\operatorname{div}[A(\mathbf{x})\nabla u(t)]$ in $L^2(\Omega)$ for almost every t so that $f=u_t-\operatorname{div}[A(\mathbf{x})\nabla u]\in L^2(0,T;L^2(\Omega))$.

Hence, the pair (u,f) is indeed a solution to (EX-CON). □

If $W\in H^2(\Omega)\cap H_0^1(\Omega)$, then by choosing $V^{(\gamma)}\equiv W$ and setting $\gamma=0$ in formula (4.10) we obtain another solution for the exact controllability problem (EX-CON). The proof of the following theorem is similar to that of Theorem 6.1 and is omitted.

Theorem 6.2. *Assume that $w\in H_0^1(\Omega)$ and $W\in H^2(\Omega)\cap H_0^1(\Omega)$. Then the functions*

$$u(t,\mathbf{x})=\sum_{i=1}^{\infty}u_i(t)e_i(\mathbf{x})\quad\textit{and}\quad f(t,\mathbf{x})=\sum_{i=1}^{\infty}f_i(t)e_i(\mathbf{x}),$$

where

$$u_i(t)=w_i\left(e^{-\lambda_i t}-\frac{e^{-\lambda_i T}(e^{\lambda_i t}-e^{-\lambda_i t})}{e^{\lambda_i T}-e^{-\lambda_i T}}\right)+W_i\left(1-e^{-\lambda_i t}+\frac{e^{-\lambda_i T}(e^{\lambda_i t}-e^{-\lambda_i t})}{e^{\lambda_i T}-e^{-\lambda_i T}}\right)$$

and

$$f_i(t)=-2\lambda_i w_i\frac{e^{-\lambda_i T}e^{\lambda_i t}}{e^{\lambda_i T}-e^{-\lambda_i T}}+\lambda_i W_i\left(1+\frac{2e^{-\lambda_i T}e^{\lambda_i t}}{e^{\lambda_i T}-e^{-\lambda_i T}}\right),$$

form a solution pair to the exact controllability problem (EX-CON).

7. One-dimensional numerical simulations

In one space dimension the eigen pairs $\{e_i\}$ are well known so that optimal solutions for (OP1) and (OP2) can be computed from series solution formulae (4.1)–(4.2) or (4.9)–(4.10), respectively.

The one-dimensional constraint equations are defined on the spatial interval $\Omega = (0, 1)$:

$$u_t - u_{xx} = f \quad \text{in } (0, T) \times (0, 1), \qquad u(t, 0) = u(t, 1) = 0 \quad \text{and} \quad u(0, x) = w(x).$$

The eigen pairs $\{(\lambda_i, e_i)\}_{i=1}^{\infty}$ are determined from

$$-e''(x) = \lambda e(x) \quad 0 \leqslant x \leqslant 1, \qquad e(0) = e(1) = 0.$$

It is well known that

$$\lambda_i = (i\pi)^2 \quad \text{and} \quad e_i(x) = \sqrt{2} \sin(i\pi x), \quad i = 1, 2, \ldots.$$

Given a target function $W(x)$, the solution to optimal control problem (OP1) is explicitly given by (4.1)–(4.2). To find the solution to (OP2) we first need to construct $W^{(\gamma)}$ and $V^{(\gamma)}$ satisfying (2.2)–(2.5); we then use solution formulae (4.9)–(4.10). Note that w_i, W_i and $V_i^{(\gamma)}(t)$ in (4.2) and (4.10) are calculated by

$$w_i = \int_0^1 w(x) e_i(x)\, dx, \qquad W_i = \int_0^1 W(x) e_i(x)\, dx \quad \text{and}$$

$$V_i^{(\gamma)}(t) = \int_0^1 V^{(\gamma)}(t, x) e_i(x)\, dx.$$

We consider two sets of data (the initial condition w, the target function W and the terminal time T):

$$\text{Data I.} \quad T = 2, \qquad w(x) = \sum_{i=1}^{5} i\, e_i(x)/\sqrt{2}, \qquad W(x) = T \sum_{i=1}^{5} i\, e_i(x)/\sqrt{2}.$$

$$\text{Data II.} \quad T = 1, \qquad w(x) = \sum_{i=1}^{1000} i\, e_i(x)/\sqrt{2},$$

$$W(x) = 1 = \sum_{i=1}^{\infty} W_i e_i(x) \quad \text{in } L^2(\Omega) \quad \text{where } W_i = \int_0^1 e_i = \sqrt{2}\,\frac{1 - (-1)^i}{i\pi}.$$

For each data set we solve (OP1) by series solution formulae (4.1)–(4.2). In each case we vary the parameter γ and plot the optimal solution $\hat{u}$ at the terminal time T (the "$*$" curve) versus the target function W (the "$-$" curve). See Figs. 1 and 3.

For each data set we solve (OP2) by series solution formulae (4.9)–(4.10). In the case of Data I, we choose

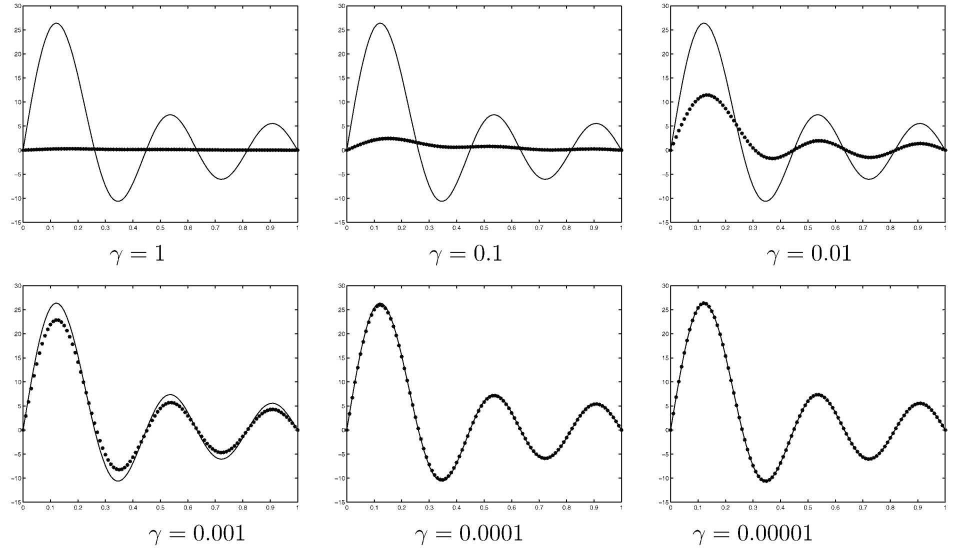

Fig. 1. Optimal solution $\hat{u}(T)$ and target W for (OP1) with Data I ($T = 2$). $*$: optimal solution $\hat{u}(T)$, $-$: target function W.

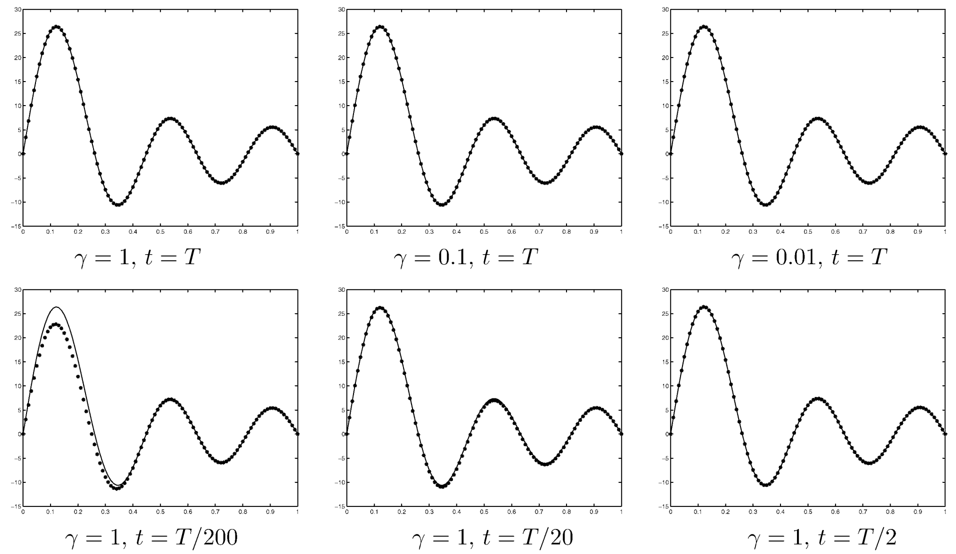

Fig. 2. Optimal solution $\hat{u}(t)$ and target W for (OP2) with Data I ($T = 2$). $*$: optimal solution $\hat{u}(t)$, $-$: target function W.

$$W^{(\gamma)}(x) = W(x) = T\sum_{i=1}^{5} i\, e_i(x)/\sqrt{2} \quad \text{and}$$

$$V^{(\gamma)}(t,x) = W(x) = T\sum_{i=1}^{5} i\, e_i(x)/\sqrt{2}$$

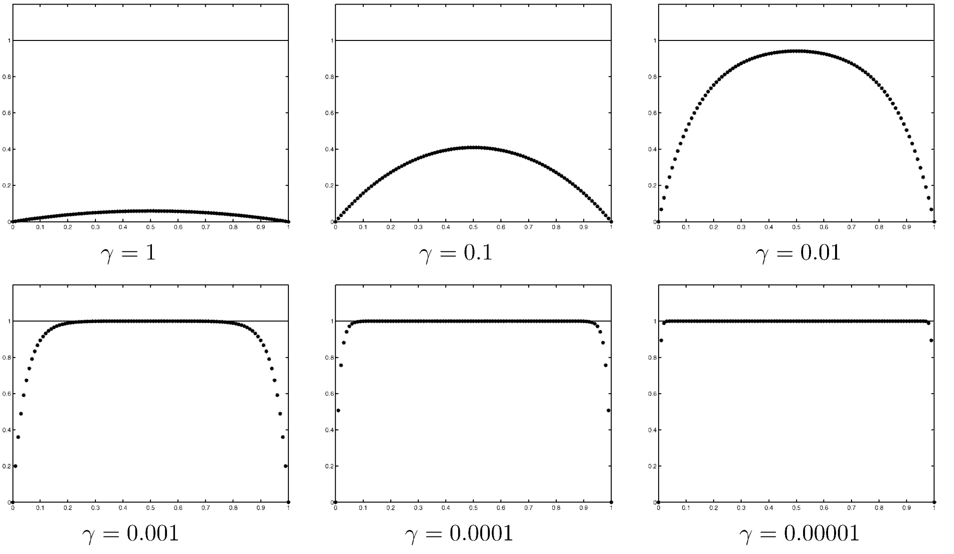

Fig. 3. Optimal solution $\hat{u}(T)$ and target W for (OP1) with Data II ($T = 1$). $*$: optimal solution $\hat{u}(t)$, $-$: target function W.

which evidently satisfy assumptions (2.2)–(2.5); in addition, formula (4.10) takes on the simpler form (5.10), i.e., $\hat{u} = \sum_{i=1}^{5} \hat{u}_i(t)\sqrt{2}\sin(i\pi x)$ where

$$\hat{u}_i = \frac{2i}{\sqrt{2}} - \frac{i}{\sqrt{2}}\left(e^{-i^2\pi^2 t} - \frac{e^{-2i^2\pi^2}(e^{i^2\pi^2 t} - e^{-i^2\pi^2 t})}{i^2\pi^2\gamma e^{2i^2\pi^2} + e^{2i^2\pi^2} - e^{-2i^2\pi^2}}\right).$$

In the case of Data II, we choose

$$W^{(\gamma)}(x) = \frac{\sqrt{2}}{\pi}\sum_{i=1}^{N_\gamma}\frac{1-(-1)^i}{i}e_i(x) \quad \text{and}$$

$$V^{(\gamma)}(t,x) = W^{(\gamma)}(x) = \frac{\sqrt{2}}{\pi}\sum_{i=1}^{N_\gamma}\frac{1-(-1)^i}{i}e_i(x),$$

where $N_\gamma \to \infty$ as $\gamma \to 0$ (e.g., N_γ is the integer part of the decimal number $[1000 + \ln(1/\gamma)]$). It can be verified that $W^{(\gamma)}$ and $V^{(\gamma)}$ satisfy assumptions (2.2)–(2.5); in addition, formula (4.10) takes on the simpler form (5.9), i.e., $\hat{u} = \sum_{i=1}^{N_\gamma} \hat{u}_i(t)\sqrt{2}\sin(i\pi x)$ where

$$\hat{u}_i = \frac{\sqrt{2}}{i\pi}[1-(-1)^i] + \left(\frac{i}{\sqrt{2}} - \frac{\sqrt{2}}{i\pi}\big[1-(-1)^i\big]\right) \times \left(e^{-i^2\pi^2 t} - \frac{e^{-i^2\pi^2}(e^{i^2\pi^2 t} - e^{-i^2\pi^2 t})}{2i^2\pi^2\gamma e^{i^2\pi^2} + e^{i^2\pi^2} - e^{-i^2\pi^2}}\right), \quad i = 1, 2, \ldots, 1000$$

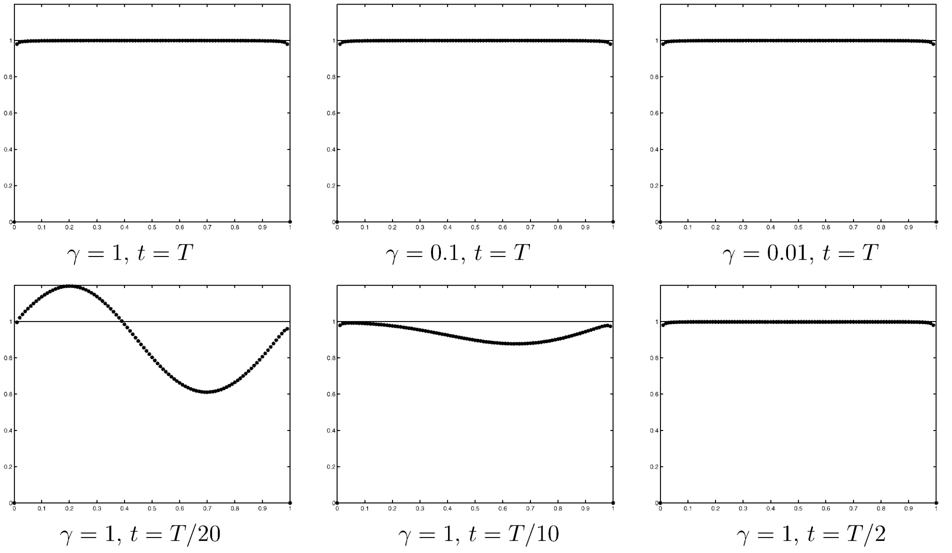

Fig. 4. Optimal solution $\hat{u}(t)$ and target W for (OP2) with Data II ($T = 1$). $*$: optimal solution $\hat{u}(t)$, $-$: target function W.

and

$$\hat{u}_i = \frac{\sqrt{2}}{i\pi}\bigl[1 - (-1)^i\bigr]\left(1 - e^{-i^2\pi^2 t} + \frac{e^{-i^2\pi^2}(e^{i^2\pi^2 t} - e^{-i^2\pi^2 t})}{2i^2\pi^2\gamma e^{i^2\pi^2} + e^{i^2\pi^2} - e^{-i^2\pi^2}}\right),$$
$$i = 1001, 1002, \ldots, N_\gamma.$$

As in the case of (OP1), for each data set we vary the parameter γ and plot the optimal solution $\hat{u}$ for (OP2) at the terminal time T (the "$*$" curve) versus the target function W (the "$-$" curve). See the first row of plots in Figs. 2 and 4. Note that unlike in the case of (OP1), the optimal solution $\hat{u}(T)$ for (OP2) matches W very well even for $\gamma = 1$. This phenomena is expected from Corollaries 5.4 and 5.5.

Moreover, in the case of (OP2), we again from Corollaries 5.4 and 5.5 anticipate optimal solution $\hat{u}(t)$ to yield good matching to W even for moderate γ and $t \ll T$. When $\gamma = 1$, we plot some snapshots of the optimal solution $\hat{u}$ (the "$*$" curve) versus the target function W (the "$-$" curve). See the second row of plots in Figs. 2 and 4.

For Data I the admissible state and the target state have matching boundary conditions (both have homogeneous boundary conditions). For Data II the admissible state and the target function have nonmatching boundary conditions. For both data sets and for sufficiently small γ, the optimal solutions expressed by the series formulae did a good job of tracking the target functions in the interior at the terminal time T, as predicted by Theorems 5.2 and 5.3. The optimal solutions of (OP2) furnish good matchings to the target state even for moderate γ and $t \ll T$, as predicted by Corollaries 5.4 and 5.5.

References

[1] L. Bal'chuk, Optimal control of systems described parabolic equations, SIAM. J. Control 7 (1969) 546–558.
[2] Y. Cao, M. Gunzburger, J. Turner, The controllability of systems governed by parabolic differential equations, J. Math. Anal. Appl. 215 (1997) 174–189.
[3] J.I. Diaz, A. Fursikov, A simple proof of the approximate controllability from the interior for nonlinear evolution problems, Appl. Math. Lett. 7 (1994) 85–87.
[4] Y.V. Egorov, Some problems in the theory of optimal control, Soviet Math. 3 (1962) 1080–1084.
[5] Y.V. Egorov, Some problems in the theory of optimal control, Zh. Vychisl. Mat. Mat. Fiz. 5 (1963) 887–904.
[6] L. Evans, Partial Differential Equations, Amer. Math. Soc., Providence, RI, 1998.
[7] H. Fattorini, Control in finite time of differential equations in Banach space, Comm. Pure Appl. Math. 19 (1966) 7–34.
[8] H. Fattorini, D. Russell, Exact controllability theorems for linear parabolic equations in one space dimension, Arch. Ration. Mech. Anal. 49 (1971) 272–292.
[9] A. Fursikov, O. Imanuvilov, On exact boundary zero controllability of two-dimensional Navier–Stokes equations, Acta Appl. Math. 36 (1994) 1–10.
[10] A. Fursikov, O. Imanuvilov, On controllability of certain systems stimulating a fluid flow, in: Flow Control, in: IMA Vol. Math. Appl., vol. 68, Springer-Verlag, New York, 1995, pp. 149–184.
[11] R. Glowinski, J.-L. Lions, Exact and approximate controllability for distributed parameter systems, Acta Numer. (1994) 269–378.
[12] R. Glowinski, J.-L. Lions, Exact and approximate controllability for distributed parameter systems, Acta Numer. (1995) 159–333.
[13] M. Gunzburger, S. Manservisi, The velocity tracking problem for Navier–Stokes flows with bounded distributed controls, SIAM J. Control Optim. 37 (1999) 1913–1945.
[14] M. Gunzburger, S. Manservisi, Analysis and approximation of the velocity tracking problem for Navier–Stokes flows with distributed control, SIAM J. Numer. Anal. 37 (2000) 1481–1512.
[15] J.-L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, Springer-Verlag, Berlin, 1971.
[16] J.-L. Lions, E. Magenes, Nonhomogeneous Boundary Value Problems and Applications, Springer-Verlag, Berlin, 1972.
[17] J.-L. Lions, Exact controllability, stabilization and perturbations for distributed systems, SIAM Rev. 30 (1988) 1–68.
[18] D. Russell, A unified boundary controllability theory for hyperbolic and parabolic partial differential equations, Stud. Appl. Math. 52 (1973) 189–211.
[19] D. Russell, Controllability and stabilizability theory for linear partial differential equations: Recent progress and open questions, SIAM Rev. 20 (1978) 639–739.
[20] D. Russell, Some remarks on numerical aspects of coefficient identification in elliptic systems, in: Optimal Control of Partial Differential Equations, Birkhäuser, Boston, MA, 1984, pp. 210–228.
[21] R. Temam, Navier–Stokes Equations, Amer. Math. Soc./Chelsea, Providence, RI, 2001.
[22] T. Tsachev, Optimal control of linear parabolic equation: The constrained right-hand side as control function, Numer. Funct. Anal. Optim. 13 (1992) 369–380.