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Subrings of Idealizer Rings

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INTRODUCTION

Given a right ideal M in a ring T, the *idealizer* of M in T is the largest subring of T which contains M as a two-sided ideal. The purpose of this paper is to demonstrate that for certain subrings R of such idealizers, the properties of R are relatively close to those of T. Specifically, we require that $TM = T$, that R is a subring of T containing M as a two-sided ideal, and that R/M is a semisimple ring. For all three standard global homological dimensions—right and left global dimension, and global weak dimension—we prove that

$$\dim(T) \leqslant \dim(R) \leqslant 1 + \dim(T).$$

When M is a finite intersection of maximal right ideals of T, and when the dimension in question is either the right global dimension or the global weak dimension, we derive conditions showing when each of the two allowable possibilities for $\dim(R)$ can occur. We also investigate when semiheredity conditions and chain conditions can be inherited by R from T. In the final two sections we investigate similar questions in the context of nonsingular rings. In particular, we derive conditions under which R can be a *splitting ring*, which means that the singular submodule of any R-module must be a direct summand. As an application, the results of this paper are used to construct a right and left Ore domain which is a splitting ring with all three global dimensions equal to 2.

N.B.—In this paper all rings have identities, and all modules and subrings are unital. Also, we use the term "semisimple" to refer to a ring or a module which is a direct sum of simple modules, rather than to a ring or a module whose Jacobson radical is zero.

1. Subidealizers

Given a right ideal M in a ring T, the idealizer of M in T is just the set $S = \{t \in T \mid tM \leqslant M\}$. In [9] and [7], it is shown that the properties of S are nearly identical with those of T provided M is a *semimaximal* right ideal of T, by which is meant that M must be a finite intersection of maximal right ideals of T. [Equivalently, T/M must be a semisimple module.] The close connections between T and S seem to derive mainly from the following three consequences of the semimaximality assumption:

(A) There is no loss of generality in assuming that $TM = T$. Specifically, T contains another semimaximal right ideal M' such that $TM' = T$ and the idealizers of M' and M coincide [9, Proposition 1.7].

(B) S/M is a semisimple ring, which follows from the observation that S/M is isomorphic to the endomorphism ring of T/M [9, Proposition 1.1].

(C) T/S is a semisimple right S-module [9, Corollary 1.5].

The aim of the present paper is to show that conditions (A) and (B) together are sufficient to ensure that the properties of S and T are relatively close, even when M is not necessarily semimaximal in T and S is not necessarily the whole idealizer of M. For ease of reference to the conditions which are required, we introduce the following definitions:

(1) A right ideal M in T is a *generative* right ideal provided $TM = T$. [Note that as a consequence $M^2 = MTM = M$.]

(2) A *subidealizer* of M (in T) is any subring of T which contains M as a two-sided ideal.

(3) If R is a subidealizer of M, then R is a *tame* subidealizer provided R/M is a semisimple ring.

The discussion above provides one class of examples of these concepts: the idealizer of a semimaximal right ideal can always be expressed as a tame subidealizer of a generative semimaximal right ideal. As a second class of examples, we observe that a maximal right ideal M of a ring T must either be generative or else a two-sided ideal. If we have a field F contained in the center of T, then the sum $F + M$ is always a tame subidealizer of M.

Proposition 1.1. *Let M be a generative right ideal of T, and let R be any tame subidealizer of M.*

(a) *T_R and ${}_RM$ are both finitely generated and projective.*

(b) *The natural map $T \otimes_R T \to T$ is an isomorphism.*

Proof. According to [9, Lemma 2.1], T_R is finitely generated projective and $T \otimes_R T \to T$ is an isomorphism. The methods used there to show that T_R is finitely generated projective can also be used to show the same for ${}_RM$.

An immediate consequence of Proposition 1.1(b) is that for any modules A_T and ${}_TB$, the natural map $A \otimes_R B \to A \otimes_T B$ is an isomorphism. From this, we infer that the following natural maps are also isomorphisms: $A \otimes_R T \to A$, $T \otimes_R B \to B$, $A \to A \otimes_R T$, $B \to T \otimes_R B$. In view of the latter two isomorphisms, we obtain $\mathrm{Hom}_R(A, C) = \mathrm{Hom}_T(A, C)$ for all C_T and $\mathrm{Hom}_R(B, D) = \mathrm{Hom}_T(B, D)$ for all ${}_TD$, as in [13, Corollary 1.3]. We note for later use that because of the isomorphisms ${}_RT \to T \otimes_R T$ and $T_R \to T \otimes_R T$, we obtain $(T/R) \otimes_R T = 0$ and $T \otimes_R (T/R) = 0$.

PROPOSITION 1.2. *Let M be a generative right ideal of T, and let R be any subidealizer of M.*

(a) $\mathrm{Tor}_n^R(A, B) \cong \mathrm{Tor}_n^T(A, B)$ *for all* A_T, ${}_TB$, *and all* $n > 0$.

(b) $\mathrm{Ext}^n_R(A, C) \cong \mathrm{Ext}^n_T(A, C)$ *for all* A_T, C_T, *and all* $n > 0$.

(c) $\mathrm{Ext}^n_R(B, D) \cong \mathrm{Ext}^n_T(B, D)$ *for all* ${}_TB$, ${}_TD$, *and all* $n > 0$.

Proof. Using the observations in the paragraph above together with the result that T_R is projective, an easy induction establishes (a) and (b). Inasmuch as T_R is flat and $T \otimes_R T \to T$ is an isomorphism, T is a left localization of R in the sense of [13], hence [13, Corollary 1.3] gives us (c).

As noted in the preceding proof, a ring T is always a left localization of any subring R which is a subidealizer of a generative right ideal M. In view of the symmetry in the results of Proposition 1.2, it is natural to ask whether T must also be a right localization of R, i.e., is ${}_RT$ flat? According to [7, Proposition 3], the answer is yes when M is semimaximal in T and R is the whole idealizer of M. On the other hand, [9, Example 7.6] shows that the answer is no in general, and the following proposition indicates that the full idealizers of semimaximal right ideals are the only tame subidealizers over which T is a right localization.

PROPOSITION 1.3. *Let M be a generative right ideal of T, and let R be any tame subidealizer of M. Then ${}_RT$ is flat if and only if M is semimaximal in T and R is the idealizer of M in T.*

Proof. Sufficiency is given by [7, Proposition 3].

Conversely, assume that ${}_RT$ is flat, and suppose that R is distinct from the idealizer S of M. Observing that S/R is a nonzero right module over the semisimple ring R/M, we see that S/R has a direct summand A which is isomorphic to a nonzero right ideal J of R/M. Inasmuch as the map

$$J \to J \otimes_R T \to (R/M) \otimes_R T \to T/M$$

coincides with the injection of J into T/M, we infer that $J \otimes_R T \neq 0$, from which it follows that $A \otimes_R T \neq 0$ and $(S/R) \otimes_R T \neq 0$. Recalling that $(T/R) \otimes_R T = 0$, we see that the map $(S/R) \otimes_R T \to (T/R) \otimes_R T$ is not injective, which contradicts the flatness of ${}_R T$. Therefore $R = S$.

Now $R/M = A_1 \oplus \cdots \oplus A_n$, where each A_i is a simple right R-module. Inasmuch as

$$T/M \cong (R/M) \otimes_R T \cong (A_1 \otimes_R T) \oplus \cdots \oplus (A_n \otimes_R T),$$

we see that T/M will be semisimple in the event that any $A_i \otimes_R T$ which is nonzero is a simple right T-module. Choosing in such a case a maximal right ideal K of R such that $A_i \cong R/K$, we have $A_i \otimes_R T \cong T/KT$, hence it suffices to show that KT is a maximal right ideal of T. If J is any proper right ideal of T which contains KT, then $1 \notin J$ and so the maximality of K forces $J \cap R = K$. We now have an injective map $R/K \to T/J$ which, when tensored with the flat module ${}_R T$, yields another injective map $T/KT \to T/J$. Thus $KT = J$, hence KT is indeed a maximal right ideal of T. Therefore T/M is a semisimple right T-module, as required.

2. Homological Dimensions

This section is devoted to comparing the global and global weak dimensions of a ring with those of a tame subidealizer of a generative right ideal. Our procedure involves looking at projective and weak dimensions of modules over the two rings, for which we use the following notation: the projective dimension of a module A over a ring S is denoted $pd_S(A)$, while the weak dimension of A is denoted $wd_S(A)$.

Proposition 2.1. *Let M be a generative right ideal of T and let R be any tame subidealizer of M. If A is any right T-module, then $pd_R(A) = pd_T(A)$ and $wd_R(A) = wd_T(A)$.*

Proof. It is immediate from Proposition 1.2 that $pd_T(A) \leqslant pd_R(A)$. On the other hand, it follows from the projectivity of T_R that any projective resolution for A_T is also a projective resolution for A_R, whence $pd_R(A) \leqslant pd_T(A)$. Thus $pd_R(A) = pd_T(A)$, and similarly $wd_R(A) = wd_T(A)$.

Theorem 2.2. *Let M be a generative right ideal of T and let R be any tame subidealizer of M. Then*

$$\text{r.gl.dim.}(T) \leqslant \text{r.gl.dim.}(R) \leqslant 1 + \text{r.gl.dim.}(T).$$

Proof. It is clear from Proposition 2.1 that r.gl.dim.$(T) \leqslant$ r.gl.dim.(R). For the other inequality, we need only consider what happens when r.gl.dim.$(T) = n < \infty$.

Case I. $n = 0$. Here M must be a direct summand of T_T and thus also a direct summand of R_R, whence $(R/M)_R$ is projective. Given any right ideal J of R, we note that J/JM is a projective right (R/M)-module, hence it must be projective as an R-module. Since JM is a right ideal of T, we obtain $pd_R(JM) = pd_T(JM) = 0$ from Proposition 2.1, from which it follows that J is projective. Therefore r.gl.di m.$(R) \leqslant 1$.

Case II. $n > 0$. Given any right ideal J of R, we obtain $pd_R(JM) = pd_T(JM) \leqslant n - 1$ from Proposition 2.1. In particular, $pd(M_R) \leqslant n - 1$, whence $pd[(R/M)_R] \leqslant n$. Inasmuch as J/JM is a projective right (R/M)-module, it follows that $pd_R(J/JM) \leqslant n$, and thus $pd_R(J) \leqslant n$. Therefore r.gl.dim.$(R) \leqslant n + 1$.

THEOREM 2.3. *Let M be a generative right ideal of T and let R be any tame subidealizer of M. Then*

$$GWD(T) \leqslant GWD(R) \leqslant 1 + GWD(T).$$

Proof. Just as in Theorem 2.2, we obtain $GWD(T) \leqslant GWD(R)$ in all cases, and $GWD(R) \leqslant 1 + GWD(T)$ whenever $GWD(T) > 0$. The only case remaining is to prove the latter inequality when $GWD(T) = 0$, and here too we may proceed as in Theorem 2.2 once we show that $(R/M)_R$ is flat. Inasmuch as T is a regular ring, we see that for any $x \in M$, xT is a direct summand of T_T. Then xT is also a direct summand of R_R, hence R/xT is projective. As a right R-module, R/M is the direct limit of the modules $\{R/xT \mid x \in M\}$, from which we conclude that $(R/M)_R$ is flat.

According to Theorems 2.2 and 2.3, there are exactly two possibilities for r.gl.dim.(R) and for $GWD(R)$, and one asks under what conditions each possibility may occur. In case R is the whole idealizer of a semimaximal right ideal of T, these dimensions almost always coincide with those of T, as proved in [10, Theorem 2.8] and [7, Theorems 5 and 6]. In the present situation, if M is assumed to be semimaximal in T, we derive conditions which say precisely when each possibility occurs. In preparation for this, we first prove the following two lemmas, which will also be needed later.

LEMMA 2.4. *Let M be a generative right ideal of T and let R be any tame subidealizer of M.*

(a) $\mathrm{Tor}_n{}^R(R/M, T) = 0$ *for all* $n > 0$.

(b) $\mathrm{Tor}_n{}^R(R/M, T/R) = 0$ *for all* $n > 0$.

(c) *If A is any right R-module, then* $\operatorname{Tor}_n{}^R(A/AM, T) = 0$ *and* $\operatorname{Tor}_n{}^R(A/AM, T/R) = 0$ *for all* $n > 0$.

Proof. (a) In light of the isomorphism $M \otimes_R T \to M$, we obtain $\operatorname{Tor}_1{}^R(R/M, T) = 0$. For $n > 1$, it follows from Proposition 1.2 that

$$\operatorname{Tor}_n{}^R(R/M, T) \cong \operatorname{Tor}_{n-1}^R(M, T) \cong \operatorname{Tor}_{n-1}^T(M, T) = 0.$$

(b) The map $(R/M) \otimes_R R \to (R/M) \otimes_R T$ is clearly injective, hence in view of (a) we obtain $\operatorname{Tor}_1{}^R(R/M, T/R) = 0$. For $n > 1$, it is immediate from (a) and the flatness of ${}_R R$ that $\operatorname{Tor}_n{}^R(R/M, T/R) = 0$.

(c) Since A/AM is a projective right (R/M)-module, these results are immediate consequences of (a) and (b).

LEMMA 2.5. *Let M be a generative right ideal of T and let R be any tame subidealizer of M. If H is a right ideal of R which contains M, then $R \cap HT = H$, $pd_R(R/H) = pd_T(T/HT)$, and $wd_R(R/H) = wd_T(T/HT)$.*

Proof. In view of Lemma 2.4, we have $\operatorname{Tor}_1{}^R(R/H, T/R) = 0$, whence the map $R/H \to (R/H) \otimes_R T$ must be injective. Thus $R \cap HT = H$.

Inasmuch as R/M is semisimple, we have a split exact sequence $0 \to H/M \to R/M \to R/H \to 0$. Tensoring with T/R, we obtain another split exact sequence

$$0 \to (H/M) \otimes_R (T/R) \to T/R \to T/(R + HT) \to 0,$$

from which we infer that $T/(R + HT)$ is isomorphic to a direct summand of $(T/R)_R$. Since T_R is projective, we thus obtain $pd_R[T/(R + HT)] \leqslant 1$, and consequently $wd_R[T/(R + HT)] \leqslant 1$ also.

Since $R \cap HT = H$, we obtain an exact sequence

$$0 \to R/H \to T/HT \to T/(R + HT) \to 0.$$

If $(T/HT)_T$ is projective, then $(T/HT)_R$ is projective by Proposition 2.1, from which we infer that R/H must be projective. On the other hand, if R/H is projective it follows immediately that $(T/HT)_T$ is projective. Thus we see that $pd_R(R/H) = 0$ if and only if $pd_T(T/HT) = 0$.

Assuming now that $pd_R(R/H)$ and $pd_T(T/HT)$ are both positive, we see from Proposition 2.1 that $pd_R(T/HT) = pd_T(T/HT) > 0$ also. In view of the above exact sequence, we obtain $pd_R(R/H) = pd_R(T/HT)$, and therefore $pd_R(R/H) = pd_T(T/HT)$.

Similarly, $wd_R(R/H) = wd_T(T/HT)$.

THEOREM 2.6. *Assume that M is a generative semimaximal right ideal of T,* *and let R be any tame subidealizer of M. Let S denote the idealizer of M in T, and set $H = \{x \in R \mid (S/R)x = 0\}$. If $R \neq T$, then*

$$\text{r.gl.dim.}(R) = \sup\{\text{r.gl.dim.}(T), 1 + pd_T(T/HT)\}.$$

Proof. If r.gl.dim.(T) is infinite, then we are done by Theorem 2.2, hence we may assume that r.gl.dim.$(T) = n < \infty$. We note that since H is a two-sided ideal of R which contains M, R/H is a semisimple ring.

Case I. $n = 0$. In view of the assumption that $R \neq T$, we see that $M < R$. Since $TM = T$, it follows that the map $R \otimes_R (R/M) \to T \otimes_R (R/M)$ is not injective, from which we conclude that $_R(R/M)$ is not flat. Thus $GWD(R) > 0$, and so r.gl.dim. $(R) > 0$. According to Theorem 2.2, we thus obtain r.gl.dim.$(R) = 1$.

Case II. $n > 0$, $pd_T(T/HT) \leqslant n - 1$. We have $pd[(R/H)_R] \leqslant n - 1$ by Lemma 2.5, and S/R is a projective right (R/H)-module, whence $pd[(S/R)_R] \leqslant n - 1$. Thus $pd(S_R) \leqslant n - 1$, and so $pd[(T/S)_R] \leqslant n$.

Consider any right ideal J of R. Due to the semimaximality of M, T/S is a semisimple right S-module, from which we infer that JT/JS is isomorphic to a direct summand of a direct sum of copies of T/S. Thus $pd_R(JT/JS) \leqslant n$. Since JS/J is a projective right (R/H)-module, we also have $pd_R(JS/J) \leqslant n - 1$, hence $pd_R(JT/J) \leqslant n$. According to Proposition 2.1, we have $pd_R(JT) \leqslant n - 1$, from which we conclude that $pd_R(J) \leqslant n - 1$. Therefore r.gl.dim.$(R) \leqslant n$, which in light of Theorem 2.2 yields r.gl.dim.$(R) = n$.

Case III. $n = 1$, $pd_T(T/HT) = 1$. Inasmuch as r.gl.dim.$(R) \leqslant 2$ by Theorem 2.2, it suffices to prove that r.gl.dim.$(R) > 1$.

According to Lemma 2.5, $(R/H)_R$ is not projective, from which it follows that $(R/H)_R$ must have a simple submodule W which is not projective. If it happens that $\text{Hom}_R(W, T) \neq 0$, then we obtain an exact sequence $0 \to W \to T \to V \to 0$. In this situation it is clear that $pd_R(V) > 1$, hence r.gl.dim.$(R) > 1$. Thus we may assume, without loss of generality, that $\text{Hom}_R(W, T) = 0$.

Letting P/R denote the sum of all submodules of $(S/R)_R$ which are isomorphic to W, we see that P/R is a fully invariant submodule of $(S/R)_R$, whence P is an (R, R)-submodule of $_RS_R$. Noting that S/R is a faithful right (R/H)-module, we infer that $P \neq R$.

We now claim that P_R cannot be projective. If it is, then there exist elements $\{p_i \mid i \in I\}$ in P and maps $\{f_i | i \in I\}$ from P_R into R_R such that for each $x \in P$, $f_i x = 0$ for all but finitely many $i \in I$, and $\sum_i p_i(f_i x) = x$. In view of the assumption that $\text{Hom}_R(W, T) = 0$, it follows that $\text{Hom}_R[(P/R)_R, T_R] = 0$, from which we infer that for each i the map f_i is just left multiplication by the element $u_i = f_i 1$. Inasmuch as the elements u_i all belong to the ideal

$N = \{n \in R \mid nP \subseteq R\}$, we conclude from the equation $\sum_i p_i(f_i 1) = 1$ that $R \subseteq PN$.

Observing that $M \subseteq N$, and recalling that $M = M^2$, we see that $M \otimes_R (R/N) = 0$, from which it follows that $\mathrm{Tor}_1{}^R(R/M, R/N) = 0$. Inasmuch as $P \subseteq S$, P/R is a projective right (R/M)-module, hence we obtain $\mathrm{Tor}_1{}^R(P/R, R/N) = 0$. It follows from this that $R \cap PN = N$, whence $N = R$ and then $P = R$, which is false.

Therefore P_R cannot be projective. Since T_R is projective, we thus obtain r.gl.dim.$(R) > 1$.

Case IV. $n > 1$, $pd_T(T/HT) = n$. According to Lemma 2.5,

$$pd[(R/H)_R] = n.$$

Inasmuch as S/R is a faithful right (R/H)-module, it follows that $(R/H)_R$ is isomorphic to a direct summand of some direct sum of copies of $(S/R)_R$, from which we obtain $pd[(S/R)_R] \geqslant n$. Since $n > 1$, it follows that $pd(S_R) \geqslant n$ also, whence $pd[(T/S)_R] \geqslant n + 1$. In light of Theorem 2.2, we thus obtain r.gl.dim.$(R) = n + 1$.

THEOREM 2.7. *Assume that M is a generative semimaximal right ideal of T, and let R be any tame subidealizer of M. Let S denote the idealizer of M in T, and set $H = \{x \in R \mid (S/R)x = 0\}$. If $R \neq T$, then*

$$GWD(R) = \sup\{GWD(T), 1 + wd_T(T/HT)\}.$$

Proof. If $GWD(T)$ is infinite, then we are done by Theorem 2.3, hence we may assume that $GWD(T) = n < \infty$.

Case I. $n = 0$. Analogous to Theorem 2.6.

Case II. $n > 0$, $wd_T(T/HT) \leqslant n - 1$. Analogous to Theorem 2.6.

Case III. $n = 1$, $wd_T(T/HT) = 1$.

Inasmuch as $GWD(R) \leqslant 2$ by Theorem 2.3, it suffices to prove that $GWD(R) > 1$.

According to Lemma 2.5, $(R/H)_R$ is not flat. Inasmuch as S/R is a faithful right (R/H)-module, we see that $(R/H)_R$ must be isomorphic to a direct summand of some direct sum of copies of $(S/R)_R$, from which it follows that $(S/R)_R$ is not flat. Therefore $\mathrm{Tor}_1{}^R(S/R, R/K) \neq 0$ for some left ideal K of R, and we claim that this K may be chosen so that $SK = K$.

Observing that $M = M^2$ and $(S/R)M = 0$, we see that $(S/R) \otimes_R M = 0$, whence $\mathrm{Tor}_1{}^R(S/R, R/M) = 0$. Inasmuch as $R/(K + M)$ is isomorphic to a direct summand of ${}_R(R/M)$, we thus obtain $\mathrm{Tor}_1^R[S/R, R/(K + M)] = 0$. In light of the exact sequence

$$0 \to M/(K \cap M) \to R/K \to R/(K + M) \to 0,$$

it follows that $\mathrm{Tor}_1^R[S/R, M/(K \cap M)] \neq 0$. Since ${}_RM$ is flat by Proposition 1.1, we have $\mathrm{Tor}_2^R(S/R, R/M) = 0$ also, from which we infer that $\mathrm{Tor}_1^R[S/R, R/(K \cap M)] \neq 0$. Thus, replacing K by $K \cap M$, we may first of all assume that $K \subseteq M$. Now $SK \subseteq M$, so that SK is a left ideal of R. Inasmuch as SK/K is a projective left (R/M)-module, we use the fact that $\mathrm{Tor}_1^R(S/R, R/M) = 0$ to see that $\mathrm{Tor}_1^R(S/R, SK/K) = 0$. Together with the condition $\mathrm{Tor}_1^R(S/R, R/K) \neq 0$, this forces $\mathrm{Tor}_1^R(S/R, R/SK) \neq 0$. We now replace K by SK, so that we have a left ideal K of S such that $K \subseteq R$ and $\mathrm{Tor}_1^R(S/R, R/K) \neq 0$.

We now claim that S_R is not flat. If it is flat, then the natural map $S \otimes_R K \to SK$ is an isomorphism. Since the map $R \otimes_R K \to S \otimes_R K \to SK$ is also an isomorphism, it follows that $R \otimes_R K \to S \otimes_R K$ is an isomorphism. However, this shows that $(S/R) \otimes_R K = 0$ and so $\mathrm{Tor}_1^R(S/R, R/K) = 0$, which is false. Therefore S_R is not flat. Since T_R is projective, we thus obtain $GWD(R) > 1$.

Case IV. $n > 1$, $wd_T(T/HT) = n$. Analogous to Theorem 2.6.

We note that the module T/HT used in Theorems 2.6 and 2.7 is somewhat easier to find if R/M happens to be a simple ring. In this case S/R is either 0 or a faithful right (R/M)-module, from which it follows that T/HT is either 0 or just T/M. In particular, when M is a maximal right ideal of T, then S/M (which is isomorphic to the endomorphism ring of T/M) must be a division ring, hence R/M must be a division ring also.

We next take up a question intermediate to the questions of when the global or global weak dimensions of R can be at most 1, namely, when R can be right or left semihereditary. Part of our procedure is based on S. U. Chase's characterization of semiheredity [3, Theorem 4.1]: A ring R is *left* semihereditary if and only if every torsionless *right* R-module is flat. [We recall that a *torsionless* R-module is simply one which can be embedded in a direct product of copies of R.]

THEOREM 2.8. *Assume that M is a generative semimaximal right ideal of T, and let R be any tame subidealizer of M. Let S denote the idealizer of M in T, and set $H = \{x \in R \mid (S/R)x = 0\}$.*

(a) *R is left semihereditary if and only if T is left semihereditary and $(T/HT)_T$ is flat.*

(b) *R is right semihereditary if and only if T is right semihereditary and $(T/HT)_T$ is projective.*

Proof. (a) If R is left semihereditary, then $GWD(R) \leqslant 1$, hence Theorem 2.7 shows that $wd_T(T/HT) = 0$. Given any torsionless right T-module A, it follows from the projectivity of T_R that A_R is torsionless too. Then A_R must be flat, hence so is $(A \otimes_R T)_T \cong A_T$. Therefore T is left semihereditary.

Conversely, assume that T is left semihereditary and that $(T/HT)_T$ is flat. We are done if $R = T$, hence we may assume that $R \neq T$, so that we obtain $GWD(R) = 1$ from Theorem 2.7. Now any torsionless right R-module A can clearly be embedded in a torsionless right T-module B, which must be flat because T is left semihereditary. Inasmuch as T_R is projective and $GWD(R) = 1$, we conclude that B_R is flat and so is A. Therefore R is left semihereditary.

(b) First assume that T is right semihereditary and that $(T/HT)_T$ is projective. Given any finitely generated right ideal J of R, it follows as in Case II of Theorem 2.6 that $pd_R(JT/J) \leqslant 1$. Inasmuch as JT is a finitely generated right ideal of T and thus is projective, we obtain $pd_R(JT) = 0$ from Proposition 2.1, hence J must be projective. Therefore R is right semihereditary.

Conversely, if R is right semihereditary then it follows as in (a) that $wd_T(T/HT) = 0$, hence by Lemma 2.5 $(R/H)_R$ is flat.

Inasmuch as S/R is faithful right (R/H)-module, it follows that $(R/H)_R$ may be embedded in a direct sum of n copies of $(S/R)_R$, for some positive integer n, whence $(R/H)_R \cong A/R^n$ for some finitely generated submodule A of $(S^n)_R$. Since $(T^n)_R$ is projective and R is right semihereditary, it follows that A must be projective. In view of the exact sequence $0 \to R^n \to A \to R/H \to 0$, we now see that $(R/H)_R$ is finitely presented. Since it is already flat it must therefore be projective, whence $(T/HT)_T$ is projective.

If J is any finitely generated right ideal of T, then since T_R is finitely generated we see that J_R is finitely generated. Since T_R is projective and R is right semihereditary, it follows that J_R is projective, and then Proposition 2.1 says that J_T is projective. Therefore T is right semihereditary.

To conclude this section, we investigate the relationship between the left global dimensions of T and R, where R is a tame subidealizer of some generative right ideal M of T. We begin with a lemma about the projective dimensions of left T-modules, which is based on the following observation: Inasmuch as ${}_RM$ is projective by Proposition 1.1, we have $pd[{}_R(R/M)] \leqslant 1$, from which it follows that $pd_R(A/MA) \leqslant 1$ for all left R-modules A.

LEMMA 2.9. *Let M be a generative right ideal of T, and let R be any tame subidealizer of M. If A is any left T-module, then*

$$pd_T(A) \leqslant pd_R(A) \leqslant 1 + pd_T(A).$$

Proof. It is immediate from Proposition 1.2 that $pd_T(A) \leqslant pd_R(A)$. Observing that $M(T/R) = 0$, we see from the remarks above that $pd[_R(T/R)] \leqslant 1$, whence $pd(_RT) \leqslant 1$. Thus $pd_R(P) \leqslant 1$ for all projective left T-modules P, from which a straightforward induction establishes that $pd_R(A) \leqslant 1 + pd_T(A)$.

THEOREM 2.10. *Let M be a generative right ideal of T, and let R be any tame subidealizer of M. Then*

$$\text{l.gl.dim.}(T) \leqslant \text{l.gl.dim.}(R) \leqslant 1 + \text{l.gl.dim.}(T).$$

Proof. It is clear from Lemma 2.9 that l.gl.dim.$(T) \leqslant$ l.gl.dim.(R).To prove the other inequality we may assume that l.gl.dim.$(T) = n < \infty$.

Case I. $n = 0$. In this case $M = eT$ for some idempotent $e \in T$, and the idealizer of M is just $eT + T(1 - e)$, from which we see that $R = eT + R(1 - e)$. It is clear from this equation that $R(1 - e)$ is a two-sided ideal of R. Observing that $R = eTe + R(1 - e)$, we infer that $R/R(1 - e) \cong eTe$, which is a semisimple ring. Since $R(1 - e)M = 0$, we see that $_RM$ is semisimple.

To show that l.gl.dim.$(R) \leqslant 1$, it suffices to prove that any essential left ideal J of R is projective. Inasmuch as $M \subseteq \text{soc}(_RR) \subseteq J$ by [6, Corollary 1.3], we have $M(R/J) = 0$ and thus $pd_R(R/J) \leqslant 1$. Therefore J is projective.

Case II. $n > 0$. If J is any left ideal of R, then TJ is a left ideal of T, hence we obtain $pd_R(TJ) \leqslant n$ from Lemma 2.9. Observing that $M(TJ/J) = 0$, we see that $pd_R(TJ/J) \leqslant 1$, from which it follows that $pd_R(J) \leqslant n$. Therefore l.gl.dim.$(R) \leqslant n + 1$.

In view of the previous results in this section, one would expect at this point a theorem showing when each of the two possibilities allowed by Theorem 2.10 can occur, at least in the case when M is semimaximal. At a minimum, when T is not semisimple and M is a generative semimaximal right ideal of T, the idealizer of M ought to have the same left global dimension as T. Unfortunately this is false, as the following example shows.

We first choose any left hereditary, left noetherian simple ring P which is not artinian. (For example, either of the rings $F(y)[x]$ or $F[y][x]$ discussed in Part III of [9] will do, as will the principal ideal domains constructed in [4, Theorems 1.4 and 2.3].) Since P is simple but not a division ring, any maximal right ideal N of P is generative. We now let F denote the center of P (which is a field) and set $T = \begin{pmatrix} F & 0 \\ P & P \end{pmatrix}$, $M = \begin{pmatrix} F & 0 \\ P & N \end{pmatrix}$. Inasmuch as N is a generative maximal right ideal of P, M must be a generative maximal right ideal of T. We finally let R be the idealizer of M in T, and note that $R = \begin{pmatrix} F & 0 \\ P & I \end{pmatrix}$, where I is the idealizer of N in P.

Setting $K = \begin{pmatrix} F & 0 \\ P & 0 \end{pmatrix}$, we note that K is a two-sided ideal of T, that $T/K \cong P$, and that ${}_T(T/K)$ is projective. Since the left ideals $A = \begin{pmatrix} 0 & 0 \\ P & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix}$ are both left (T/K)-modules, it follows from the left hereditary assumption on P that any submodule of ${}_TA$ or ${}_TB$ must be projective. Observing that all proper submodules of ${}_TK$ are contained in A, we also have that all submodules of ${}_TK$ are projective. Since ${}_TT = K \oplus B$, we thus conclude that T is left hereditary. On the other hand, T cannot be semisimple because it contains nonzero nilpotent ideals, for example A. Therefore l.gl.dim.$(T) = 1$.

According to Theorem 2.10, l.gl.dim.$(R) \leqslant 2$. We claim that

$$\text{l.gl.dim.}(R) = 2,$$

which we prove by showing that R contains a left ideal which is not projective, namely A. To see this, it clearly suffices to prove that ${}_IP$ is not projective.

If on the contrary ${}_IP$ is projective, then there exists a nonzero homomorphism $f: {}_IP \to {}_II$. In view of the remarks after Proposition 1.1, we have $(P/I) \otimes_I P = 0$, hence $(P/I) \otimes_I fP = 0$ and so $fP = P(fP)$. Thus fP is a nonzero left ideal of P which is contained in I, hence the two-sided ideal $W = \{x \in I \mid Px \subseteq I\}$ is nonzero.

Now $\text{soc}(P_P) = 0$ because P is simple but not artinian, hence we see that N must be an essential right ideal of P. The simplicity of P also implies that the right singular ideal of P is zero, from which we infer that $WN \neq 0$. Thus WN is a nonzero two-sided ideal of P, hence we obtain $WN = P$ and then $I = P$, which is impossible. Therefore ${}_IP$ is in fact not projective, and we have shown that l.gl.dim.$(R) = 2$.

Thus we see that the left global dimension of the idealizer of a generative maximal right ideal of T can be larger than l.gl.dim.(T). We can also use this example to show that the idealizer of a generative maximal right ideal of a left noetherian ring need not be left noetherian, as follows.

With K, A, and B as above, we use the left noetherian assumption on P to see that ${}_TA$ and ${}_TB$ are noetherian. Inasmuch as $A \oplus B$ is a maximal left ideal of T, it follows that T must be left noetherian. However, R is left semihereditary by Theorem 2.8 but not left hereditary, hence R cannot be left noetherian.

3. Chain Conditions

In this section we investigate when a subidealizer of T can inherit chain conditions from T. We first look at right-hand chain conditions, where the situation is relatively straightforward.

THEOREM 3.1. *Assume that M is a generative semimaximal right ideal of T,* *and let R be any tame subidealizer of M. Also, let S denote the idealizer of M in T.*

(a) *R is right noetherian if and only if T is right noetherian and $(S/R)_R$ is finitely generated.*

(b) *R is right artinian if and only if T is right artinian and $(S/R)_R$ is finitely generated.*

Proof. (a) If R is right noetherian, then the finitely generated right R-module T_R must be noetherian, whence T is a right noetherian ring. Since $(S/R)_R$ is a submodule of $(T/R)_R$, it is clear that $(S/R)_R$ is finitely generated.

Conversely, assuming that T is right noetherian and that $(S/R)_R$ is finitely generated, we see from [9, Theorem 2.2] that S is right noetherian. Clearly S_R is finitely generated. Given any right ideal J of R, we observe that JS and JM are right ideals of S, from which we infer that JS and JM are finitely generated as right R-modules. Now JS/JM is a finitely generated right (R/M)-module and so is noetherian, hence J/JM is finitely generated. Therefore J must be finitely generated, whence R is right noetherian.

(b) If R is right artinian, then the finitely generated right R-module T_R must be artinian, hence T is a right artinian ring. Since R is also right noetherian, it follows from (a) that $(S/R)_R$ is finitely generated.

Conversely, assuming that T is right artinian and that $(S/R)_R$ is finitely generated, we see from [9, Theorem 2.2] that S is right artinian. Inasmuch as T is also right noetherian, it follows from (a) that R is right noetherian. Now $(R/M)_R$ is already artinian, and M_S has composition series, hence it suffices to show that any composition factor C of M_S has finite length as an R-module.

Case I. $CM = 0$. Here C is a right module over R/M and so is semisimple. Since R is right noetherian, C_R is finitely generated and thus is a finite direct sum of simple R-modules.

Case II. $CM \neq 0$. Here the set $A = \{x \in C \mid xM = 0\}$ is a proper S-submodule of C, whence $A = 0$. Then given any nonzero $x \in C$, we see that xM is a nonzero S-submodule of C, hence $xM = C$ and thus $xR = C$. Therefore C_R is simple.

As we saw in the example at the end of Section 2, the left noetherian condition need not carry over from T to the idealizer of a generative semimaximal right ideal. It turns out, however, that the left artinian condition does carry over. This is due to the fact that in an artinian ring, the idealizer of a semimaximal right ideal can also be expressed as the idealizer of a semimaximal left ideal, as we now show.

LEMMA 3.2. *Let J be the Jacobson radical of T, and assume that T/J is* *semisimple. Assume that M is a semimaximal right ideal of T, and let S denote the idealizer of M in T. Then T contains a generative semimaximal left ideal K such that S is also the idealizer of K in T.*

Proof. Inasmuch as M is semimaximal in T, we have $J \subseteq M$, and then [9, Lemma 3.1] shows that S/J is the idealizer of M/J in T/J. If we can express S/J as the idealizer of some generative semimaximal left ideal K/J of T/J, then K will be a generative semimaximal left ideal of T and [9, Lemma 3.1] will show that S is the idealizer of K in T. Thus we may confine our attention to T/J, i.e., we may assume that T is semisimple.

Now $M = eT$ for some idempotent $e \in T$. Clearly $S = eT + T(1 - e)$, hence S is also the idealizer of the left ideal $T(1 - e)$. Inasmuch as T is a semisimple ring, $T(1 - e)$ is a semimaximal left ideal of T, hence [9, Proposition 1.7] says that $T(1 - e)$ may be enlarged to a generative semimaximal left ideal K of T such that S is also the idealizer of K.

THEOREM 3.3. *Assume that M is a generative semimaximal right ideal of T, and let R be any tame subidealizer of M. Let S denote the idealizer of M in T, and set $H = \{s \in S \mid sM = 0\}$. Then R is left artinian if and only if T is left artinian and ${}_R[S/(R + H)]$ is finitely generated.*

Proof. Assume first that T is left artinian and that ${}_R[S/(R + H)]$ is finitely generated. In light of Lemma 3.2, it follows immediately from Theorem 3.1 that S is left artinian, whence ${}_SM$ must have a composition series. Inasmuch as ${}_R(R/M)$ is already artinian, it thus suffices to show that any composition factor C of ${}_SM$ has finite length as an R-module.

Case I. $MC = 0$. Here C is a left module over R/M, hence it suffices to show that ${}_RC$ is finitely generated. In view of the hypothesis on ${}_R[S/(R+H)]$, it is clear that ${}_R(S/H)$ is finitely generated. Observing that C is a simple left (S/H)-module, we conclude from this that ${}_RC$ must be finitely generated.

Case II. $MC \neq 0$. Just as in Case II of Theorem 3.1(b), ${}_RC$ is simple.

Conversely, assuming that R is left artinian, we see that ${}_RM$ is an artinian module, whence ${}_SM$ must be artinian. Inasmuch as M is semimaximal in T, S/M is a semisimple ring, from which we now infer that S is left artinian. Observing that M is a faithful left module over the left artinian ring S/H, we infer that ${}_S(S/H)$ can be embedded in some finite direct sum of copies of ${}_SM$. Since R is left noetherian, we conclude from this that ${}_R(S/H)$ must be finitely generated, whence ${}_R[S/(R + H)]$ is finitely generated.

If P denotes the prime radical of T, then P is contained in the Jacobson radical of T and hence in M. Thus ${}_RP$ is artinian and so ${}_TP$ is artinian, hence it suffices to show that T/P is left artinian. Inasmuch as M/P is a generative semimaximal right ideal of T/P and R/P is a tame subidealizer of M/P, we may thus assume that $P = 0$, i.e., we may assume that T is semiprime.

Inasmuch as R is left artinian, its Jacobson radical J is nilpotent, and the ring R/J is semisimple. Now JM is a nilpotent right ideal of T and thus $JM = 0$, from which we see that ${}_RM$ is semisimple. Since R is left artinian, we obtain a decomposition ${}_RM = A_1 \oplus \cdots \oplus A_n$, where each A_i is a simple left R-module. Recalling that T_R is flat and that $TM = T$, we infer that $T = TA_1 \oplus \cdots \oplus TA_n$, hence T will be a semisimple ring (and thus left artinian) if we show that each TA_i is a simple left T-module. Given any nonzero $x \in TA_i$, we use the relations $TM = T$ and $MT = M$ to see that Mx is a nonzero R-submodule of A_i, from which it follows that $Mx = A_i$ and then $Tx = TA_i$. Therefore ${}_T(TA_i)$ is indeed simple.

4. Nonsingular Rings

This section has two objectives, the first of which is to establish the basic relationships between singular and nonsingular modules over a nonsingular ring T and over a subidealizer R of an essential right ideal of T. Secondly, we investigate a situation under which the inheritance of properties from T by R is left-right symmetric: Specifically, we show that under certain conditions R can be realized as a subidealizer of a left ideal in a ring which is Morita-equivalent to T.

Our notation for singular submodules coincides with that used in [6]: $\mathcal{S}(P)$ denotes the collection of essential right ideals of a ring P, $Z(A)$ or $Z_P(A)$ denotes the singular submodule of a P-module A, and $Z_r(P)$ and $Z_l(P)$ denote the right and left singular ideals of P. In case P is a right nonsingular ring, we use S^0 to stand for the localization functor associated with the singular torsion theory. In the present paper we only have need for S^0P, which is just the maximal right quotient ring of P [6, p. 18].

Proposition 4.1. *Let T be a right nonsingular ring, M an essential right ideal of T, R any subidealizer of M.*

(a) $\mathcal{S}(T) = \{K \leqslant T_T \mid K \cap R \in \mathcal{S}(R)\}$.

(b) $\mathcal{S}(R) = \{J \leqslant R_R \mid JM \in \mathcal{S}(T)\}$.

(c) $Z_R(A) = Z_T(A)$ *for any right T-module A.*

(d) $Z_r(R) = Z(T_R) = 0$.

(e) *R_R is essential in T_R, and $S^0R = S^0T$.*

Proof. (a) Consider any $K \in \mathcal{S}(T)$, and suppose that $(K \cap R) \cap A = 0$ for some right ideal A of R. Then $K \cap AM = 0$ and so $AM = 0$, because

AM is a right ideal of T. Inasmuch as $M \in \mathscr{S}(T)$ and $Z_r(T) = 0$, it follows that $A = 0$, and thus $K \cap R \in \mathscr{S}(R)$.

Now let K be any right ideal of T such that $K \cap R \in \mathscr{S}(R)$, and suppose that $K \cap A = 0$ for some right ideal A of T. Then $A \cap M$ is a right ideal of R satisfying $(K \cap R) \cap (A \cap M) = 0$, whence $A \cap M = 0$. Inasmuch as $M \in \mathscr{S}(T)$, it follows that $A = 0$, and thus $K \in \mathscr{S}(T)$.

(b) If J is a right ideal of R such that $JM \in \mathscr{S}(T)$, then $JM \in \mathscr{S}(R)$ by (a), hence $J \in \mathscr{S}(R)$.

Now let $J \in \mathscr{S}(R)$ and suppose that $JM \cap A = 0$ for some right ideal A of T. Then $(J \cap A)M = 0$, hence $J \cap A = 0$. Now $A \cap M$ is a right ideal of R satisfying $J \cap (A \cap M) = 0$, hence $A \cap M = 0$ and so $A = 0$. Therefore $JM \in \mathscr{S}(T)$.

(c) is immediate from (a) and (b).

(d) According to (c) we have $Z(T_R) = 0$, and then $Z(R_R) = 0$ also.

(e) Since $M \in \mathscr{S}(T)$ and T is an essential right T-submodule of S^0T, we see that M_T is essential in $(S^0T)_T$, whence S^0T/M is a singular right T-module. Also S^0T is a nonsingular right T-module, hence in light of (c) we see that $(S^0T)_R$ is nonsingular while $(S^0T/M)_R$ is singular, from which we infer that M_R is essential in $(S^0T)_R$. Consequently R_R is essential in $(S^0T)_R$, and in particular R_R is essential in T_R. Now S^0T is a regular, right self-injective ring [because $Z_r(T) = 0$], hence [6, Proposition 1.16] says that $S^0T \doteq S^0R$.

PROPOSITION 4.2. *Assume that M is a finitely generated, projective, essential, generative right ideal in a right nonsingular ring T, and let R be any tame subidealizer of M. Set $V = \{v \in S^0T \mid vM \leqslant M\}$, and let S denote the idealizer of M in T.*

(a) *V is a ring which is Morita-equivalent to T.*

(b) *M is a finitely generated, projective, generative left ideal of V.*

(c) *R is a tame subidealizer of M in V, and S is the idealizer of M in V.*

(d) *If ${}_TT$ is essential in ${}_T(S^0T)$, then V is a left nonsingular ring and M is an essential left ideal of V.*

Proof. (a) The natural map $\phi: V \to \mathrm{End}_T(M)$ is injective because $M \in \mathscr{S}(T)$ and $(S^0T)_T$ is nonsingular. Since $(S^0T)_T$ is injective, ϕ is also surjective and so is an isomorphism. Now M_T is finitely generated projective by hypothesis, and it is a generator because it is generative, hence V must be Morita-equivalent to T.

(b) It is obvious that M is a left ideal of V. We see by Proposition 1.1 that ${}_RM$ is finitely generated, and R is clearly a subring of V, hence ${}_VM$ must be finitely generated.

Since M_T and T_R are both finitely generated projective, M_R must be finitely generated projective, hence there exist elements $m_1, \ldots, m_k \in M$ and maps $f_1, \ldots, f_k \in \mathrm{Hom}_R(M_R, R_R)$ such that $m_1(f_1x) + \cdots + m_k(f_kx) = x$ for all $x \in M$. Recalling that M is an idempotent two-sided ideal of R, we see that the maps f_i are endomorphisms of M_R. In view of the discussion following Proposition 1.1, each f_i is an endomorphism of M_T and hence is left multiplication by some $v_i \in V$. Since $M \in \mathscr{S}(T)$ and $(S^0T)_T$ is nonsingular, we now infer that $m_1v_1 + \cdots + m_kv_k = 1$, whence M is a generative left ideal of V.

Inasmuch as R is a subidealizer of a generative left ideal of V, we can use the discussion following Proposition 1.1 to see that $V \otimes_R M \cong M$. According to Proposition 1.1, ${}_RM$ is projective, hence we conclude that ${}_VM$ is projective also.

(c) It is clear that R is a tame subidealizer of M in V, and that S is a subring of V which is contained in the idealizer of M. If v is any element of the idealizer of M in V, then we have $Mv \subseteq M$, hence it follows from the identity $TM = T$ that $v \in T$. However, $vM \leqslant M$ because $v \in V$, hence we see that $v \in S$. Therefore S is the idealizer of M in V.

(d) We claim that ${}_RM$ must be essential in ${}_R(S^0T)$. Given any submodule A of ${}_R(S^0T)$ satisfying $M \cap A = 0$, we see that $M(T \cap TA) = 0$. In view of the relation $TM = T$, it follows that $T \cap TA = 0$, and then $TA = 0$ because ${}_TT$ is essential in ${}_T(S^0T)$. Thus $A = 0$, and so ${}_RM$ is essential in ${}_R(S^0T)$. As a consequence ${}_RM$ is essential in ${}_RV$, hence M is certainly an essential left ideal of V.

As another consequence we see that ${}_VM$ is essential in ${}_V(S^0T)$, hence ${}_VV$ must be essential in ${}_V(S^0T)$. Since S^0T is a regular ring [because $Z_r(T) = 0$], it now follows as in [6, Proposition 1.16] that $Z_l(V) = 0$.

THEOREM 4.3. *Assume that T is a semiprime right and left Goldie ring with $GWD(T) \leqslant 1$, and let M be a finitely generated, essential, generative, semimaximal right ideal of T. Let R be any tame subidealizer of M, and let S denote the idealizer of M in T.*

(a) *R is a semiprime right and left Goldie ring.*

(b) *If $R = S$, then* r.gl.dim.(R) = r.gl.dim.(T), l.gl.dim.(R) = l.gl.dim.(T), *and* $GWD(R) = GWD(T)$.

(c) *If $R \neq S$, then* r.gl.dim.$(R) = \sup\{2$, r.gl.dim.$(T)\}$, l.gl.dim.$(R) = \sup\{2$, l.gl.dim.$(T)\}$, *and* $GWD(R) = 2$.

(d) *R is right noetherian if and only if T is right noetherian and $(S/R)_R$ is finitely generated.*

(e) *R is left noetherian if and only if T is left noetherian and $_R(S/R)$ is finitely generated.*

Proof. Inasmuch as T is a semiprime right and left Goldie ring, it follows from [11, Theorem 1.7] that T is right and left nonsingular as well as right and left finite-dimensional. According to [12, Corollary 1, p. 228], it follows from this that every finitely generated flat right or left T-module is projective, hence, due to the assumption $GWD(T) \leqslant 1$, we see that T is right and left semihereditary. In particular, we note that M_T must be projective.

Since T is a semiprime right and left Goldie ring, its right and left maximal quotient rings must coincide, i.e., S^0T is also the maximal left quotient ring of T. It follows that $_TT$ is essential in $_T(S^0T)$, hence we see from Proposition 4.2 that the ring V described there is Morita-equivalent to T, that M is a finitely generated, projective, essential, generative left ideal of V, and that S is the idealizer of M in V. Also, the Morita-equivalence implies that V is a semiprime right and left Goldie ring with $GWD(V) \leqslant 1$. Thus, with the sole exception of semimaximality, we see that V, M, R, S satisfy the left-right duals of all our hypotheses on T, M, R, S.

Inasmuch as M is semimaximal in T, the ring S is a tame subidealizer of M. Thus, according to Proposition 1.3, M will be a semimaximal left ideal of V provided we show that V_S is flat. In view of Proposition 4.1, we see that $Z_l(S) = 0$ and that $_SS$ is essential in $_SV$. Since S is left semihereditary by Theorem 2.8, it now follows from [6, Lemma 2.2] that V_S is flat.

(a) According to Proposition 4.1, $Z_r(R) = 0$, and by symmetry $Z_l(R) = 0$ also. Since T_R is flat, any independent family $\{A_i\}$ of nonzero left ideals of R gives rise to an independent family $\{TA_i\}$ of nonzero left ideals of T, hence it follows from the finite-dimensionality of $_TT$ that $_RR$ must be finite-dimensional. Symmetrically, R_R is finite-dimensional also. Noting that any right annihilator ideal of R must be $\mathscr{S}$-closed in the sense of [6, p. 14], we see from [6, Theorem 1.24] that R has ACC on right annihilators. Therefore R is a right Goldie ring, and likewise also a left Goldie ring. If N is any nilpotent right ideal of R, then NM is a nilpotent right ideal of T, hence $NM = 0$ and so $N = 0$. Therefore R is semiprime.

(b) If $R = T$, then we are done, hence we may assume that $R \neq T$. It follows that M is a proper essential right ideal of T, hence M cannot be a direct summand of T, i.e., $(T/M)_T$ is not projective. Since M_T is finitely generated $(T/M)_T$ cannot be flat either, hence $GWD(T) > 0$ and r.gl.dim.$(T) > 0$. According to Theorems 2.6 and 2.7, r.gl.dim.$(R) =$ r.gl.dim.(T) and $GWD(R) = GWD(T)$. By symmetry, l.gl.dim.$(R) =$ l.gl.dim.(V), hence l.gl.dim.$(R) =$ l.gl.dim.(T).

(c) Set $H = \{x \in R \mid (S/R)x = 0\}$, which is a proper two-sided ideal of R. Since $R \cap HT = H$ by Lemma 2.5, we see that HT is a proper right ideal of T. Now $HT \in \mathscr{S}(T)$ because $M \leqslant HT$, hence HT cannot be a direct summand of T, i.e., T/HT is not projective. Observing that T/HT is isomorphic to a direct summand of T/M (because T/M is semisimple), and that T/M is a finitely presented right T-module with $pd_T(T/M) \leqslant 1$, we see that T/HT too is a finitely presented right T-module with $pd_T(T/HT) \leqslant 1$. Since T/HT is not projective it thus cannot be flat, hence we obtain $wd_T(T/HT) = 1$ and $pd_T(T/HT) = 1$. According to Theorems 2.6 and 2.7, we now have r.gl.dim.$(R) = \sup\{2, \text{r.gl.dim.}(T)\}$ and $GWD(R) = 2$. By symmetry, l.gl.dim.$(R) = \sup\{2, \text{l.gl.dim.}(V)\}$, hence l.gl.dim.$(R) = \sup\{2, \text{l.gl.dim.}(T)\}$.

(d) is automatic from Theorem 3.1.

(e) By symmetry and (d), R is left noetherian if and only if V is left noetherian and ${}_R(S/R)$ is finitely generated. This is enough because V is left noetherian if and only if T is left noetherian.

5. Splitting Rings

A ring P is said to be a (*right*) *splitting ring* provided that for every right P-module A, $Z(A)$ is a direct summand of A. As noted in [2, Proposition1.12], this is equivalent to the requirement that $\text{Ext}_P{}^1(A, C) = 0$ for all nonsingular A_P and all singular C_P. According to [7, Theorem 10], the idealizer of an essential, generative, semimaximal right ideal in a right nonsingular splitting ring is again a splitting ring, hence we ask when a subidealizer in a splitting ring can be a splitting ring. One preparatory lemma is required first.

LEMMA 5.1. *Let M be an essential generative right ideal in a right nonsingular ring T, and let R be any tame subidealizer of M. If A is any nonsingular right R-module, then*

(a) *The natural map $A \to A \otimes_R T$ is injective.*

(b) *$Z(A \otimes_R T)$ can be embedded in a direct sum of copies of T/M.*

(c) *If C is any right (R/M)-module, then $\text{Ext}_R{}^n(A, C) = 0$ for all $n > 0$.*

(d) *If C is any right T-module, then $\text{Ext}_R{}^n(A, C) \cong \text{Ext}_T{}^n(A \otimes_R T, C)$ for all $n > 0$.*

Proof. (a) According to [6, Proposition 1.18], A may be embedded in some direct product P of copies of S^0R. Inasmuch as $S^0R = S^0T$, P is also a right T-module, hence we can express the embedding as a composition $A \to A \otimes_R T \to P \otimes_R T \to P$. Thus $A \to A \otimes_R T$ must be injective.

(b) Since $\mathrm{Tor}_1{}^R(A/AM, T) = 0$ by Lemma 2.4, we obtain an exact sequence

$$0 \to AM \otimes_R T \to A \otimes_R T \to (A/AM) \otimes_R T \to 0.$$

In view of (a) we see that A can be embedded in a right T-module, hence AM is a right T-module. Thus $AM \otimes_R T \cong AM$, which is nonsingular, hence the map

$$Z(A \otimes_R T) \to A \otimes_R T \to (A/AM) \otimes_R T$$

must be injective. Inasmuch as A/AM is a projective right (R/M)-module, we see that $(A/AM) \otimes_R T$ can be embedded in a direct sum of copies of T/M, from which the required embedding of $Z(A \otimes_R T)$ follows.

(c) In view of the projectivity of T_R, we have $\mathrm{Tor}_p{}^R(T, R/M) = 0$ for all $p > 0$. As we noted above, AM is a right T-module, hence [1, Proposition 4.1.2, p. 117] says that $\mathrm{Tor}_1{}^R(AM, R/M) \cong \mathrm{Tor}_1{}^T[AM, T \otimes_R (R/M)]$. However, $T \otimes_R (R/M) = 0$ because M is generative, hence we obtain $\mathrm{Tor}_1{}^R(AM, R/M) = 0$. Observing that $(A/AM) \otimes_R M = 0$, we see that $\mathrm{Tor}_1{}^R(A/AM, R/M) = 0$ also, and thus $\mathrm{Tor}_1{}^R(A, R/M) = 0$. Inasmuch as ${}_RM$ is projective, we also obtain $\mathrm{Tor}_p{}^R(A, R/M) = 0$ for all $p > 1$.

According to [1, Proposition 4.1.3, p. 118], we now have $\mathrm{Ext}_R{}^n(A, C) \cong \mathrm{Ext}^n_{R/M}(A/AM, C)$ for all $n > 0$, and the latter terms are all zero because R/M is a semisimple ring.

(d) This will follow immediately from [1, Proposition 4.1.3, p. 118] provided we show that $\mathrm{Tor}_p{}^R(A, T) = 0$ for all $p > 0$.

We have $\mathrm{Tor}_1{}^R(A/AM, T) = 0$ by Lemma 2.4. As noted above, AM is a right T-module, hence $\mathrm{Tor}_1{}^R(AM, T) \cong \mathrm{Tor}_1{}^T(AM, T) = 0$ by Proposition 1.2, and thus $\mathrm{Tor}_1{}^R(A, T) = 0$. Inasmuch as $pd({}_RT) \leqslant 1$ by Lemma 2.9, we also have $\mathrm{Tor}_p{}^R(A, T) = 0$ for all $p > 1$.

THEOREM 5.2. *Let M be an essential generative right ideal in a right nonsingular ring T, and let R be any tame subidealizer of M. If T is a splitting ring, and if $\mathrm{Ext}_T{}^1(T/M, C) = 0$ for all singular C_T, then R is a splitting ring.*

Proof. Given any nonsingular A_R and any singular C_R, we must show that $\mathrm{Ext}_R{}^1(A, C) = 0$. Inasmuch as $\mathrm{Ext}_R{}^1(A, C/CM) = 0$ by Lemma 5.1, it suffices to show that $\mathrm{Ext}_R{}^1(A, CM) = 0$. Recalling that $M = M^2$, it follows that there is no loss of generality in assuming that $CM = C$.

Now $C \cong P/J$ for some direct sum P of copies of M_R and some R-submodule J of P. There exists an exact sequence

$$0 \to J/JM \to P/JM \to C \to 0,$$

and we have $\mathrm{Ext}_R^2(A, J/JM) = 0$ by Lemma 5.1, hence all that remains is to show that $\mathrm{Ext}_R^1(A, P/JM) = 0$.

Obviously P/J and J/JM are singular, whence P/JM must be singular. Since P is a right T-module, JM is a T-submodule of P, hence P/JM is a right T-module. According to Proposition 4.1, P/JM is also singular as a T-module, from which we see that

$$\mathrm{Ext}_T^1[(A \otimes_R T)/Z(A \otimes_R T), P/JM] = 0$$

(because T is a splitting ring).

In view of Lemma 5.1, there exists an exact sequence

$$0 \to Z(A \otimes_R T) \to V \to W \to 0,$$

where V is a direct sum of copies of T/M. According to our hypotheses we have $\mathrm{Ext}_T^1(T/M, P/JM) = 0$, whence $\mathrm{Ext}_T^1(V, P/JM) = 0$. Inasmuch as T is a splitting ring, [14, Theorem 1.3] says that the injective dimension of the singular T-module P/JM is at most 1, from which we infer that $\mathrm{Ext}_T^2(W, P/JM) = 0$. In view of the exact sequence above, we obtain $\mathrm{Ext}_T^1[Z(A \otimes_R T), P/JM] = 0$, and thus $\mathrm{Ext}_T^1(A \otimes_R T, P/JM) = 0$. A final application of Lemma 5.1 now shows that $\mathrm{Ext}_R^1(A, P/JM) = 0$, and we are done.

At this point we note one special case where Theorem 5.2 may be applied, namely when all singular right T-modules are injective. (See [6, Chapter III] for an investigation of rings with this property.) In this case T is obviously a splitting ring, and the Ext condition is automatic.

As in the cases of right global dimension and global weak dimension, the question of when a tame subidealizer of a right ideal M becomes a splitting ring can be answered precisely in case M is a semimaximal right ideal. In order to accomplish this, we need a sharpened version of Lemma 5.1(b), which we obtain from the following two lemmas.

LEMMA 5.3. *Assume that M is a generative semimaximal right ideal of T, and let R be any tame subidealizer of M. Let S denote the idealizer of M in T, and set $H = \{x \in R \mid (S/R)x = 0\}$.*

(a) *H is a two-sided ideal of S.*

(b) *T/S and HT/H are both semisimple right R-modules.*

Proof. (a) It is clear that H is a two-sided ideal of R, and we note that $M \subseteq H$. We have $H/M = (H/M)^2$ because R/M is semisimple, which together with the idempotence of M ensures that H is idempotent. Inasmuch as $SH \subseteq R$, we see from this that $SH = H$.

Since S/M is a semisimple right (R/M)-module, we obtain a decomposition $S/M = (R/M) \oplus A$ of right (R/M)-modules. Noting that $A \cong (S/R)_R$, we see that $H/M = \{y \in R/M \mid Ay = 0\}$. Inasmuch as H/M is a left ideal of S/M, we infer from this that $(H/M)A$ is a nilpotent left ideal of S/M. Now S/M is a semisimple ring because M is semimaximal in T, hence we obtain $(H/M)A = 0$. In view of the equation $S/M = (R/M) \oplus A$, it follows that H/M is a right ideal of S/M, whence $HS = H$.

(b) We can write M as a finite intersection $M_1 \cap \cdots \cap M_k$ of maximal right ideals of T. Setting $C_i = \{t \in T \mid tM \leqslant M_i\}$ for each i, we note that the C_i are submodules of T_R and that $S = C_1 \cap \cdots \cap C_k$. Thus to show that $(T/S)_R$ is semisimple, it suffices to prove that each T/C_i is a simple right R-module. Given any $t \in T \backslash C_i$, we see from the maximality of M_i that $tM + M_i = T$, from which it follows that $tR + C_i = T$. Therefore $(T/C_i)_R$ is indeed simple.

Inasmuch as $HS = H$ by (a), $(HT/H)_R$ is an epimorphic image of some direct sum of copies of $(T/S)_R$. As we have just shown that $(T/S)_R$ is semisimple, it follows that $(HT/H)_R$ is semisimple also.

LEMMA 5.4. *Assume that M is an essential, generative, semimaximal right ideal in a right nonsingular ring T, and let R be any tame subidealizer of M. Let S denote the idealizer of M in T, and set* $H = \{x \in R \mid (S/R)x = 0\}$. *If A is any nonsingular right R-module, then* $Z(A \otimes_R T)$ *can be embedded in a direct sum of copies of* T/HT.

Proof. Noting that $(R/H)M = 0$, we see from Lemma 2.4 that $\operatorname{Tor}_1^R(R/H, T) = 0$, hence the map $H \otimes_R T \to HT$ is an isomorphism. Since this map is the composition of the map $H \otimes_R T \to HT \otimes_R T$ with the isomorphism $HT \otimes_R T \to HT$, we find that the map $H \otimes_R T \to HT \otimes_R T$ is an isomorphism. Recalling from Proposition 1.2 that $\operatorname{Tor}_1^R(HT, T) \cong \operatorname{Tor}_1^T(HT, T) = 0$, we infer from this that $\operatorname{Tor}_1^R(HT/H, T) = 0$.

Inasmuch as A is nonsingular, it follows from [6, Proposition 1.18] that we may assume A to be a submodule of some direct sum P of copies of S^0R, and we use Proposition 4.1 to see that P is also a right T-module. Now AHT/AH is an epimorphic image of some direct sum of copies of HT/H, and $(HT/H)_R$ is semisimple by Lemma 5.3, hence we infer that AHT/AH must be isomorphic to a direct summand of a direct sum of copies of HT/H. Since $\operatorname{Tor}_1^R(HT/H, T) = 0$, it follows that $\operatorname{Tor}_1^R(AHT/AH, T) = 0$, and so the map $AH \otimes_R T \to AHT \otimes_R T$ is injective. Of course $AHT \otimes_R T \cong AHT$, which is a nonsingular right R-module because P_R is nonsingular, hence we infer that $AH \otimes_R T$ is nonsingular.

Observing that $(A/AH)M = 0$, we obtain $\mathrm{Tor}_1^R(A/AH, T) = 0$ from Lemma 2.4, hence we get an exact sequence

$$0 \to AH \otimes_R T \to A \otimes_R T \to (A/AH) \otimes_R T \to 0.$$

Inasmuch as $Z(AH \otimes_R T) = 0$, the map

$$Z(A \otimes_R T) \to A \otimes_R T \to (A/AH) \otimes_R T$$

must be injective. In view of the fact that A/AH is a projective right (R/H)-module, we see that $(A/AH) \otimes_R T$ may be embedded in a direct sum of copies of T/HT, and the required embedding of $Z(A \otimes_R T)$ follows.

THEOREM 5.5. *Assume that M is an essential, generative, semimaximal right ideal in a right nonsingular ring T, and let R be any tame subidealizer of M. Let S denote the idealizer of M in T, and set $H = \{x \in R \mid (S/R)x = 0\}$. Then R is a splitting ring if and only if T is a splitting ring and* $\mathrm{Ext}_T^1(T/HT, C) = 0$ *for all singular C_T.*

Proof. First assume that T is a splitting ring and that $\mathrm{Ext}_T^1(T/HT, C) = 0$ for all singular C_T. Given any nonsingular A_R and any singular C_R, proceed as in Theorem 5.2 to the point where

$$\mathrm{Ext}_T^1[(A \otimes_R T)/Z(A \otimes_R T), P/JM] = 0.$$

Invoking Lemma 5.4, we obtain an exact sequence

$$0 \to Z(A \otimes_R T) \to V \to W \to 0,$$

where V is a direct sum of copies of T/HT. In view of our hypothesis on T/HT, the remainder of the argument proceeds as in Theorem 5.2, and we find that R is indeed a splitting ring.

Conversely, assume that R is a splitting ring. Given any nonsingular right T-module A and any singular right T-module C, we see from Proposition 4.1 that A_R is nonsingular and C_R is singular, whence $\mathrm{Ext}_R^1(A, C) = 0$. Then $\mathrm{Ext}_T^1(A, C) = 0$ by Proposition 1.2, hence T is a splitting ring.

According to Lemma 2.4, $\mathrm{Tor}_1^R(S/R, T) = 0$, hence we obtain an exact sequence

$$0 \to R \otimes_R T \to S \otimes_R T \to (S/R) \otimes_R T \to 0.$$

The composition

$$R \otimes_R T \to S \otimes_R T \xrightarrow{f} T \to R \otimes_R T$$

(where f is just multiplication) is clearly the identity on $R \otimes_R T$, hence the exact sequence above splits and we find that $[(S/R) \otimes_R T]_T$ is isomorphic to a direct summand of $(S \otimes_R T)_T$.

If C is any singular right T-module, then we have $\mathrm{Ext}_R{}^1(S, C) = 0$ because R is a splitting ring. In view of Lemma 5.1, it follows that $\mathrm{Ext}_T{}^1(S \otimes_R T, C) = 0$ and hence that $\mathrm{Ext}_T{}^1[(S/R) \otimes_R T, C] = 0$. Inasmuch as S/R is a faithful right (R/H)-module, we see that $(R/H)_R$ must be isomorphic to a direct summand of a direct sum of copies of $(S/R)_R$. Thus T/HT is isomorphic to a direct summand of a direct sum of copies of $(S/R) \otimes_R T$, and so $\mathrm{Ext}_T{}^1(T/HT, C) = 0$.

We conclude by illustrating the usefulness of the preceding results in constructing examples. According to [14, Theorem 2.2], the right global dimension of a right splitting ring is at most 2, and examples have been constructed in [5] and [6, Example 5.11] of splitting rings with global dimension 2. However, both of these examples have nonzero socle, whereas the present example will be a right and left Ore domain which is a left splitting ring as well as a right splitting ring. Also, both global dimensions of this ring are 2, as well as the global weak dimension.

To begin with, choose any field F of characteristic 0 which is infinite-dimensional over the rationals (which we shall denote by Q). We consider F to be a differential field with the zero derivation, and according to [8, Theorem, p. 771] F may be extended to a universal differential field K. Next let T denote the ring of differential polynomials over K as constructed in [4, Theorem 1.4]: T is a principal right and left ideal domain, T is a simple ring but not a division ring, and all simple right T-modules are injective. It is easy to see that the same arguments, with appropriate changes of sign, show that all simple left T-modules are injective. As indicated in [6, pp. 54, 55], all singular right T-modules are injective, and likewise all singular left T-modules are injective.

Choosing any maximal right ideal M of T, we use the fact that T is not a division ring to see that M is an essential right ideal of T. In particular, $M \neq 0$, hence the simplicity of T ensures that M is also generative. Since Q is clearly contained in the center of T, we see that the ring $R = Q + M$ is a tame subidealizer of M. According to Theorem 5.2, R must be a right splitting ring.

The ring T is obviously a semiprime right and left Goldie ring with all three global dimensions equal to 1. Since M must be a principal right ideal of T, all the hypotheses of Theorem 4.3 are satisfied, and we see in particular that R is a right and left Goldie ring. Of course R is a domain because T is, and so R must be a right and left Ore domain. Inasmuch as the derivation on F is zero, it follows from the construction of T that F is contained in the center of T and thus in the idealizer of M. Therefore R is not equal to the idealizer of M, hence Theorem 4.3 says that $\text{r.gl.dim.}(R) = \text{l.gl.dim.}(R) = GWD(R) = 2$.

If we let V be as in Proposition 4.2, then it follows from the Morita-equivalence that all singular left V-modules are injective. As proved in Theorem 4.3, V is a left nonsingular ring and M is a finitely generated, projective, essential, generative, semimaximal left ideal of V, hence it follows from Theorem 5.2 that R is also a left splitting ring.

Finally, letting S denote the idealizer of M in T, we see that $F + M$ is a subring of S, whence $(F + M)/M$ is a subring of S/M. Inasmuch as F is infinite-dimensional over Q, we see that S/M must be infinite-dimensional over R/M, from which we conclude that neither $(S/R)_R$ nor ${}_R(S/R)$ can be finitely generated. According to Theorem 4.3, R is neither left nor right noetherian.

To recapitulate, we have constructed a right and left Ore domain R which is neither right nor left noetherian, such that R is a right and left splitting ring, while the right and left global dimensions of R and its global weak dimension are all equal to 2.

References

1. H. Cartan and S. Eilenberg, "Homological Algebra," Princeton University Press, Princeton, New Jersey, 1956.
2. V. C. Cateforis and F. L. Sandomierski, The singular submodule splits off, *J. Algebra* **10** (1968), 149–165.
3. S. U. Chase, Direct products of modules, *Trans. Amer. Math. Soc.* **97** (1960), 457–473.
4. J. H. Cozzens, Homological properties of the ring of differential polynomials, *Bull. Amer. Math. Soc.* **76** (1970), 75–79.
5. J. Fuelberth and M. L. Teply, A splitting ring of global dimension two, *Proc. Amer. Math. Soc.* **35** (1972), 317–324.
6. K. R. Goodearl, Singular torsion and the splitting properties, *Mem. Amer. Math. Soc.* No. 124 (1972).
7. K. R. Goodearl, Idealizers and nonsingular rings, *Pacific J. Math.* **48** (1973), 395–402.
8. E. R. Kolchin, Galois theory of differential fields, *Amer. J. Math.* **75** (1953), 753–824.
9. J. C. Robson, Idealizers and hereditary noetherian prime rings, *J. Algebra* **22** (1972), 45–81.
10. J. C. Robson, Idealizer rings, *in* "Ring Theory" (R. Gordon, Ed.), pp. 309–317, Academic Press, New York, 1972.
11. F. L. Sandomierski, Semisimple maximal quotient rings, *Trans. Amer. Math. Soc.* **128** (1967), 112–120.
12. F. L. Sandomierski, Nonsingular rings, *Proc. Amer. Math. Soc.* **19** (1968), 225–230.
13. L. Silver, Noncommutative localizations and applications, *J. Algebra* **7** (1967), 44–76.
14. M. L. Teply, Homological dimension and splitting torsion theories, *Pacific J. Math.* **34** (1970), 193–205.