On computational environments of topological spaces

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ABSTRACT

As has been disclosed by K. Martin, a large number of important topological spaces do not have any continuous domain as their computational model. So it is of interest to study new kinds of pragmatic computational environments so as to model more topological spaces. In this paper we focus on bounded complete continuous posets with enough maximal points, which are shown to be a good choice for computational environments of Tychonoff spaces with no directed complete model. It is proved that the maximal point space of a Choquet complete weak domain is also Choquet complete. Furthermore, it is proved that X is a Tychonoff space iff X has a bounded complete weak domain. And it is also shown that Hausdorff compactifications of Tychonoff spaces can be realized via some of their computational environments.

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1. Introduction

The study of domain theoretic models of topological spaces has remained an area of interest and active research for at least twenty years, and the earliest work may date back to Scott [25], Kamimura and Tang [12, 13], and some others. Herein, a continuous domain D is called a model of a topological space X, if \(X \cong \max(D)\), where \(\max(D)\) is the maximal point space of D with the relative Scott topology.

One of the striking applications for domain theoretic models is to provide computational environments for topological spaces. If a topological space X has a continuous domain D as its model, then the order structure on D can effectively describe the computational way of its maximal point space and hence that of X. And one can also find that, through a series of important papers by Edalat [5–8], domain theoretic models are capable of providing settings in which many classical mathematical branches, such as the theory of dynamical systems, measure theory and fractal theory, may be developed.

In the study of domain theoretic models, a fundamental problem is: which kinds of topological spaces have models? There has been much work [2–8, 10–24] about this problem, among which the most recent and significant works have been done by Martin [19–24] and Bennett [2, 3]. In [22], Martin proved that any topological space having models must be Choquet complete. From this theorem, it follows that many important topological spaces one is familiar with, including all metric spaces with no complete metric, have no model and thus cannot be computed within continuous directed complete environments.

In order to provide domain theoretic computational environments for more topological spaces, Liang and Keimel [16] proposed the notion of order environments of topological spaces, which are just a counterpart of domain environments in the non-dcpo setting. An order environment of a topological space X is a continuous poset P satisfying \((\max(P), \sigma(P)|_{\max(P)} \cong X)\) and \(\sigma(P)|_{\max(P)} = \lambda(P)|_{\max(P)}\). It is proved in [16] that every Tychonoff topological space has order environments.

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† Not long after this paper had been submitted, Professor Jihua Liang, the second author, passed away on March 21, 2007. Professor Liang had greatly promoted Domain Theory in China. Her early disappearance is a loss for the whole Domain Theory community.

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Let \( P \) be a continuous poset.

Let \( D \) be a continuous poset. Then \( \text{Idl} \) is a continuous poset. An abstract basis is a set \( I \in D \) such that \( \forall x \in D \), \( x \ll y \Rightarrow \exists z \in P \) such that \( x \neq z \) and \( x \ll z \ll y \). (1) \( \{ \ll x : x \in D \} \) is a base for the Scott topology \( \sigma(D) \), and \( \{ \ll x \setminus F : x \in D, F \subseteq D \text{ finite}\} \) is a base for the Lawson topology \( \lambda(D) \).

2. Order and topology

Let \( (D, \leq) \) be a partially ordered set (poset, for short). A nonempty subset \( A \subseteq D \) is directed if for any nonempty finite subset \( F \subseteq A \), there is an upper bound \( x \in A \setminus F \). The supremum of \( A \), when existing, is denoted by \( \vee A \). \( D \) is bounded complete if any subset of \( D \) with a upper bound has a supremum, or equivalently, any nonempty subset of \( D \) has an infimum. For \( A \subseteq D \), let \( \uparrow A = \{ y \in D : \exists x \in A \text{ s.t. } x \leq y \} \), and \( \downarrow A = \{ y \in D : \exists x \in A \text{ s.t. } y \leq x \} \). \( A \) is an upper subset of \( D \) if \( A = \uparrow A \), and a lower subset if \( A = \downarrow A \). An ideal of \( D \) is a directed lower subset. An element \( x \in D \) is maximal if \( \uparrow x = \{ x \} \). \( D \) is the set of all maximal elements in \( D \). \( D \) is directed complete if any directed subset of it has a supremum. For \( x, y \in D \), \( x \ll y \) if for any directed subset \( A \subseteq D \) with \( y \leq \vee A \), there exists \( d \in A \) such that \( x \leq d, x \in D \) is compact if \( x \ll y \). \( K(D) = \{ x \in D : x \ll x \} \) is the set of all compact elements in \( D \). For any \( x \in D \), \( \downarrow x = \{ y \in D : y \ll x \} \) and \( \uparrow x = \{ y \in D : x \ll y \} \).

A poset \( D \) is continuous if for any \( x \in D \), \( \downarrow x \) is directed and \( x = \uparrow x \). \( B \subseteq D \) is called a base of \( D \) if for any \( x \in D \), \( \uparrow x \cap B = x \). \( D \) is called \( \omega \)-continuous if \( D \) has a countable base. It is clear that a poset is continuous iff it has a base. \( D \) is an algebraic poset if \( K(D) \) is a base of \( D \). A Scott domain is a continuous bounded complete and directed complete poset, and a continuous bounded complete poset is a poset which is both continuous and bounded complete.

A subset \( U \) in a poset \( D \) is Scott open if it is an upper subset and \( \forall x \in D \), \( x \ll y \Rightarrow \exists z \in P \) such that \( x \neq z \) and \( x \ll z \ll y \). The topology \( \sigma(D) \) on \( D \) is the topology just taking \( \{ \ll x : x \in D \} \) as a subbase. The Lawson topology \( \lambda(D) \) on \( D \) is the topology just taking \( \{ \ll x : x \in D \} \) as a subbase. \( D \) is Lawson compact if \( \sigma(D) \) is a base of \( D \). A Scott domain is a continuous bounded complete and directed complete poset, and a continuous bounded complete poset is a poset which is both continuous and bounded complete.

### Theorem 2.1 ([11])

Let \( P \) be a continuous poset.

1. The approximation relation \( x \ll y \) on \( P \) has the interpolation property: \( x \ll y \) and \( x \neq y \Rightarrow \exists z \in P \) such that \( x \neq z \) and \( x \ll z \ll y \).
2. \( \{ \ll x : x \in D \} \) is a base for the Scott topology \( \sigma(D) \), and \( \{ \ll x \setminus F : x \in D, F \subseteq D \text{ finite}\} \) is a base for the Lawson topology \( \lambda(D) \).
3. \( D \) with the Lawson topology is Hausdorff.

#### Definition 2.2 ([11])

An abstract basis is a set \( B \) with a transitive order \( \prec \) such that \( M \prec x \Rightarrow \exists y \in B, M \prec y \prec x \) for any \( x \in B \) and any finite \( M \subseteq B \).

It is well known that for a continuous poset \( D \), \( (D, \ll) \) is an abstract basis.

A subset \( I \) of an abstract basis \( (B, \prec) \) is called an ideal if \( I \) is a directed and lower subset of \( B \) with respect to \( \prec \). For \( x \in B \), let \( I_x = \{ y \in B : y \prec x \} \). For an abstract basis \( (B, \prec) \), let \( \text{Idl}(B, \prec) \) be the poset of all ideals ordered by set-theoretical inclusion. \( \text{Idl}(B, \prec) \) is called the ideal completion of \( (B, \prec) \). From [11], we have

#### Proposition 2.23

\( (B, \prec) \) is an abstract basis.

1. For any \( I \in \text{Idl}(B, \prec) \), \( x \in I \Rightarrow \exists x_1 \in I \) such that \( x \prec x_1 \).
2. \( \forall x \in B, I_x \) is an ideal.
3. For any two ideals \( I_1, I_2, I_1 \ll I_2 \) iff there are \( x_1, x_2 \in B \) such that \( x_1 \prec x_2 \) and \( I_1 \subseteq \downarrow \prec x_1 \subseteq \downarrow \prec x_2 \subseteq I_2 \).

#### Proposition 2.24

Let \( D \) be a continuous poset. \( \text{Idl}(D, \ll) \) is a continuous dcpo, and for any \( x, y \in D, x \ll y \) iff \( I_x \ll I_y \). Furthermore, for \( I \in \text{Idl}(D, \ll) \), \( \{ I_x : x \in I \} \) is a Scott open neighborhood system of \( I \) in \( \text{Idl}(D, \ll) \).

#### Proposition 2.25

Let \( D \) be a continuous bounded complete poset. \( \text{Idl}(D, \ll) \) is a Scott domain.

#### Definition 2.6 ([20])

Let \( X \) be a topological space. \( X \) is said to have a model if there is a continuous domain \( D \) such that \( X \cong \text{max}(D) \), where \( \text{max}(D) \) is the maximal point spaces of \( D \) with the relative Scott topology inherited from \( (D, \sigma(D)) \).

#### Definition 2.7 ([9])

A topological space \( X \) has a Hausdorff compactification (compactification, for short), if there is a compact Hausdorff space \( Y \) such that \( X \) is homeomorphic to some dense subspace of \( Y \).

#### Definition 2.8 ([9])

A topological space \( X \) is a Tychonoff space if it has a Hausdorff compactification.
For a topological space $Y$, the upper space $UY$ of $Y$ is the set of all nonempty compact subsets of $Y$ ordered by reverse inclusion. For a subset $A \subseteq Y$, $A^\circ$ and $\bar{A}$ are the interior and closure of $A$, respectively. $O(Y)$ is the collection of open subsets of $Y$.

**Theorem 2.9 ([14]).** Let $Y$ be a compact Hausdorff topological space and $UY$ the upper space.

1. $UY$ is a continuous domain, and for $\forall \kappa_1, \kappa_2 \in UY$, $\kappa_1 \ll \kappa_2$ iff $\kappa_2 \subseteq \kappa_1^\circ$.
2. $UY$ is a Scott domain. So $UY$ is Lawson compact.
3. $UY$ is a model of $Y$. The mapping $\epsilon_Y : Y \rightarrow UY, y \mapsto \{y\}$ is just a homeomorphic embedding onto $\max(UY) = \{\{y\} : y \in Y\}$.

**3. On Choquet completeness of continuous posets**

For the sake of convenience, a continuous poset $D$ will be called a *weak domain* if for all $x \in D$, $\uparrow x \cap \max(D) \neq \emptyset$, or alternatively, $D = \downarrow \max(D)$.

If we view the order in $D$ as an information order, maximal elements as ideal information and non-maximal elements as partial information, then $\uparrow x \cap \max(D) \neq \emptyset$ for any $x \in D$ just requires that every element in $D$ must partially describe some ideal information.

The Lawson condition is an important trick when dealing with model problems. In [17], Liang and Kou introduced the hull property, which is closely related to the Lawson condition.

**Definition 3.1 ([17]).** A poset $D$ is said to satisfy the hull property if for any $x \in D$, $m \in \max(D), x \not\ll m \Rightarrow \uparrow x_0 \cap \uparrow m_0 = \emptyset$ for some $x_0 \ll x, m_0 \ll m$.

**Proposition 3.2 ([17]).** (1) A continuous dcpo $D$ satisfies the Lawson condition iff $D$ satisfies the hull property.
(2) A continuous poset $D$ satisfies the Lawson condition iff $D$ satisfies the hull property.

Note that for continuous posets, the Lawson condition does not necessarily imply the hull property.

**Example 3.3.** Let $D$ be a poset consisting of two monotone increasing sequences $\{a_i : i = 1, 2, \ldots, n, \ldots, \infty\} \cup \{b_i : i = 1, 2, \ldots, n, \ldots\}$ with $a_i \leq b_i$ for $i = 1, 2, \ldots, n, \ldots$. Then $D$ does not satisfy the hull property, but the Lawson condition.

The equivalence between the Lawson condition and the hull property also holds in weak domains.

**Proposition 3.4.** A weak domain $D$ satisfies the Lawson condition if it satisfies the hull property.

**Proof.** Only the necessity needs to be proved. Let $m \in \max(D)$ with $x \not\ll m$. Then there is $x_0 \ll x$ with $x_0 \not\ll m$, or alternatively, $m \in (D \setminus \uparrow x_0) \cap \max(D)$. Since $D$ is continuous and satisfies the Lawson condition, there exists $m_1 \in D$ such that $m \in \uparrow m_1 \cap \max(D) \subset (D \setminus \uparrow x_0) \cap \max(D)$. Now fix an element $m_0 \in D$ with $m_1 \ll m_0 \ll m$. Then $\uparrow m_0 \cap \max(D) \subset (D \setminus \uparrow x_0) \cap \max(D)$, which implies $\uparrow m_0 \cap \uparrow x_0 \cap \max(D) = \emptyset$. Note that in $D$, $\downarrow \max(D) = D$ and hence it follows $\uparrow m_0 \cap \uparrow x_0 = \emptyset$. □

**Corollary 3.5.** Lawson compact continuous domains always have the hull property.

Choquet completeness is considered as an important technique in the model problem, for it naturally contains the two essential notions of approximation and completeness in domain theory (cf. [20]). In the following, we consider Choquet completeness of weak domains.

**Definition 3.6 ([23]).** Let $(X, \tau)$ be a topological space and $\tau_x = \{(U, x) : x \in U \in \tau\}$. Then $X$ is said to be Choquet complete if there is a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of functions

$$\alpha_n : \tau_x^n \rightarrow \tau$$

such that

(i) For each $(U_1, x_1), \ldots, (U_n, x_n) \in \text{dom}(\alpha_n), x_n \in \alpha_n((U_1, x_1), \ldots, (U_n, x_n)) \subseteq U_n$, and
(ii) For any sequence $\{V_n, x_n\}_{n=1}^{\infty}$ in $\tau_x$ with $V_{n+1} \subseteq \alpha_n((V_1, x_1), \ldots, (V_n, x_n))$, for all $n \geq 1$, we have $\cap_{n \geq 1} V_n \neq \emptyset$.

Completely metrizable spaces, Cech-complete spaces and locally compact Hausdorff spaces are all Choquet complete. Furthermore, in [20], Martin proved that locally compact sober spaces and hence continuous domains with Scott topology are Choquet complete. The continuous poset $D$ in Example 3.3 above is, however, not Choquet complete.

**Proposition 3.7.** A continuous poset $D$ is Choquet complete iff for every sequence $x_1 \ll x_2 \ll \cdots \ll x_n \ll \cdots$ in $D$, $\cap_{n \geq 1} \uparrow x_n = \emptyset$.

**Proof.** $\Leftarrow$: See the proof of Theorem 4.3 in [23].
$\Rightarrow$: Now let $\{\alpha_n : \tau_x^n \rightarrow \tau\}_{n=1}^{\infty}$ be a sequence of functions satisfying (i) and (ii) in Definition 3.6. Then for a given way-below sequence $x_1 \ll x_2 \ll \cdots \ll x_n \ll \cdots$, consider the sequence $\{V_n, y_n\}_{n=1}^{\infty}$, where $V_n = \uparrow x_n$ and $y_n = x_{n+1}$ for every $n = 1, 2, \ldots$. It can be readily checked that for this sequence $\{V_n, y_n\}_{n=1}^{\infty}$ and any $n \geq 1, V_{n+1} \subseteq \alpha_n((V_1, y_1), \ldots, (V_n, y_n))$. Then we have $\cap_{n \geq 1} V_n \neq \emptyset$. Hence $\cap_{n \geq 1} \uparrow x_n = \cap_{n \geq 1} \uparrow x_n \cap \cap_{n \geq 1} V_n \neq \emptyset$. □
A poset is said to be countable chain complete if every countable chain has a supremum. Then we have

**Corollary 3.8.** Every countable chain complete continuous poset is Choquet complete.

From Proposition 3.7 and a similar procedure of proof for Theorem 5.1 in [22], we immediately have

**Theorem 3.9.** If $D$ is a Choquet complete weak domain, then $(\text{max}(D), \sigma(D)|_{\text{max}(D)})$ is Choquet complete.

**Example 3.10.** The reverse of the preceding theorem does not hold. In fact, let $X$ be the set of natural numbers with the discrete topology and $Y = X \cup \{\infty\}$ be the one-point compactification of $X$. Let $D_X = \{x \in U : x \in X \neq \emptyset\}$ and $\kappa_n = \{k \in X : k \geq n\} \cup \{\infty\}$ for $n = 1, 2, \ldots$. As is discussed later, $D_X$ with the reverse containment order is a weak domain and $\text{max}(D_X) = X$. Thus $\text{max}(D_X)$ is Choquet complete. Meanwhile, note that $(\kappa_n)_{n=1}^\infty$ is a sequence of non-empty closed-and-open subsets in $Y$ such that $\kappa_n \supseteq \kappa_{n+1}$ for every $n \geq 1$ and $X \cap (\kappa_\infty \kappa_n) = \emptyset$. Hence, from Proposition 3.7 one knows $D_X$ is not Choquet complete.

**Example 3.11.** Let $D$ be the collection of all proper subsets of the reals $\mathcal{R}$ with the set-inclusion order.

1. $D$ is an algebraic poset with the base $K(D) = \{x : x \subseteq \mathcal{R} \text{ is finite}\}$, and $\text{max}(D) = \{\mathcal{R} \setminus \{a : a \in \mathcal{R}\} : \downarrow \text{max}(D) = D\}$. So $D$ is a weak domain.

2. From Proposition 3.7, one can verify that $D$ is Choquet complete and bounded complete.

3. Let $x_n = \mathcal{R} \setminus \{n, n+1, n+2, \ldots\}$ for $n = 1, 2, \ldots$. Then $\{x_n : n = 1, 2, \ldots\}$ is a monotone chain without upper bounds in $D$. So the reverse of Corollary 3.8 does not hold.

### 4. On weak domain environments and compactifications of topological spaces

**Definition 4.1.** A topological space $X$ is said to have a weak domain environment, if there is a weak domain $D$ such that $X \cong (\text{max}(D), \sigma(D)|_{\text{max}(D)})$ and $D$ has the hull property. $D$ is then said to be a weak domain environment of $X$.

The notion of weak domain environments is a little stronger than that of order environment defined in [16].

**Example 4.2 ([6]).** Let $(X, d)$ be a metric space, and let $BX$ be the formal ball model of $X$, i.e., the set of $X \times [0, \infty)$ with the order by: for $(x, r), (y, s) \in BX$,

$$(x, r) \leq (y, s) \iff d(x, y) \leq r - s.$$  

Then $(BX, \leq)$ is a weak domain environment of $X$. Furthermore, $(BX, \leq)$ is a model of $X$ if $d$ is a complete metric.

**Corollary 4.3.** Let $(X, d)$ be a metric space. If $BX$ is Choquet complete, then $X$ is completely metrizable.

**Proof.** Since $BX$ is a Choquet complete weak domain, it follows from Theorem 3.9 that $(\text{max}(BX), \sigma(BX)|_{\text{max}(BX)})$ is Choquet complete. Then $X$ is a Choquet complete metric space and hence is completely metrizable.

Let $X$ be a Tychonoff space and $Y$ a compactification of $X$. For the sake of simplicity, we just view $X$ as a dense subspace of $Y$. Let $D_X = \{\kappa \in U : \kappa \cap X \neq \emptyset\}$ with the reverse inclusion order.

**Lemma 4.4.** $(D_X, \leq)$ is a bounded complete weak domain environment of $X$.

**Proof.** (1) Note that $D_X$ is a lower subset of $U$ and $U$ is a Scott domain, so $D_X$ is a bounded complete continuous poset. It is clear that $\text{max}(D_X) = \{\{x\} : x \in X\}$, and so $D_X$ is a weak domain. Moreover, in $D_X$, it still holds that $\kappa_1 \ll \kappa_2$ iff $\kappa_2 \subseteq \kappa_1^\uparrow$ for any $\kappa_1, \kappa_2 \in D_X$.

2. Let $e_X$ be such a mapping as

$$e_X : X \rightarrow D_X, \quad x \mapsto \{x\}.$$  

Then $e_X$ is an injection with $e_X(X) = \text{max}(D_X)$.

3. $e_X$ is an embedding with respect to $(D_X, \sigma(D_X))$. Firstly, $e_X$ is continuous. In fact, $\forall \kappa \in D_X$ and $x \in e_X^{-1}(\kappa)$ imply that $\kappa \ll \{x\}$. Then $x \in \kappa^\uparrow$ and $\kappa^\uparrow \cap X \subseteq e_X^{-1}(\kappa)$. Secondly, $e_X|_{\text{max}(D_X)} : X \rightarrow \text{max}(D_X)$ is an open mapping. In fact, fix an open subset $U \in \Theta(X)$ and let $V \in \Theta(Y)$ with $U = V \cap X$. Then for $\forall \{x\} \in e_X(U), x \in V$. Since $Y$ is compact and Hausdorff, there exists $W \in \Theta(Y)$ such that $x \in W \subseteq \overline{W} \subseteq V$. Note that $\overline{W} \in D_X$ and $\{x\} \in (\uparrow W) \cap \text{max}(D_X) \subseteq e_X(U)$. Thus $e_X|_{\text{max}(D_X)}$ is an open mapping.

4. $D_X$ has the hull property. Fix $x \in X$ and $\kappa \in D_X$ with $\kappa \not\subseteq \{x\}$. Then $x$ is not in $\kappa$. By the regularity separation of $Y$, there exist $U, V \in \Theta(Y)$ such that $x \in U, \kappa \subseteq V$ and $\overline{U} \cap \overline{V} = \emptyset$. Then $\{x\} \subseteq \uparrow U, \kappa \subseteq \uparrow V, \uparrow U \cap \uparrow V = \emptyset$ and $\overline{U} \cap \overline{V} = D_X$. Note that $D_X$ in the preceding construction is generally not directed complete. In fact, it can be readily verified that $D_X$ is directed complete iff $X$ is compact.

**Lemma 4.5.** Let $D$ be a bounded complete weak domain with the hull property and $E = \text{Id}(D, \ll)$ be the ideal completion of $(D, \ll)$. Then

1. The mapping

$$e : (D, \sigma(D)) \rightarrow (E, \sigma(E)), \quad x \mapsto \downarrow x$$

is an embedding.

2. For $\forall x \in \text{max}(D), \downarrow x \in \text{max}(E)$. Hence $e(\text{max}(D)) \subseteq \text{max}(E)$ and $\text{max}(D)$ is a Tychonoff space with respect to the relative Scott (Lawson) topology.
Lemma 4.11. A topological space X is Choquet complete if for every non-empty Tychonoff space Y with a bounded complete algebraic weak domain environment.

Proof. Let X be a second countable Tychonoff space and Y one of its Hausdorff compactifications. Then X is Choquet complete if there exist x ∈ max(D) and e(x, max(D)) ≤ c_λ(E)({|x : x ∈ max(D)|}). Since D is a bounded complete continuous poset, E is a Scott domain and hence has the hull property. Now for all v ∈ max(E) and v ∈ λ(E) with l ∈ V, there exists l ∈ E such that l = l ∩ max(E) ≤ v ∩ max(E). Since l < l, there is some x ∈ I with l < x < l. Fix an x ∈ max(D) ∩ x. Then x < x ∈ max(D) ∩ x. Hence, l ∈ c_λ(E)({|x : x ∈ max(D)|}). □

Note that {↓↓ : x ∈ max(D)} is generally a proper subset of max(Id(D, <)) for a bounded complete weak domain D with the hull property. In fact, we have

Proposition 4.12. For a bounded complete weak domain D,

1. Id(D, <) = {↓↓ : x ∈ D} if D is directed complete;
2. max(Id(D, <)) = {↓↓ : x ∈ max(D)} if D is directed complete.

Proof. (1) If D is directed complete, it is clear that Id(D, <) = {↓↓ : x ∈ D}. Now suppose Id(D, <) = {↓↓ : x ∈ D}. Then l = l ∩ max(E) holds. For an ideal I with respect to ≤ in D, let l = ∪_x∈l x. Then I and l has the same least upper bound if it exists. Since l ∈ Id(D, <), there is a a ∈ D with a = l. Then a is an upper bound of l in I. In that D is bounded complete, l_1 and hence l has the least upper bound in D.

(2) If D is directed complete, it is also clear that max(Id(D, <)) = {↓↓ : x ∈ max(D)}, and so we prove the necessity. For an ideal I with respect to ≤ in D, let I = ∪_x∈I x. Then I and l has the same least upper bound if it exists. Since l ∈ Id(D, <) and Id(D, <) is a continuous domain with max(Id(D, <)) = {↓↓ : x ∈ max(D)}, there is m ∈ max(D) such that l ⊆ m. Then m is an upper bound of l in I. In that D is bounded complete, l_1 and hence l has the least upper bound in D. □
Theorem 4.13. Let $X$ be a Tychonoff space and $Y$ a Hausdorff compactification of $X$. Then $\max(Iddl(D_X, \ll)) = cl_{\lambda(Iddl(D_X, \ll))}(\{ \uparrow x : x \in \max(D_X) \})$ is a Hausdorff compactification of $X$.

Proof. For convenience, we denote $Iddl(D_X, \ll)$ and $Iddl(UY, \ll)$ by $E_X$ and $E_Y$, respectively. Note that $\max(E_X) = \max(E_Y)$. In fact, it is clear that $\max(E_X) \subseteq \max(E_Y)$. Meanwhile, for $\forall l \in E_Y$ and $\forall x \in l$, there is $k \in I$ with $k \ll k_1$. Then $k_1 \subseteq k^o$ and so $k \cap X \supset k^o \cap X \neq \emptyset$. It follows that $k \in D_X$ and $l \in E_X$. Thus $\max(E_Y) \subseteq \max(E_X)$.

Since $UY$ is a Scott domain, $E_Y$ has the hull property and is isomorphic to $UY$. Then $\max(E_Y) \cong \max(UY) \cong Y$ is Hausdorff compact. Hence $\max(E_Y)$ is closed in $(E_Y, \lambda(E_Y))$. From Lemma 4.5 it follows $cl_{\lambda(E_Y)}(\{ \uparrow x : x \in \max(D_Y) \}) \subseteq \max(E_Y)$. Thus $\max(E_X) = cl_{\lambda(E_Y)}(\{ \uparrow x : x \in \max(D_X) \})$ is compact. Since $X \cong \{ \uparrow x : x \in \max(D_X) \}$, $\max(E_X)$ is a Hausdorff compactification of $X$. □

Lemma 4.14. Let $(D, \ll)$ be a weak domain with the hull property and for any ideal $l \in Iddl(D, \ll)$, $l$ will be maximal in $Iddl(D, \ll)$ if there is a net $(m_s : s \in S) \subseteq \max(D)$ such that for any $x \in I$ and $y \in D$, $\downarrow y \setminus \emptyset$ implies $\{ m_s : s \in S \}$ to be eventually in $\uparrow x \setminus \uparrow y$. Then $\max(E) = cl_{\lambda(Iddl(D, \ll))}(\{ \uparrow x : x \in \max(D) \})$ and is compact.

Proof. Let $(m_s : s \in S, m_t \in \max(D))$ be a net in $\{ \uparrow x : x \in \max(D) \}$ converging to $l$ with respect to $\lambda(Iddl(D, \ll))$. Then for any $x \in I$ and $y \in D$ with $\downarrow y \setminus \emptyset$, $\ll I$ and $\downarrow y \notin I$ in $Iddl(D, \ll)$. Thus $\uparrow (\uparrow x) \setminus \uparrow (\uparrow y)$ is an open neighborhood of $l$ in $\lambda(Iddl(D, \ll))$. Then there is an $s_0 \in S$ such that $m_{s_0} \in \uparrow (\uparrow x) \setminus \uparrow (\uparrow y)$ whenever $s \geq s_0$. Note that $\downarrow m_s \in \uparrow (\uparrow x) \setminus \uparrow (\uparrow y)$ implies $x \ll m_s$ and $\downarrow m_s \notin \uparrow (\uparrow y)$ implies $m_s \neq y$. Thus $m_s \in \uparrow x \setminus \uparrow y$ for $s \geq s_0$. So $l \in \max(Iddl(D, \ll))$ and then $cl_{\lambda(Iddl(D, \ll))}(\{ \uparrow x : x \in \max(D) \}) \subseteq \max(Iddl(D, \ll))$. With Lemma 4.11, $cl_{\lambda(Iddl(D, \ll))}(\{ \uparrow x : x \in \max(D) \}) = \max(Iddl(D, \ll))$.

Since $D$ is a weak domain with the hull property, $Iddl(D, \ll)$ is a Scott domain and so is compact with respect to $\lambda(Iddl(D, \ll))$. Thus $\max(Iddl(D, \ll))$ is compact. □

Theorem 4.15. If $X$ has a weak domain environment $D$, then an ideal $l \in Iddl(D, \ll)$ will be maximal in $Iddl(D, \ll)$ if there is a net $(m_s : s \in S) \subseteq \max(D)$ such that for any $x \in I$ and $y \in D$, $\downarrow y \setminus \emptyset$ implies $\{ m_s : s \in S \}$ to be eventually in $\uparrow x \setminus \uparrow y$. Then $\max(Iddl(D, \ll))$ is a Hausdorff compactification of $X$.

Proof. By Lemma 4.14, $\max(Iddl(D, \ll))$ is compact. $X \cong (\max(D), \sigma(D)|_{\max(D)})$ and so $\max(Iddl(D, \ll))$ is a Hausdorff compactification of $X$. □

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References