Minimum cycle factors in quasi-transitive digraphs

Jørgen Bang-Jensen\textsuperscript{a}, Morten Hegner Nielsen\textsuperscript{b,*}

\textsuperscript{a} Department of Mathematics and Computer Science, University of Southern Denmark, DK-5230 Odense, Denmark
\textsuperscript{b} Department of Mathematics and Statistics, University of Winnipeg, Winnipeg, MB, Canada

Received 8 June 2006; received in revised form 26 November 2007; accepted 4 December 2007
Available online 9 January 2008

Abstract

We consider the minimum cycle factor problem: given a digraph \(D\), find the minimum number \(k_{\min}(D)\) of vertex disjoint cycles covering all vertices of \(D\) or verify that \(D\) has no cycle factor. There is an analogous problem for paths, known as the minimum path factor problem. Both problems are \(\mathcal{NP}\)-hard for general digraphs as they include the Hamilton cycle and path problems, respectively.

In 1994 Gutin [G. Gutin, Polynomial algorithms for finding paths and cycles in quasi-transitive digraphs, Australas. J. Combin. 10 (1994) 231–236] proved that the minimum path factor problem is solvable in polynomial time, for the class of quasi-transitive digraphs, and so is the Hamilton cycle problem.

As the minimum cycle factor problem is analogous to the minimum path factor problem and is a generalization of the Hamilton cycle problem, it is therefore a natural question whether this problem is also polynomially solvable, for quasi-transitive digraphs.

We conjecture that the problem of deciding, for a fixed \(k\), whether a quasi-transitive digraph \(D\) has a cycle factor with at most \(k\) cycles is polynomial, and we verify this conjecture for \(k = 3\).

We introduce the notion of an irreducible cycle factor and show how to convert a given cycle factor into an irreducible one in polynomial time when the input digraph is quasi-transitive. Finally, we show that even though this process will often reduce the number of cycles considerably, it does not always yield a minimum cycle factor.

\(c\)\(\copyright\) 2007 Elsevier B.V. All rights reserved.

Keywords: Complementary cycles; Cycle factor; Quasi-transitive digraph; Network flow; Polynomial algorithm; Irreducible cycle factor

1. Introduction

A digraph is \textit{semicomplete} if it has no pair of non-adjacent vertices and is \textit{quasi-transitive} if the presence of arcs \(xy\) and \(yz\) implies an arc between \(x\) and \(z\) (if we require the arc to go from \(x\) to \(z\) then \(D\) is \textit{transitive}). Thus, quasi-transitive digraphs generalize semicomplete digraphs which, in turn, generalize tournaments. Quasi-transitive digraphs were introduced in [5] and by now a lot is known about the structure of these digraphs, see e.g. [1]. In particular they have a certain recursive structure (see Theorem 2) which enables the development of polynomial algorithms for a number of generally \(\mathcal{NP}\)-hard problems. A recent paper [3] generalizes some of the known results (and improves on some of the complexity results) for quasi-transitive digraphs and related classes. Even though the problems treated

\textsuperscript{*} Corresponding author. Tel.: +1 204 786 9492.
E-mail addresses: jbj@imada.sdu.dk (J. Bang-Jensen), m.nielsen@uwinnipeg.ca (M.H. Nielsen).

1572-5286/$ - see front matter \(c\)\(\copyright\) 2007 Elsevier B.V. All rights reserved.
in [3] have similarities to the minimum cycle factor problem, the techniques used there (and in previous papers) do not seem to suffice for solving this problem. We shall see why in the following sections.

As is well known, most problems are virtually hopeless to attack for completely general digraphs. The simple assumption on local structure defining quasi-transitivity leads to a class of digraphs having sufficient structure to make many problems solvable, but at the same time leaves some problems, e.g. the present one, quite non-trivial. Note that, for the more restricted class of tournaments, the minimum cycle factor problem is trivial.

Two cycles in a digraph $D$ are complementary, if they are disjoint and cover all vertices of $D$. The existence of complementary cycles has been studied for various classes of digraphs in the literature. In [13] it was shown that, with a single exception, every 2-strong tournament on $n \geq 6$ vertices contains complementary cycles of length 3 and $n - 3$ and this was extended to $k$ and $n - k$ for all $k \in \{3, \ldots, n - 3\}$ in [14]. In [7] it was shown that a 2-strong locally semicomplete digraph of order $n \geq 8$ contains a pair of complementary cycles unless it is the second power of an odd cycle in which case no such cycles exist. In [12] those strong but not 2-strong tournaments which do not have a pair of complementary cycles were characterized. See [1, Section 6.10] for further results on complementary cycles in digraphs.

In this paper we describe an $O(n^3)$ algorithm for deciding whether a quasi-transitive digraph of order $n$ has a cycle factor with at most three cycles. We shall use, as a subroutine, an algorithm from [4] for the complementary cycles problem in semicomplete digraphs.

2. Terminology and preliminaries

Terminology not defined below is consistent with [1].

For a digraph $D = (V, A)$, the order (size) of $D$ is the cardinality of $V(A)$. We will denote by $n$ ($m$) the order (size) of the digraph $D$ under consideration and by $V(D)$ ($A(D)$) the set of vertices (arcs) of $D$. An arc from $x$ to $y$ will be denoted by $xy$ or $x \rightarrow y$ and we say that $x$ dominates $y$. We write $R \rightarrow S (R \leftrightarrow S)$ for disjoint vertex subsets or digraphs $R, S$ if $r \rightarrow s (r \leftrightarrow s)$ and $s$ does not dominate $r$ for every choice of vertices $r \in R, s \in S$.

The underlying graph $UG(D)$ of a digraph $D$ is the undirected graph with the same vertices as $D$ and which has an edge $xy$ for each pair $x, y \in V(D)$ such that $x \rightarrow y$ or $y \rightarrow x$ (or both).

Paths and cycles will always be directed. The girth of $D$, denoted by $g(D)$, is the length of a shortest cycle in $D$. An oriented graph is a digraph without cycles of length two.

For each $x \in V(D)$, we denote by $N^+(x)$ ($N^-(x)$) the set of those vertices $y \in V(D)$ for which $x \rightarrow y$ ($y \rightarrow x$), that is, the set of out-neighbours (in-neighbours) of $x$. Two vertices $x, y$ in a digraph $D$ are similar if $N^+(x) = N^+(y)$ and $N^-(x) = N^-(y)$, that is, they have the same in- and out-neighbours. For a digraph $D = (V, A)$ and a set $X \subseteq V$, $D \langle X \rangle$ is the subdigraph induced by $X$. When we are considering a vertex $x$ on some cycle $C$ we denote by $x^-$ ($x^+$) the predecessor (successor) of $x$ on $C$. Notice that we do not use the subscript $C$ as it will always be clear from the context which cycle we are considering. For a pair of distinct vertices $x, y$ on a cycle $C$, $C[x, y]$ is the subpath of $C$ from $x$ to $y$.

A $k$-path-$q$-cycle subdigraph ($k$-path-$q$-cycle factor) of a digraph $D$ is a (spanning) collection $F$ of $k$ paths and $q$ cycles, all disjoint. When $k = 0$, $F$ is a cycle subdigraph (and a cycle factor if it is spanning) and when $q = 0$, $F$ is a path subdigraph (and a path factor or path cover if it is spanning). We denote by $pc(D)$ ($pc(D)$) the path covering number (path-cycle covering number) of $D$, that is, the minimum (minimum positive) number of paths in a path factor (path-cycle factor) of $D$. Note that $1 \leq pc(D) \leq pc(D)$ always holds. A $p$-cycle factor ($p$-path factor) in $D$ is minimum if $p = \min\{j \mid D$ has a $j$-cycle factor $\}$ ($p = pc(D)$).

**Definition 1.** For every digraph, $D$, with at least one cycle and every non-negative integer, $i$, define

$$\eta_i(D) = \min\{j \mid D$ has a $j$-path-$i$-cycle factor $\}$$

$$M_i(D) = \{j \mid D$ has a $j$-path-$i$-cycle factor $\}$$

$$\beta(D) = |V(D)| - g(D).$$

Thus $\eta_0(D) = pc(D)$ and $\eta_i(D) = 0$ if and only if $D$ has an $i$-cycle factor, so for general digraphs the computation of $\eta_i(D)$ is $NP$-hard already for $i = 0, 1$. As we shall see later, this is not so for quasi-transitive digraphs. Observe
also, that for a quasi-transitive digraph \( D \) (containing a cycle), \( \beta(D) \) can be computed in \( \mathcal{O}(n^2) \), since the girth of \( D \) is two or three.

For a digraph, \( R \), with vertex set \( V(R) = \{u_1, u_2, \ldots, u_r\} \), and digraphs, \( H_1, H_2, \ldots, H_r \), let \( D = R[H_1, H_2, \ldots, H_r] \) be the digraph with vertex set \( V(D) = V(H_1) \cup \cdots \cup V(H_r) \) in which \( xy \in A(D) \) if and only if \( x \in V(H_i), \) \( y \in V(H_j) \) and \( u_i u_j \in A(R) \), where \( i \neq j \), or \( i = j \) and \( xy \in A(H_i) \). In other words, \( D \) is obtained from \( R \) by substituting the digraph \( H_i \) for vertex \( u_i \), for each \( i = 1, 2, \ldots, r \).

We denote by \( \bar{K}_p \) the digraph on \( p \) vertices and no arcs. When we consider \( \bar{K}_p \) as a subdigraph of another digraph \( D \), we also call it an independent subset of size \( p \) in \( D \). A digraph \( D \) is an extended semicomplete digraph if there is a semicomplete digraph \( R \) such that \( D = R[\bar{K}_{n_1}, \ldots, \bar{K}_{n_r}] \), for some choice of integers, \( n_1, \ldots, n_r > 0 \).

3. Auxiliary results

We shall make use of several results on generalizations of tournaments.

Theorem 2 ([5]). Let \( D \) be a quasi-transitive digraph.

(a) If \( D \) is not strong then \( D = T[H_1, H_2, \ldots, H_r] \) for some transitive oriented graph \( T \), where the \( H_i \) are the strong components of \( D \).

(b) If \( D \) is strong then \( D = S[Q_1, Q_2, \ldots, Q_s] \) for some strong semicomplete digraph \( S \), where the \( Q_i \) are the subdigraphs of \( D \) such that \( UG(Q_i) \) are the connected components of \( UG(D) \). Each \( Q_i \) is either a non-strong quasi-transitive digraph or a single vertex and if \( q_i \to q_j \to q_i \) is a 2-cycle in \( S \) then each of \( Q_i \) and \( Q_j \) is a single vertex.

Note that the above decomposition of a quasi-transitive digraph is unique as the (strongly) connected components of a (di)graph are unique. Below we shall refer to this decomposition as the canonical decomposition of \( D \). If we further recursively decompose each \( H_j \) \( (Q_i) \) above and continue the recursion until we reach a digraph which is either transitive oriented or semicomplete then we obtain the so-called canonical total decomposition of \( D \). Using elementary graph algorithms we can find the canonical (total) decomposition of a given quasi-transitive digraph in time \( \mathcal{O}(n^4) \) (\( \mathcal{O}(n^3) \)).

The following two theorems were proved by Gutin.

Theorem 3 ([10]). In time \( \mathcal{O}(n^4) \) we can find a minimum path factor in a quasi-transitive digraph.

The algorithm of Theorem 3 is recursive and will find the path covering number of every quasi-transitive subdigraph \( W \) which is encountered as one recursively decomposes the quasi-transitive digraph \( D \) into transitive subdigraphs and semicomplete subdigraphs according to Theorem 2. Thus in total time \( \mathcal{O}(n^4) \) we can determine all of these numbers and have the corresponding path covers available.

Theorem 4 ([10]). In time \( \mathcal{O}(n^4) \) we can either find a Hamilton cycle in a given quasi-transitive digraph \( D \) or a proof that no such cycle exists in \( D \).

Theorem 5 ([11]). An extended semicomplete digraph is Hamiltonian if and only if it is strong and contains a cycle factor. If it exists a Hamilton cycle can be found in time \( \mathcal{O}(n^{2.5}) \).

The following partial strengthening (which is not hard to prove) of Theorem 5 will be useful later.

Theorem 6. If a strong extended semicomplete digraph has a \( k \)-cycle factor \( C \) \( (k > 1) \) then it has a \( (k-1) \)-cycle factor which can be constructed from \( C \) in time \( \mathcal{O}(n^2) \).

Definition 7. Let \( D = S[Q_1, Q_2, \ldots, Q_s] \) be a strong quasi-transitive digraph. Then \( E(D) \) is the (spanning) strong extended semicomplete subdigraph of \( D \) given by

\[
E(D) = S[\bar{K}_{n_1}, \bar{K}_{n_2}, \ldots, \bar{K}_{n_s}], \quad \text{where } n_i = |V(Q_i)|.
\]

Theorem 8 ([10]). A strong quasi-transitive digraph \( D = S[Q_1, Q_2, \ldots, Q_s] \) is Hamiltonian if and only if \( E(D) \) contains a cycle which covers at least \( pc(Q_i) \) vertices from the independent set \( V(\bar{K}_{n_i}) \), \( i = 1, 2, \ldots, s \).
This result plays a very important role in our proofs, so a few words are in order here. These should help the reader to follow our arguments later. Let \( D = S[Q_1, Q_2, \ldots, Q_s] \) be a strong quasi-transitive digraph.

(a) First observe that in an extended semicomplete digraph \( D' = S[\overline{K}_{n_1}, \ldots, \overline{K}_{n_s}] \) any two vertices in \( \overline{K}_{n_i} \) are similar.

Hence, a cycle \( C \) in \( D' \) covering a set \( X \) of \( 1 \leq p_i < n_i \) vertices of \( \overline{K}_{n_i} \) can be converted into a new cycle \( C' \) covering \( p_i \) vertices of \( \overline{K}_{n_i} \) by replacing \( X \) by an arbitrary different set \( Y \) of \( p_i \) vertices from \( \overline{K}_{n_i} \).

(b) If \( C \) is a cycle in \( E(D) \) which covers \( p_i \geq \text{pc}(Q_i) \) vertices of \( \overline{K}_{n_i} \), \( i = 1, 2, \ldots, s \), then, by replacing, for each \( i = 1, 2, \ldots, s \), the \( p_i \) vertices from \( C \) in \( \overline{K}_{n_i} \) by the \( p_i \) paths of a \( p_i \)-path cover of \( Q_i \), we obtain a Hamiltonian cycle of \( D \).

(c) If \( C \) is a cycle of \( D \) which intersects at least two \( Q_i \)'s then, by contracting inside each \( Q_i \) every maximal subpath of \( C \) (maximal wrt. being a subdigraph of \( Q_i \)), we obtain a cycle of \( E(D) \). Here, by contracting a maximal subpath \( x_1 x_2 \ldots x_r \) of \( C \) inside \( Q_i \), we mean replacing it by the vertex \( x_1 \). Note that \( x_1 \) and \( x_r \) have the same adjacencies to \( V(D) - V(Q_i) \).

Sometimes the following non-algorithmic characterization of Hamiltonicity of quasi-transitive digraphs is more useful than Theorem 8.

**Theorem 9 (\cite{5}).** A strong quasi-transitive digraph \( D = S[Q_1, Q_2, \ldots, Q_s] \) is Hamiltonian if and only if it has a cycle factor \( C \) such that no cycle of \( C \) is a cycle of some \( Q_i \).

**Theorem 10.** Let \( D = S[\overline{K}_{n_1}, \ldots, \overline{K}_{n_s}] \) be a strong extended semicomplete digraph such that at least one \( n_i \) is larger than one (that is, \( D \) is not semicomplete). Then \( D \) has a 2-cycle factor if and only if it has a cycle factor.

**Proof.** One direction is trivial. To prove the other, suppose that \( D \) has a cycle factor \( \mathcal{F} \) and assume that \( \mathcal{F} \) has more than two cycles. Then \( D \) has a Hamiltonian cycle \( C \), by Theorem 5. Let \( x \) and \( y \) be two vertices of \( C \) which belong to the same \( \overline{K}_{n_i} \). Then \( x \rightarrow y^+ \) and \( y \rightarrow x^+ \) (by the remark (a) above) and now \( C[y^+, x]y^+ \) and \( C[x^+, y]x^+ \) form a 2-cycle factor of \( D \). \( \square \)

We shall also make extensive use of the following structural characterization of longest cycles in extended semicomplete digraphs.

**Theorem 11 (\cite{6}).** Let \( D = R[\overline{K}_{n_1}, \ldots, \overline{K}_{n_r}] \) be a strong extended semicomplete digraph. For \( i = 1, 2, \ldots, r \), let \( m_i \) denote the maximum number of vertices from \( V(\overline{K}_{n_i}) \) which can be covered by a cycle subdigraph of \( D \). Then every longest cycle of \( D \) contains precisely \( m_i \) vertices from \( V(\overline{K}_{n_i}) \), \( i = 1, 2, \ldots, r \).

**Theorem 12 (\cite{11}. Theorem 5.7.7 and the proof of Theorem 6.11.2).** In \( \mathcal{O}(n^3) \) we can find a longest cycle \( C \) in an extended semicomplete digraph \( D \). Furthermore, \( \text{pc}(D - C) = \eta_1(D) \), unless \( C \) is a Hamilton cycle.

**Theorem 13 (\cite{4}).** In \( \mathcal{O}(n^3) \) we can find complementary cycles in a given semicomplete digraph or verify that no such cycles exist.

**Lemma 14 (\cite{8}).** Let \( D \) be an extended semicomplete digraph with a path \( P \) and a cycle \( C \) disjoint from \( P \). There exists a path \( P' \) in \( D \) with \( V(P') = V(P) \cup V(C) \) and such that for some \( x \in V(C) \), \( C[x, x^-] \) is a subpath of \( P' \). Furthermore, given \( P, C \) one can construct \( P' \) in time \( \mathcal{O}(|V(P)| \cdot |V(C)|) \).

**Corollary 15 (\cite{8}).** For every extended semicomplete digraph \( D \), \( \text{pc}(D) = \text{pcc}(D) \).

The following lemma is well known (see e.g. \cite[Corollary 3.11.7]{1}).

**Lemma 16.** Given a digraph \( D \), in \( \mathcal{O}(n^{\frac{3}{2}}) \) we can find a cycle factor in \( D \) or verify that none exists.

Before we go on to solving the minimum cycle factor problem, we shall make a remark on what separates this problem from previously solved problems. Consider a strong quasi-transitive digraph \( D \) with canonical decomposition \( D = S[Q_1, Q_2, \ldots, Q_s] \). A recurrent tool in \( \cite{3} \) and previous papers on quasi-transitive digraphs is using Theorem 5 and Lemma 14 to reduce, in a given path-cycle subdigraph of \( D \), the number of cycles intersecting more than one \( Q_i \) (the so-called large cycles below); in a sense that will become clear below, such cycles can be viewed as being part of
the structure of (an appropriate extension of) \( S \), rather than the \( Q_i \)s. In [3], for instance, this allows the authors to find a cheapest cycle in a quasi-transitive digraph, that is one cycle with minimum possible cost of its vertices. Generally, this method allows us to get rid of many large cycles, but it does not tell us how to remove small cycles, i.e. those contained in a \( Q_i \) and thus being part of the structure of the individual \( Q_i \)s. So when we consider a (potentially non-minimum) cycle factor in \( D \), which may contain many small cycles, we have no immediate way of checking whether it is possible to reduce the number of small cycles, and hence verify whether the cycle factor is actually minimum. For this reason the small cycles constitute a main obstacle in the present problem.

4. Characterizing the number of cycles of a minimum cycle factor in a quasi-transitive digraph

We start with a technical definition which will be used for cycle factors in this section and for general path-cycle factors in Section 8. At the end of that section we will also supply some motivation for the definition.

**Definition 17.** Let \( \mathcal{F} = P_1 \cup \cdots \cup P_p \cup C_1 \cup \cdots \cup C_q \) be a \( p \)-path-\( q \)-cycle factor of a digraph \( D \). We say that \( \mathcal{F} \) is reducible if there exists a \( p' \)-path-\( q' \)-cycle factor \( \mathcal{F}' = P_1' \cup \cdots \cup P_p' \cup C_1' \cup \cdots \cup C_q' \) of \( D \) such that each of the following hold:

(a) \( p' \leq p \) and \( q' \leq q \) and \( p' + q' < p + q \)

(b) for every \( i \in \{1, 2, \ldots, q\} \) either \( V(C_i) \subseteq \bigcup_{j=1}^{p'} V(P_j) \) or there is a \( j \in \{1, 2, \ldots, q'\} \) such that \( V(C_i) \subseteq V(C_j') \)

(c) \( \bigcup_{j=1}^{p'} V(P_j) \subseteq \bigcup_{j=1}^{p'} V(P_j') \).

Such an \( \mathcal{F}' \) is called a **reduction** of \( \mathcal{F} \). If no reduction of \( \mathcal{F} \) exists then \( \mathcal{F} \) is said to be irreducible.

Finally, if \( D = R(Q_1, Q_2, \ldots, Q_r) \) is quasi-transitive then those cycles of \( \mathcal{F} \) that are contained in a \( Q_i \) are called **small cycles** and all other cycles of \( \mathcal{F} \) are called **large cycles**.

It is clear that every minimum cycle factor is irreducible. In the following \( D \) will always denote a quasi-transitive digraph with canonical decomposition \( D = R(Q_1, Q_2, \ldots, Q_r) \), where \( R \) is either a strong semicomplete digraph or a transitive oriented digraph, depending on whether \( D \) is strong or not.

**Lemma 18.** Let \( \mathcal{F} \) be an irreducible cycle factor in \( D \). Then the following holds:

(a) If \( D \) is non-strong then \( \mathcal{F} \) has no large cycle.

(b) If \( D \) is strong then \( \mathcal{F} \) has precisely one large cycle.

**Proof.** Part (a) is clear since \( R \) is acyclic when \( D \) is non-strong. To prove (b), suppose that \( D \) is strong and that \( \mathcal{F} = C_1 \cup \cdots \cup C_q \) is an irreducible \( q \)-cycle factor, where the cycles are ordered so that \( C_1, \ldots, C_q \) are all the large cycles (possibly \( s = 0 \)). Let \( D^* \) denote the strong extended semicomplete digraph that one obtains from \( D \) by contracting, for each \( j \in \{1, \ldots, q\} \), every maximal subpath of \( C_j \) inside every \( Q_j \) and then deleting all remaining arcs inside \( Q_j \). This way each small cycle is contracted into a vertex and \( C_1, \ldots, C_s \) are converted into a cycle subdigraph \( \mathcal{F}^* \) of \( D^* \). Let \( C^* \) be a longest cycle of \( D^* \). By Theorem 11 we can construct \( C^* \) such that it covers at least one vertex from every \( V(Q_i) \) and every vertex of \( \mathcal{F}^* \) and possibly some of the other vertices (those corresponding to small cycles in \( \mathcal{F} \)). Now, by re-substituting the contracted paths (including contracted small cycles) we obtain a new cycle subdigraph \( \mathcal{F}' \) containing one large cycle, \( C' \), corresponding to \( C^* \), and possibly some small cycles all of which are also small cycles of \( \mathcal{F} \). If \( s = 0 \) or \( s > 1 \) then it is easy to see that \( \mathcal{F}' \) has fewer cycles than \( \mathcal{F} \) and that the vertices of each cycle of \( \mathcal{F} \) are covered by one cycle of \( \mathcal{F}' \), i.e. \( \mathcal{F}' \) is a reduction of \( \mathcal{F} \), a contradiction.”

**Corollary 19.** Every minimum cycle factor in a strong quasi-transitive digraph contains exactly one large cycle.

Recall the definition of the strong extended semicomplete digraph \( E(D) \) from Section 3. Let \( \mathcal{C} \) be the set of cycle subdigraphs of \( E(D) = R[\overline{K}_{m_1}, \ldots, \overline{K}_{m_r}] \) and let \( m_i(D) = \max_{S \in \mathcal{C}} \{|V(S) \cap V(\overline{K}_{m_i})|\} \), for all \( i = 1, \ldots, r \). By Theorem 11, every longest cycle in \( E(D) \) contains exactly \( m_i(D) \) vertices of \( H_i \).

**Lemma 20.** If \( D \) is a strong quasi-transitive digraph containing a cycle factor then \( D \) has a minimum cycle factor in which the (unique) large cycle \( C \) intersects \( Q_i \) in exactly \( m_i(D) \) paths for each \( i = 1, 2, \ldots, r \). That is, by contracting each maximal subpath of \( C \) which lies inside \( Q_i \) (for every \( i = 1, 2, \ldots, r \)), we obtain a longest cycle of \( E(D) \).
Proof. Let \( \mathcal{F} \) be a minimum cycle factor in \( D \) with the large cycle \( C_0 \) (by Corollary 19, \( C_0 \) is unique). For each \( i = 1, \ldots, r \), let \( p_i \) be the number of maximal subpaths of \( C_0 \) inside \( Q_i \) and note that, by the definition of \( m_i(D) \), we have \( p_i \leq m_i(D) \). If \( Q_i \) contains no small cycle from \( \mathcal{F} \) (i.e., \( p_i \geq pc(Q_i) \)) then \( Q_i \) clearly contains an \( m_i(D) \)-path cover. If \( Q_i \) contains \( c_i \geq 1 \) small cycles from \( \mathcal{F} \), we claim that \( p_i = m_i(D) \). Suppose not and delete one arc in a small cycle in \( Q_i \) (thus turning the cycle into a path) until we either get \( m_i(D) \) paths or have used all small cycles. In the latter case, also delete an appropriate number of arcs in the current \( p_i + c_i \) paths until we have an \( m_i(D) \)-path factor in \( Q_i \). In both cases, by doing this in each \( Q_i \) and replacing \( C_0 \) by a large cycle entering \( Q_i \) exactly \( m_i(D) \) times, we obtain a cycle factor with fewer cycles than \( \mathcal{F} \), a contradiction. It follows from the arguments above that \( \mathcal{F} \) and any longest cycle \( C \) in \( E(D) \) we can construct a minimum cycle factor with the desired property: for the \( m_i(D) \) vertices of \( V(\overline{\mathcal{K}(V(Q_i))}) \cap C \), substitute \( m_i(D) \) paths of \( Q_i \) and keep the small cycles of \( \mathcal{F} \). □

A canonical minimum cycle factor is one for which the unique large cycle intersects each \( Q_i \) in exactly \( m_i(D) \) paths. By Lemma 20, every strong quasi-transitive digraph with a cycle factor has a canonical minimum cycle factor.

Let \( I(D) = \{ i \mid m_i(D) < pc(Q_i) \} \) and note that for every \( i \), \( Q_i \) has the same number \( c_i \) of small cycles with respect to every canonical minimum cycle factor (for \( i \not\in I(D) \) this number is zero). Since, for every \( t \geq 0 \) and every digraph \( G \), we have \( \eta_{t+1}(G) \geq \eta_t(G) - 1 \), we see that, for \( i \in I(D) \), we have \( c_i = \min\{ j \mid \eta_j(Q_i) = m_i(D) \} \).

Hence, we have the following characterization of the number, \( k_{\min}(D) \), of cycles in a minimum cycle factor:

**Theorem 21.** For every strong quasi-transitive digraph, \( D \), containing a cycle factor, we have

\[
k_{\min}(D) = 1 + \sum_{i \in I(D)} \min\{ j \mid \eta_j(Q_i) = m_i(D) \}.
\]

Furthermore, every cycle factor of \( D \) has at least \( 1 + \sum_{i \in I(D)} (pc(Q_i) - m_i(D)) \) cycles.

It follows from Theorem 21 that we could determine \( k_{\min}(D) \) in polynomial time, provided we could calculate \( \eta_j(Q_i) \), for every \( j \in \{0, 1, \ldots, |V(Q_i)\rceil - 1\} \) and every \( i = 1, 2, \ldots, r \), in polynomial time. However, it is not clear whether our approach (using network flows) below can be extended to the general case \( (j \geq 3) \), since it depends on the sets \( M_j(Q_i) \) having a sufficiently simple structure (cf. the remarks preceding Theorem 30).

**Problem 4.1.** Determine the complexity of computing \( k_{\min}(D) \) and finding a minimum cycle factor of a quasi-transitive digraph \( D \).

When \( k_{\min}(D) \) is small, in particular when \( k_{\min}(D) \in \{1, 2, 3\} \), the problem is polynomially solvable: the case \( k_{\min}(D) = 1 \) is the Hamiltonian cycle problem, which is polynomial by Theorem 4, and the cases \( k_{\min}(D) = 2, 3 \) will be covered in the next sections.

**Conjecture 22.** For each fixed \( k \) there is a polynomial algorithm which determines whether a given quasi-transitive digraph \( D \) has a cycle factor with at most \( k \) cycles and, if so, finds a minimum cycle factor of \( D \).

5. Checking whether \( k_{\min}(D) = 2 \)

A connected, non-strong quasi-transitive digraph has a cycle factor with two cycles if and only if its canonical decomposition is \( D = P_2[Q_1, Q_2] \), where \( P_2 \) is the path on two vertices and \( Q_i \) is Hamiltonian for \( i = 1, 2 \). Thus by Theorem 4, if \( D \) is non-strong we can determine in time \( O(n^4) \) whether \( k_{\min}(D) = 2 \). Hence we may assume below that we are given a non-Hamiltonian strong quasi-transitive digraph \( D = S[Q_1, Q_2, \ldots, Q_s] \) which contains a cycle factor (recall Lemma 16), i.e., \( k_{\min}(D) > 1 \).

It follows from Theorem 21 that \( k_{\min}(D) = 2 \) if and only if \( D \) has a canonical minimum cycle factor with precisely one large and one small cycle and this will happen if and only if for some \( i \in \{1, 2, \ldots, s\} \), \( pc(Q_i) = \eta_0(Q_i) \leq m_j(D) \) for all \( j \neq i \) and \( pc(Q_i) - 1 = \eta_1(Q_i) = m_i(D) \).

**Lemma 23.** For every strong digraph \( D \) containing a cycle, \( pc(D) - 1 \leq \eta_1(D) \leq pc(D) \). Furthermore, given a \( p \)-path cover of \( D \), we can construct a \( p' \)-path-1-cycle factor of \( D \) with \( p' \leq p \) in time \( O(n^2) \).
Proof. Clearly, $\eta_1(D) \geq pc(D) - 1$ holds for every digraph $D$, since any $\eta_1(D)$-path-1-cycle factor gives rise to a path cover of $D$ with $\eta_1(D) + 1$ paths by deleting one arc from the cycle. For the other inequality we claim that, given a $p$-path cover we can construct a $k$-path-1-cycle factor with $k \leq p$, thus showing that $\eta_1(D) \leq pc(D)$. Let $P_i = v_i^{(i)}v_i^{(i+1)} \cdots v_i^p$, $i \in \{1, 2, \ldots, p\}$, be the paths in a $p$-path cover $P$. Since $D$ is strong, each of the vertices $v_i^{(i)}$ has an out-neighbour $w_i$ and if, for some $i$, $w_i$ is on $P_i$, we easily obtain a $k$-path-1-cycle factor with $k \leq p$. So suppose no $i$ has $w_i$ on $P_i$. Now it is easy to see that there is a $q$-subset $Q$ of $P$ ($1 < q \leq p$) such that $D(V(Q))$ has a cycle $C$ containing an arc out of every $v_i^{(i)}$ with $P_i \in Q$. Now, $C$ together with the subpaths $P_i - C (P_i \in Q)$ and the paths from $P - Q$ constitute a $p'$-path-1-cycle factor of $D$ with $p' \leq p$, i.e., $\eta_1(D) \leq p$. Now the second inequality of the claim follows by letting $p = pc(D)$. We leave the complexity claim to the reader (all we need to do is find a cycle using arcs out of $q$ of the vertices $v_i^{(i)}$ and delete that cycle). □

Lemma 24. For every non-strong quasi-transitive digraph $D$ containing a cycle, $\eta_1(D) \leq pc(D)$.

Proof. Consider the canonical decomposition $D = T[Q_1, \ldots, Q_t]$ of $D$ and recall that $T$ is a transitive oriented graph. Clearly, the cycle of any $\eta_1(D)$-path-1-cycle factor of $D$ is contained in some strong $Q_i$. By Lemma 23, $\eta_1(Q_i) \leq pc(Q_i)$. Hence, starting from a minimum path factor of $D$, which must intersect $Q_i$ in $p_i \geq pc(Q_i)$ paths, we can replace these $p_i$ paths by an $\eta_1(Q_i)$-path-1-cycle factor of $Q_i$ without increasing the overall number of paths of the factor in $D$, since the transitivity of $T$ allows every path that has none of its end vertices in $Q_i$ to bypass $Q_i$. □

By the definition of $pc(D)$, a digraph $G$ contains a $k$-path factor if and only if $k$ belongs to the interval $M_0(D) = \{pc(G), pc(G) + 1, \ldots, |V(G)|\}$. The following lemma shows that a similar property holds for $k$-path-1-cycle factors.

Lemma 25. For every digraph $D$ containing a cycle, $M_1(D) = \{\eta_1(D), \eta_1(D) + 1, \ldots, \beta(D)\}$. Also, given an $\eta_1(D)$-path-1-cycle factor and a shortest cycle of $D$, in $O(n^2)$ we can find $k$-path-1-cycle factors for all $k \in M_1(D)$.

Proof. If $D$ has a $k$-path-1-cycle factor, $k$ cannot be smaller than $\eta_1(D)$ or larger than $\beta(D)$.

Clearly, for $k \in \{\eta_1(D), \beta(D)\}$ $D$ has a $k$-path-1-cycle factor. Now, suppose that $D$ has a $k$-path-1-cycle factor for all $k \in \{\eta_1(D), \eta_1(D) + 1, \ldots, k_0\}$ for some $k_0 < \beta(D)$. We shall show that $D$ has a $(k_0 + 1)$-path-1-cycle factor, proving the lemma by induction.

Consider a $k_0$-path-1-cycle factor, $C \cup P_1 \cup P_2 \cup \ldots \cup P_{k_0}$.

If one of the $k_0$ paths contains more than one vertex then deleting one arc on that path produces a $(k_0 + 1)$-path-1-cycle factor. So assume that each $P_i$ consists of just one vertex.

If $C$ contains a chord $(u, v)$, then deleting from $C$ the two arcs $(u, u^-(v^-, v)$ and adding $(u, v)$, again produces a $(k_0 + 1)$-path-1-cycle factor. So assume also that this is not the case.

Note that, since $k_0 < \beta(D)$, $C$ is not a shortest cycle in $D$ and consider any cycle $C'$ satisfying $|C'| < |C|$. Let

$$a = |V(C') \cap V(C)|$$
$$b = |V(C') \cap V(C \cup P_1)|$$
$$d = \max\{1, \text{number of connected components of } C \cap (V(C) \setminus V(C'))\}.$$

We claim that $d \leq b$. This clearly holds if $a = 0$, since then $d = 1$ and $b = |C'|$. For $a > 0$, $C' \cap V(C)$ consists of at least $d$ disjoint paths, since otherwise $C'$ cannot partition $C$ into $d$ pieces (remember that $C$ is induced), and no two of these paths are connected by an arc in $D$. Therefore, $C'$ must have at least $d$ vertices not on $C$, i.e. $d \leq b$.

This proves the claim.

Now, consider the path-1-cycle factor of $D$ given by the cycle $C'$, the $d$ paths $C - V(C')$ and the paths (vertices) $(\bigcup_{i=1}^{k_0} P_i) - V(C')$. By the claim just proven, it contains $d + k_0 - b \leq k_0$ paths and these paths contain $|C| - a + k_0 - b = k_0 + |C| - |C'| > k_0$ vertices. So, by deleting an appropriate number of arcs from these paths, we obtain a $(k_0 + 1)$-path-1-cycle factor. The complexity claim is left to the reader. □

Since, for every subpath $x \rightarrow y \rightarrow z$ of a cycle on at least four vertices in a quasi-transitive digraph, $x$ is adjacent to $z$, it is not hard to see that in a quasi-transitive digraph we can find a shortest cycle in time $O(n^2)$ (we have to check
for a 2-cycle). Hence if the digraph in question is quasi-transitive we do not need the assumption in Lemma 25 that we are also given a shortest cycle.

We define the number \( pcc^*(D) \) to be \( pcc(D) \) if \( D \) has no cycle factor and zero otherwise. By Corollary 15, every extended semicomplete digraph \( D \) has \( pc(D) = pcc(D) \), and since we clearly have \( \eta_1(D) \geq pcc^*(D) \) for any digraph, we get the following corollary of Lemma 23 and Theorem 12.

**Corollary 26.** If \( D \) is a strong extended semicomplete digraph then \( \eta_1(D) = pcc^*(D) \) and we can find an \( \eta_1(D) \)-path-1-cycle factor in \( D \) in \( O(n^3) \).

By Theorem 3, we can find a minimum path factor in a quasi-transitive digraph in polynomial time and, as the following lemma shows, the same holds for an \( \eta_1 \)-path-1-cycle factor.

**Lemma 27.** Let \( D = R[Q_1, Q_2, \ldots, Q_r] \) be a quasi-transitive digraph. In time \( O(n^4) \) we can compute \( \eta_1(D) \) and construct an \( \eta_1(D) \)-path-1-cycle factor of \( D \).

**Proof.** As is the case with Gutin’s algorithm [10] for the path covering problem, our algorithm is recursive and calculates \( \eta_1(H) \) for every quasi-transitive digraph of \( D \) for which \( \eta_1(H) \) is part of the canonical total decomposition of \( D \).

We start with the case when \( D \) is strong. Let us first consider the possible structure of a \( k \)-path-1-cycle factor \( \mathcal{F} \) of \( D \): either the unique cycle \( C \) of \( \mathcal{F} \) is a large cycle (Type A) or it is a small cycle (Type B). Consider the usual operation of contracting each maximal subpath of \( C \) and of each path of \( \mathcal{F} \) inside \( Q_i \), for all \( i \), and finally deleting all remaining arcs inside each \( Q_i \). After this, the resulting path-cycle subdigraph \( \mathcal{F}' \) (considered as a subdigraph of \( E(D) \)) is a \( k \)-path-1-cycle subdigraph of \( E(D) \) if \( \mathcal{F} \) is of Type A and a \( (k+1) \)-path subdigraph otherwise (with one of the paths being just the vertex obtained by contracting the small cycle). From the definition of \( M(D) \) and the fact that \( \mathcal{F} \) is spanning it follows that if \( \mathcal{F} \) is of Type A, then \( \mathcal{F}' \) must contain at least \( pc(Q_i) \) vertices of \( Q_i \), for \( i = 1, 2, \ldots, r \). If it is of Type B then \( k \) of the \( k \)-path subdigraphs \( \mathcal{F}' \) must use at least \( \eta_1(Q_j) \) and no more than \( \beta(Q_j) \) vertices of \( Q_j \), if \( C \) is contained in \( Q_j \); furthermore, for all \( i \neq j \), \( \mathcal{F}' \) must contain at least \( pc(Q_i) \) vertices of \( Q_i \).

We can also go the other way: given a \( k \)-path-1-cycle subdigraph \( \mathcal{H}' \) in \( E(D) \), which contains \( k_i \geq pc(Q_i) \) vertices of \( V(Q_i) \), for each \( i \), we can construct a \( k \)-path-1-cycle factor of \( D \) by replacing the \( k_i \) vertices of \( \mathcal{H}' \) by a \( k_i \)-path cover of \( Q_i \). Similarly, suppose that we are given a \( k \)-path subdigraph of \( E(D) \) which covers at least \( k_i \geq pc(Q_i) \) vertices of \( V(Q_i) \), for each \( i \neq j \), and some number \( f \), between \( \eta_1(Q_j) \) and \( \beta(Q_j) \), of vertices in \( Q_j \). By Lemma 25, we can then construct a \( k \)-path-1-cycle factor of \( D \) by replacing vertices by subpaths, as above, and taking a cycle \( C \) in \( Q_j \) which, together with \( f \) paths, forms an \( f \)-path-1-cycle factor of \( Q_j \).

Thus we can determine \( \eta_1(D) \), provided we can solve the following two problems, since \( \eta_1(D) = \min\{k_0(D), k_1(D)\} \) and where \( k_0(D), k_1(D) \) are defined below.

1. find the minimum integer \( k_1(D) \) such that \( E(D) \) has a \( k_1(D) \)-path-1-cycle subdigraph which contains at least \( pc(Q_i) \) vertices of \( V(Q_i) \) for each \( i \).
2. find the minimum integer \( k_0(D) \) (over all \( j \in \{1, 2, \ldots, r\} \)) such that \( E(D) \) has a \( k_0(D) \)-path subdigraph which contains at least \( pc(Q_i) \) vertices of \( V(Q_i) \) for each \( i \neq j \) and between \( \eta_1(Q_j) \) and \( \beta(Q_j) \) vertices of \( Q_j \).

Recall that all vertices inside an independent set of an extended semicomplete digraph are similar. Hence every \( k_1(D) \)-path-1-cycle subdigraph \( \mathcal{F} \) of \( E(D) \) which contains at least \( pc(Q_i) \) vertices of \( V(Q_i) \) for each \( i \) corresponds to a \( k_1(D) \)-path-1-cycle factor in some induced subdigraph of \( E(D) \), namely the one we obtain by deleting the vertices of each \( K_{n_j} \), which are not covered by \( \mathcal{F} \). A similar observation holds for \( k_0(D) \)-path subdigraphs of \( E(D) \). Thus, by Corollaries 26 and 15, we have

\[
\begin{align*}
k_1(D) &= \min \left\{ pcc^*(R[K_{n_1}, \ldots, K_{n_r}]) : \begin{array}{c}
pc(Q_i) \leq i_j \leq n_j, \\
j = 1, 2, \ldots, r
\end{array} \right\} \\
k_{0, p}(D) &= \min \left\{ pcc(R[K_{n_1}, \ldots, K_{n_r}]) : \begin{array}{c}
pc(Q_j) \leq i_j \leq n_j, \\
i_j \neq p
\end{array} \right\} \\
k_0(D) &= \min \{k_{0, 1}(D), k_{0, 2}(D), \ldots, k_{0, r}(D)\}.
\end{align*}
\]

By Theorem 3 and the remark following it, we can determine \( pc(Q_i) \) and compute minimum path coverings of all \( Q_i \)s at this level and all lower levels in the canonical total decomposition of \( D \) in time \( O(n^4) \).
As explained by Gutin [10] (see also [1, exercise 3.60]), given the numbers \(pc(Q_i)\) and \(\eta_1(Q_i)\), for each \(i = 1, 2, \ldots, r\), one can find each of the numbers \(k_1(D), k_0, p(D)\) \((p = 1, 2, \ldots, r)\) in time \(O(r^3)\), using a minimum flow algorithm on an appropriate flow network. Thus the time to find \(k_0, p(D)\) is \(O(r^4)\). In fact, it is possible to calculate all the numbers \(k_0, p(D)\), \(p = 1, 2, \ldots, r\), in total time \(O(r^3)\) by exploiting the fact that the flow corresponding to \(k_0, p\) can be modified into a flow corresponding to \(k_0, p + 1\) in only \(O(r^2)\) time (we have to change the lower and upper bounds on only two of the vertices of \(R\) each time and only by a small constant amount).

Now, suppose that \(D\) is non-strong, i.e., \(R\) is a transitive oriented graph, and the cycle of an \(\eta_1(D)\)-path-1-cycle factor, \(F_1\), is contained in some \(Q_i\). Then \(F_1\) intersects \(Q_j\) in at least \(pc(Q_j)\) paths, for \(j \neq i_0\), and between \(\eta_1(Q_i)\) and \(\beta(Q_i)\) paths in \(Q_i\). Suppose that \(x \rightarrow x^+Py^− \rightarrow y\) is a subpath of some path \(\bar{P}\) of \(F_1\) such that \(x, y \notin Q_j\) and \(V(x^+Py^−) \subseteq V(Q_j)\). Since \(R\) is transitive, \(x \rightarrow y\), so we can remove the subpath \(x^+Py^−\) from \(\bar{P}\) without increasing the number of paths in \(F_1\). Therefore, we may assume that \(F_1\) has exactly \(pc(Q_j)\) paths in \(Q_j\), for \(j \neq i_0\), and exactly \(\eta_1(Q_i)\) (which may be zero) paths’ in \(Q_i\). Thus, we have

\[
\eta_1(D) = \min_{i_0 \in \{1, \ldots, r\}} \left\{ pc(R[K_{n_1}, \ldots, K_{n_r}]) : \begin{array}{l}
n_i = pc(Q_i), i \neq i_0 \\
n_{i_0} = \eta_1(Q_{i_0})
\end{array} \right\}
\]

\[
= \min_{i_0 \in \{1, \ldots, r\}} \left\{ pcc(R[K_{n_1}, \ldots, K_{n_r}]) : \begin{array}{l}
n_i = pc(Q_i), i \neq i_0 \\
n_{i_0} = \eta_1(Q_{i_0})
\end{array} \right\}
\]

where the last equality follows from the acyclicity of \(R\). Given the numbers \(\eta_1(Q_{i_0})\), the latter number can be computed by a flow algorithm in \(O(n^3)\) by “reusing” flows as above.

Having found \(pc(Q_i)\)-path factors and \(\eta_1(Q_i)\)-path-1-cycle factors of the individual \(Q_j's\) as well as the minimum flow (computed as above), we can construct an \(\eta_1(D)\)-path-1-cycle factor of \(D\) in \(O(n^3)\) using Theorem 5 and Lemmas 14 and 23.

So after finding all minimum path coverings in \(O(n^4)\), the remaining computation time \(T(n)\) is given by

\[T(n) = O(n^3) + \sum_{i=1}^{r} T(|V(Q_i)|) \in O(n^4).\]

We are now in a position to describe our algorithm.

**Theorem 28.** Let \(D = S\{Q_1, Q_2, \ldots, Q_s\}\) be a strong quasi-transitive digraph of order \(n\). In time \(O(n^4)\) we can find a minimum cycle factor with at most two cycles (hence \(k_{min} \leq 2\)) or verify that every cycle factor of \(D\) has at least three cycles.

**Proof.** Combining Theorem 3 and Lemma 27, in total time \(O(n^4)\) we can find the numbers \(\eta_1(Q)\) and \(pc(Q)\) (and corresponding factors) for all quasi-transitive digraphs \(Q\) obtained during the construction of the canonical total decomposition of \(D\). Now, use Theorems 11 and 12 to find a longest cycle \(C\) in \(E(D)\) and the numbers \(m_i(D), i = 1, 2, \ldots, s\) in total time \(O(n^3)\). Let \(I(D) = \{i \mid m_i(D) < pc(Q_i)\}\), that is, \(I(D)\) is the set of those \(i\) for which \(C\) covers less than \(pc(Q_i)\) vertices of \(V(Q_i)\). If \(|I(D)| = 0\), \(D\) has a Hamilton cycle, by Theorem 8, implying that \(k_{min} = 1\) and we can construct the desired cycle in time \(O(n)\) by inserting the minimum path factors of the \(Q_j's\) (with some arcs deleted if \(m_i(D) > pc(Q_i)\)). If \(|I(D)| > 1\), then \(D\) has no 2-cycle factor, since at least two small cycles are needed together with one large cycle to cover those \(V(Q_i)\) with \(i \in I(D)\). Finally, if \(|I(D)| = 1\), say \(I(D) = \{i\}\), then \(D\) has complementary cycles if and only if \(pc(Q_i) = m_i(D) + 1\) and \(\eta_1(Q_i) = m_i(D)\). If so, we can construct complementary cycles by replacing the \(m_j(D)\) independent vertices of \(C\) by the \(m_j(D)\) paths of an \(m_j(D)\)-path-1-cycle factor in \(Q_j\) (for \(j = i\)), respectively an \(m_j(D)\)-path cover in \(Q_j\) (for \(j \neq i\)).

**6. Checking whether \(k_{min}(D) = 3\)**

If \(D\) is not strong then it is easy to see that we can reduce the problem of checking whether \(k_{min}(D) = 3\) to at most three of the problems discussed above. So suppose below that the quasi-transitive digraph \(D = S\{Q_1, Q_2, \ldots, Q_s\}\) in question is strong and has a cycle factor (recall Lemma 16) and that we have verified that \(k_{min}(D) > 2\). Below we show how to check whether \(k_{min}(D) = 3\).

**Lemma 29.** Let \(Q = R\{Q_1, Q_2, \ldots, Q_r\}\) be a quasi-transitive digraph. If \(F\) is either
(a) an irreducible $p$-path-$q$-cycle factor of $Q$ with $p > 0$ or
(b) an $\eta_q(Q)$-path-$q$-cycle factor of $Q$ with $q > 0$ and $0 < \eta_q(Q) < \eta_{q-1}(Q)$

then every cycle of $\mathcal{F}$ is small.

**Proof.** It suffices to consider the case when $Q$ is strong, since $R$ is acyclic otherwise. Let $\mathcal{F}$ be a $k$-path-$q$-cycle factor of $Q$ with $k > 0$ and suppose that $\mathcal{F}$ contains a large cycle. By contracting each maximal subpath of $\mathcal{F}$ inside every $Q_i$ we obtain a $k'$-path-$q'$-cycle subdigraph $\mathcal{F}'$ of $E(Q)$, where $k' \geq k$ (small cycles become paths consisting of just one vertex), $q' > 0$ and every cycle of $\mathcal{F}'$ comes from a large cycle of $\mathcal{F}$. Let $C$ be a cycle of $\mathcal{F}'$. As $k > 0$, at least one path $P'$ among the $k'$ paths of $\mathcal{F}'$ is obtained from a path $P$ of $\mathcal{F}$ by the above contraction. Applying Lemma 14 to merge $P'$ and $C$ into one path $P''$, we get a $k$-path-$(q'-1)$-cycle subdigraph $\mathcal{F}''$ which covers the same vertices as $\mathcal{F}'$. Substituting back the maximal subpaths that we contracted above, we obtain a $k$-path-$(q'-1)$-cycle factor of $Q$ which satisfies properties (b) and (c) of Definition 17. This shows that $\mathcal{F}$ cannot satisfy either of (a) and (b) above. □

It follows from Theorem 21, Corollary 19, and the assumptions on $D$ that $k_{\min}(D) = 3$ if and only if $D$ has a canonical minimum cycle factor with precisely one large and two small cycles. This will happen if and only if one of the following holds:

(a) for some pair $i, j \in \{1, 2, \ldots, s\}$, $i \neq j$, we have $pc(Q_p) - 1 = \eta_1(Q_p) = m_p(D)$, for $p = i, j$, and $pc(Q_p) = \eta_0(Q_p) \leq m_p(D)$, for all $p \neq i, j$.

(b) for some $i \in \{1, 2, \ldots, s\}$, we have $\eta_1(Q_i) > \eta_2(Q_i) = m_1(D)$ and $pc(Q_p) = \eta_0(Q_p) \leq m_p(D)$ for all $p \neq i$.

Here the strict inequality in (b) holds because $k_{\min}(D) > 2$.

Using the fact that in a quasi-transitive digraph every cycle of length at least four has a chord, it is not hard to show that $\{\eta_2(D), \eta_2(D) + 1, \ldots, |V(D)| - 6\} \subseteq M_2(D)$. The largest possible number $p$ of paths in a $p$-path-2-cycle factor of $D$ is at most $|V(D)| - 4$. However, it is not difficult to construct an example of a quasi-transitive digraph having no $\left(|V(D)| - 5\right)$-path-2-cycle factor (see Fig. 1). This shows that, contrary to the situation for $M_0(D)$ and $M_1(D)$, $M_2(D)$ need not be an interval. Note also that there is an $O(n^4)$ algorithm for checking whether $M_2(D)$ contains one or both of the numbers $|V(D)| - 5$ and $|V(D)| - 4$; namely, for each 2-cycle $C$, check whether $D$ is has a 2-cycle, respectively a strong component of order at least three.

The next theorem is the main result of this section. Note that the proof only holds for a minimum cycle factor with three cycles.

**Theorem 30.** Let $D = S[Q_1, Q_2, \ldots, Q_s]$ be a strong quasi-transitive digraph of order $n$. In time $O(n^5)$ we can find a minimum cycle factor with three cycles (i.e. $k_{\min} = 3$) or verify that every cycle factor of $D$ has at least four cycles.

**Proof.** We use the same approach as in the proof of Theorem 28 and may assume that, using the algorithms for checking the existence of a cycle factor, checking Hamiltonicity and checking whether $k_{\min} = 2$, we have found that $D$ has a cycle factor and that $k_{\min} \geq 3$. Thus, $|I(D)| > 0$ and if $|I(D)| > 2$, $D$ has no 3-cycle factor, since at least three small cycles are needed together with one large cycle in order to cover those $Q_i$s with $i \in I(D)$.

Suppose that $I(D) = \{j_1, j_2\}$, i.e. $pc(Q_i) > m_1(D)$, for $i = j_1, j_2$. By Theorem 21, $D$ has a 3-cycle factor if and only if $\eta_1(Q_i) = m_1(D)$ for $i = j_1, j_2$. Since we have already found the $\eta_1$-values and corresponding factors (while
checking whether \( k_{\min}(D) = 2 \), we can check this in constant time. If the answer is yes, we can produce a 3-cycle factor in linear time when given minimum path covers of all \( Q_i \)'s, an \( m_p \)-path-1-cycle factor of \( Q_p \) \((p = j_1, j_2)\), and a longest cycle of \( E(D) \).

Now suppose \( I(D) = \{ j \} \), and note that \( \eta_1(Q_j) > m_j(D) \), since \( D \) has no 2-cycle factor. As \( \eta_{t+1}(G) \geq \eta_t(G) - 1 \), for every digraph \( G \) and \( t \geq 0 \), \( D \) has a 3-cycle factor if and only if \( \eta_1(Q_j) = m_j(D) + 1 \) and \( \eta_2(Q_j) = m_j(D) \). So assume that \( \eta_1(Q_j) = m_j(D) + 1 \) (we may verify this in constant time when we have run the algorithm for checking whether \( k_{\min}(D) = 2 \)). It remains to argue that within the claimed time bound we can check whether \( Q_j \) has two disjoint cycles and, if so, either compute \( \eta_2(Q_j) \) (and a corresponding factor) or verify that \( \eta_2(Q_j) \geq \eta_1(Q_j) \). In the latter case \( k_{\min}(D) > 3 \), by Theorem 21. \( \square \)

We need the following claim.

**Claim 31.** Let \( Q = R[H_1, H_2, \ldots, H_r] \) be a quasi-transitive digraph of order \( n \). In time \( O(n^5) \) we can decide which of the following holds and, if (C) holds, find an \( \eta_2(Q) \)-path-2-cycle factor \( F_2(Q) \) of \( Q \):

(A) \( Q \) does not have two disjoint cycles
(B) \( \infty > \eta_2(Q) \geq \eta_1(Q) \).
(C) \( \eta_2(Q) = \eta_1(Q) - 1 \).

**Proof.** It is not hard to show that there exists an \( O(n^5) \) algorithm for finding two disjoint cycles in \( Q \) (if they exist). Assume below that we have found such cycles and have computed all \( \eta_1 \)- and \( pc \)-values in the canonical total decomposition of \( Q \) in \( O(n^5) \).

Suppose, \( \eta_1(Q) > 0 \) (i.e., \( Q \) is non-Hamiltonian) and use Theorem 28 to check in \( O(n^4) \) whether \( \eta_2(Q) = 0 \). If so, (C) holds, so suppose that \( \eta_2(Q) > 0 \) and \( \eta_1(Q) > 1 \). Then Lemma 29 implies that to check whether (C) holds we need only consider \( k \)-path-2-cycle factors for which both cycles are small, i.e., the two cycles are either in different \( H_i \)'s or in the same \( H_i \). We consider these two possibilities below and apply network flows in a manner similar to what we did in the proof of Lemma 27.

(i) If the two cycles of some \( \eta_2(Q) \)-path-2-cycle factor are in two different \( H_i \), \( H_j \) then either (B) holds or we have that \( \eta_1(H_p) < pc(H_p), p = i, j \), and \( \eta_2(Q) \) equals the following number, where \( J'_1(Q) = \{ i : \eta_1(H_i) > pc(H_i) \} \):

\[
k' = \min_{p,q \in J'_1(H), p \neq q} \left\{ \min_{p \in \eta_1(H)} \left\{ \min_{p \in \eta_1(H)} \left\{ pcc(R[\overline{K}_{i_1}, \ldots, \overline{K}_{i_n}]) : \left\{ \begin{array}{l}
\eta_1(H_p) \leq t_p \leq \beta(H_p) \\
\eta_1(H_q) \leq t_q \leq \beta(H_q) \\
\pc(H_j) \leq i_j \leq n_j, j \neq p, q
\end{array} \right\} \right\} \right\}.
\]

We can compute \( k' \) in \( O(n^4) \) by reusing flows as before. In fact, as we have seen above, if \( Q \) is non-strong we need only let \( i_j = pc(H_j) \), respectively \( i_j = \eta_1(H_j) \) in this flow calculation.

(ii) If the two cycles of some \( \eta_2(Q) \)-path-2-cycle factor are in the same \( H_i \) then either (B) holds or \( \eta_2(H_i) < \eta_1(H_i) \) (in which case \( \eta_1(H_i) > 0 \)). First use Theorem 28 to check whether \( \eta_2(H_i) = 0 \). If \( \eta_2(H_i) \neq 0 \) then either (C) fails or \( 0 < \eta_2(H_i) < \eta_1(H_i) \), so we may apply Lemma 29 on \( H_i \) to recursively compute an \( \eta_2(H_i) \)-path-2-cycle factor in \( H_i \) or verify that \( \eta_2(H_i) \geq \eta_1(H_i) \). If instead \( \eta_2(H_i) = 0 \), we compute a \( pc(Q - H_i) \)-path factor in \( Q - H_i \). Now, let \( J'_2(Q) = \{ i : \eta_2(H_i) < \eta_1(H_i) \} \) and observe that either (B) holds or \( \eta_2(Q) \) equals \( \min \{ k', k'' \} \), where

\[
k'' = \min_{\mu \in J'_2(Q)} \left\{ \min_{p \in \eta_2(H_p)} \left\{ pcc(R[\overline{K}_{i_1}, \ldots, \overline{K}_{i_n}]) : \left\{ \begin{array}{l}
\pc(H_j) \leq i_j \leq n_j, j \neq p \\
p \in M_2(H_p) - \{ 0 \}
\end{array} \right\} \right\} \right\}.
\]

We claim that both of the numbers \( k' \) and \( k'' \) can be obtained in total time \( O(n^4) \). First note that, since we have the numbers \( pc(H_j) \) (and corresponding factors) available, for each \( j \in \{ i \mid \eta_2(H_i) = 0 \} \), we can find \( pc(Q - H_j) \) (and a corresponding factor) in time \( O(n^3) \) using a minimum flow algorithm on the network obtained from \( R \). Thus we obtain \( \{ pc(Q - H_p) : \eta_2(H_p) = 0 \} \), and hence \( k'' \), in \( O(n^4) \). Furthermore, as mentioned above, we can find each set \( M_2(H_p) \) in \( O(n^3) \), hence all of them in \( O(n^4) \). After that, we can compute \( k'' \) in \( O(n^4) \), using the fact that \( M_2(H_i) \) consists of at most two intervals, namely \( \{ \eta_2(H_i), \eta_2(H_i) + 1, \ldots, |V(H_i)| - 2 \} \) and \( \{ |V(H_i)| - 2 \} \). Again, if \( Q \) is non-strong we need only let \( i_j \) assume the value \( pc(H_j) \), respectively \( \eta_2(H_j) \) in the computation of \( k'' \).

It remains to argue that we can also handle the base case of the recursion. By Theorem 2, in the canonical total decomposition of \( Q \) the base case is either a transitive oriented (in particular, acyclic) digraph \( T \) or a strong...
7. Finding complementary cycles

Clearly, if a digraph $D$ has complementary cycles then $k_{\min}(D) \leq 2$. As can be seen from the following two lemmas and Theorem 13, we can actually check whether a quasi-transitive digraph $D$ has complementary cycles also in the case where $D$ is Hamiltonian (i.e. $k_{\min} = 1$).

Lemma 32. For every non-strong, traceable quasi-transitive digraph $D$, the graph $\overline{UG(D)}$ is disconnected.

Proof. Let $S_1, S_2, \ldots, S_k$ ($k > 1$) be the strong components of $D$, numbered such that $S_i \supsetneq S_{i+1}$, $i = 1, \ldots, k-1$. That this is possible follows from the assumption that $D$ is traceable and the easy fact (see [5]) that if $A$ and $B$ are two distinct strong components of a quasi-transitive digraph with $(A, B) \neq \emptyset$ then $A \nsubseteq B$. Also, by quasi-transitivity, the strong component digraph, $SC(D)$, is transitive, so the vertices of $V(S_1)$ are completely adjacent to every vertex in $V(D - S_1) = V(S_2) \cup \cdots \cup V(S_k) \neq \emptyset$, implying that there is no edge between $\overline{UG(S_1)}$ and $\overline{UG(D - S_1)}$ in $\overline{UG(D)}$. □

Lemma 33. Let $C$ be a Hamilton cycle in a quasi-transitive digraph $D = S[Q_1, \ldots, Q_s]$. If, for some $i \in \{1, \ldots, s\}$, $C$ intersects $Q_i$ in at least two paths, then $D$ has a 2-cycle factor which can be found in time $O(n)$. If such an $i$ does not exist then $D \cong S$.

Proof. Suppose that there exists an $i$ as in the lemma and let $x_1x_2\ldots x_k$ and $y_1y_2\ldots y_l$ be two disjoint maximal paths in $C \cap V(Q_i)$. Replacing $(x_1^{-}, x_1)$ by $(x_1^{-}, y_1)$ and $(y_1^{-}, y_1)$ by $(y_1^{-}, x_1)$ then yields a 2-cycle factor in $D$. Checking whether such an $i$ exists can be done in $O(n)$.

If there is no such $i$ then $C$ passes through each $Q_j$ exactly once, so each $Q_j$ is traceable. Now, by Theorem 2, $Q_j$ is either a non-strong quasi-transitive digraph or a single vertex but, by Lemma 32 and the construction of $Q_j$, it cannot be non-strong. Hence, every $Q_j$ is just a single vertex, implying that $D \cong S$. □

Combining Lemma 33 with Theorems 13 and 28 we obtain the following result.

Theorem 34. In $O(n^4)$ we can find complementary cycles in a given quasi-transitive digraph or verify that no such cycles exist.

8. Constructing an irreducible cycle factor

As we have already indicated, it seems difficult in general to determine $k_{\min}(D)$ for a given quasi-transitive digraph. In this section we shall show how to convert any given cycle factor into an irreducible one in polynomial time.

Given a quasi-transitive digraph $D = R[H_1, H_2, \ldots, H_r]$, we define the corresponding flow network $N_R$ to be the digraph with vertex set $V(N_R) = V(R) \cup \{s, t\}$ and arc set $A(N_R) = A(R) \cup \{sv : v \in V(R)\} \cup \{tv : v \in V(R)\}$. In addition, $N_R$ will have prescribed lower and upper bounds on the amount of flow that can pass through each vertex. For terminology and results on flows in networks, see e.g. [1].

We will now show how to use network flows to check whether a given cycle factor in a strong quasi-transitive digraph $D$ is irreducible. This is done by checking recursively, according to the canonical total decomposition of $D$, for possible reductions. We start with an example which illustrates the method.

Let $D$ be the quasi-transitive digraph in Fig. 2 and let $\mathcal{F} = C_1 \cup \cdots \cup C_7$ be the cycle factor which consists of the six small cycles and one large cycle which pick up all the maximal subpaths shown inside $Q_1, \ldots, Q_5$ (in the order
Let $D$ and let $F = P_1 \cup \cdots \cup P_p \cup C_1 \cup \cdots \cup C_{q_0} \cup \cdots \cup C_{q_r}$ be a $p$-path-$q$-cycle factor of $D$ where $C_1, \ldots, C_{q_0}$ are the large cycles (possibly $q_0 = 0$). For each $i = 1, 2, \ldots, r$ denote by $F_i$ the path-cycle factor of $Q_i$ consisting of those small cycles starting in $v_i$.
cycles of $\mathcal{F}$ that are in $Q_i$ and those maximal subpaths of $P_1 \cup \cdots \cup P_p \cup C_1 \cup \cdots \cup C_{q_0}$ which are contained in $Q_i$. Denote by $q_i$, $p_i$ the number of cycles, respectively paths, of $\mathcal{F}_i$ and let $J = \{i \mid q_i > 0\}$. Finally, denote by $Q'_i$ the subdigraph of $Q_i$ induced by the vertices on the paths of $\mathcal{F}_i$. Assign lower and upper bounds $l(v_i) = pc(Q'_i)$ and $u(v_i) = n_i$ to $N_R$. Suppose that $\mathcal{F}$ is not a Hamiltonian path in $D$. Then $\mathcal{F}$ is irreducible if and only if each of the following holds:

(a) If $p = 0$ and $D$ is strong then $\mathcal{F}$ has precisely one large cycle, $C$, and, for every $i \in J$, $C$ intersects $Q_i$ in precisely $m_i(D)$ paths. If $p > 0$ then all cycles of $\mathcal{F}$ are small.
(b) $\mathcal{F}_i$ is an irreducible $p_i$-path-$q_i$-cycle factor of $Q_i$, for every $i \in J$.
(c) There is no feasible integer flow of value less than $p$ in $N_R$ and every feasible flow of value $p$ in $N_R$ has value $l(v_i)$ on vertex $v_i$ whenever $i \in J$.

**Proof.** Suppose that $\mathcal{F}$ is irreducible.

It follows from the proofs of Lemmas 18, 20 and 29 that (a) must hold. If (b) does not hold then let $i \in J$ be chosen so that $\mathcal{F}_i$ is reducible. Thus $Q_i$ contains a $k$-path-$h$-cycle factor $\mathcal{F}'_i$ with $k \leq p_i$, $h \leq q_i$, $k + h < p_i + q_i$ and which satisfies properties (b) and (c) of Definition 17. If $k = p_i$ then $h < q_i$, so by replacing $\mathcal{F}$ the original $p_i$ paths ($q_i$ cycles) of $\mathcal{F}_i$ by the $p_i$ paths ($h$ cycles) of $\mathcal{F}'_i$ we obtain a reduction of $\mathcal{F}$ (note that, since (a) holds, the $p_i$ paths of $\mathcal{F}_i$ either all lie on the same large cycle of $\mathcal{F}$ or on paths of $\mathcal{F}$, so $\mathcal{F}$ satisfies (b) and (c) of Definition 17). Suppose, instead, $k < p_i$ and note that, by (c) of Definition 17, the $k$ paths of $\mathcal{F}'_i$ contain all vertices from the $p_i$ paths of $\mathcal{F}_i$. Hence, by deleting $p_i - k - 1$ arbitrary arcs on the paths of $\mathcal{F}'_i$ and one arc on one of the small cycles of $\mathcal{F}'_i$, we obtain a new path-cycle factor $\mathcal{F}''_i$ of $Q_i$ which has exactly $p_i$ paths and $h - 1 < q_i$ cycles. By the same arguments as above, substituting $\mathcal{F}''_i$ for $\mathcal{F}_i$ again yields a reduction of $\mathcal{F}$. Thus, (b) must hold.

Now, suppose that (c) does not hold and let $x$ be a feasible integer flow (wrt. the given lower and upper bounds) of value $d \leq p$ in $N_R$ which shows that (c) fails. That is, either $d < p$ or $x(v_i) > l(v_i)$, for some $i \in J$.

Consider first the case $p \leq d > 0$. By standard flow decomposition (see [1, Section 3.3]) we can decompose $x$ into $d$ path flows of value one along $(s, t)$-paths and some cycle flows. By the choice of lower and upper bounds, the total flow along these paths and cycles through the vertex $v_i$ is at least $pc(Q'_i)$ and at most $n_i$. This implies that by introducing $x(v_i)$ copies of $v_i$ we can convert these paths and cycles into a path-cycle subdigraph $\mathcal{F}^*$ of $E(D)$ with $d$ paths and $s \geq 0$ cycles. If $s > 0$ then $D$ is necessarily strong and, by Lemma 14, we can convert $\mathcal{F}^*$ into a $d$-path subdigraph $\mathcal{F}^{**}$ of $E(D)$ in time $O(n^2)$. If $s = 0$, put $\mathcal{F}^{**} = \mathcal{F}^*$. By replacing, for each $i$, the $x(v_i)$ copies of $v_i$ in $\mathcal{F}^{**}$ by a collection of $x(v_i)$ paths of $Q_i$ which cover all vertices of $Q'_i$ and all vertices of zero or more of the small cycles from $\mathcal{F}_i$ and finally adding all small cycles of $\mathcal{F}$ that were not covered already, we obtain a $d$-path-$e$-cycle factor $\mathcal{F}^{(1)}$ of $D$ with $e \leq q$ and which satisfies (b) and (c) of Definition 17. Thus, if $d < p$ then $\mathcal{F}^{(1)}$ is a reduction of $\mathcal{F}$. If $d = p$ then, since $x$ violates (c), we must have $x(v_i) > l(v_i)$, for some $i$, so by the construction above, at least one small cycle of $\mathcal{F}$ was merged into a path of $\mathcal{F}^{(1)}$. Hence, $e < q$ and again $\mathcal{F}^{(1)}$ is a reduction of $\mathcal{F}$.

Next, consider the case $p > d = 0$. Since $d = 0$, $x$ is a feasible circulation in $N_R$ and, since $p > 0$, some $v_i$ has a positive lower bound, so $N_R$ contains a cycle and $D$ must therefore be strong. By (a), $\mathcal{F}$ has no large cycle and if it has no small cycle either (i.e., it is a path factor) then every $l(v_i)$ is positive so $x$ can be decomposed into flows of value one along a spanning set of (not necessarily disjoint) cycles of $N_R$. As above, by introducing $x(v_i)$ copies of $v_i$ for every $i \in \{1, 2, \ldots, r\}$, we can convert these cycles into a cycle subdigraph $\mathcal{F}^*$ inducing a strong subgraph of $E(D)$ with at least one cycle and, applying Theorem 5, we can merge the cycles of $\mathcal{F}^*$ into one cycle $\hat{C}$. By replacing, for each $i$, the $x(v_i)$ copies of $v_i$ on $\hat{C}$ by an $x(v_i)$-path factor of $Q_i$, we obtain a Hamilton cycle of $D$. Thus, $D$ has a Hamilton path which constitutes a reduction of $\mathcal{F}$, since $p > 1$ (recall that $\mathcal{F}$ is not a Hamilton path). This contradicts the irreducibility of $\mathcal{F}$, so $\mathcal{F}$ must have at least one small cycle. As we did above, from the decomposition of $x$ into cycle flows of value one we can construct a cycle subdigraph $\mathcal{F}^{(1)}$ of $D$ consisting of large cycles and covering all vertices from the paths of $\mathcal{F}$ and possibly (every vertex of) some small cycles of $\mathcal{F}$. $\mathcal{F}^{(1)}$ is not spanning, because that would imply (by Theorem 5) that $D$ has a Hamilton path, i.e., a reduction of $\mathcal{F}$. On the other hand, if $C$ is a small cycle from $\mathcal{F}$ whose vertices are not in $V(\mathcal{F}^{(1)})$ then (by Lemma 14) $D$ has a 1-path-$q$-cycle factor whose path has vertex set $V(C) \cup V(\mathcal{F}^{(1)})$ and whose $\hat{q} < q$ cycles are the remaining small cycles of $\mathcal{F}$ not covered by $\mathcal{F}^{(1)}$. Again, this path-cycle factor would be a reduction of $\mathcal{F}$.

Finally, suppose that $p = 0$. Clearly, no flow has value less than $p$. If $D$ is non-strong then $N_R$ is acyclic and, since $x$ has value 0, $x(v_i) = 0 = l(v_i)$, for all $i$, so (c) is satisfied, contradicting our assumption. Hence $D$ is strong. Since
Lemma 35. \( F \) has exactly one large cycle which intersects every \( Q_i \) \( m_i(D) > 0 \), for all \( i \in J \). Hence, every \( l(v_i) \) is positive and since \( x \) has value zero it is a circulation and can be decomposed into flows of value one along a spanning set of (not necessarily disjoint) cycles of \( N_R \). By introducing \( x(v_i) \) copies of \( v_i \), we can convert these cycles into a cycle subdigraph \( F^* \) inducing a strong subdigraph of \( E(D) \) with at least one cycle and, applying Theorem 5, we can merge the cycles of \( F^* \) into one cycle \( \hat{C} \). By replacing, for each \( i \), the \( x(v_i) \) copies of \( v_i \) on \( \hat{C} \) by a collection of \( x(v_i) \) paths of \( Q_i \) which cover all vertices of \( Q_i \) and all vertices of zero or more of the small cycles from \( F_i \), and finally adding all small cycles of \( F \) that were not covered already, we obtain an \( e \)-cycle factor \( F^{(1)} \) of \( D \) with \( e \leq q \) and which satisfies (b) and (c) of Definition 17. But, since \( x \) violates (c), \( x(v_i) > l(v_i) \) for some \( i \), so at least one small cycle of \( F \) was merged into the large cycle of \( F^{(1)} \); hence \( e < q \) and \( F^{(1)} \) is a reduction of \( F \).

Thus, (c) holds and we have proven the necessity of (a), (b), and (c). To prove the sufficiency of (a), (b), and (c) assume that \( F \) is reducible; we shall show that at least one of (a), (b), and (c) fails.

Suppose that (a) holds, let \( F^{(1)} \) be a \( p' \)-path-\( q' \)-cycle factor which is a reduction of \( F \) and consider the path-cycle subdigraph \( F^{(2)} \) which we obtain from \( F^{(1)} \) by deleting all of its small cycles. By (b) and (c) of Definition 17, no vertex of \( V(Q'_j), \ i = 1, 2, \ldots, r \) belongs to a small cycle of \( F^{(1)} \). Hence, the collection of paths and large cycles of \( F^{(1)} \) (hence also \( F^{(2)} \)) must intersect \( Q_i \) in at least \( pc(Q'_j) = l(v_i) \) paths. Therefore, by sending one unit of flow along each of the paths and cycles in \( N_R \) which we obtain from the paths and cycles in \( F^{(2)} \) after contracting each \( Q_i \) into one vertex \( v_i \), we obtain a feasible flow \( x' \) (wrt. the given lower and upper bounds) of value \( p' \) in \( N_R \). Now, either (c) does not hold, and we are done, or \( p' = p \) and \( x'(v_i) = l(v_i) \), for every \( i \in J \). It remains to show that (b) does not hold in this case.

Suppose first that \( p = p' = 0 \). If \( D \) is non-strong then \( F \) has only small cycles, so in some \( Q_i \), \( F^{(1)} \) has fewer small cycles than \( F \). Hence, \( F^{(1)} \) restricted to \( Q_i \) is a reduction of \( F_i \), showing that (b) does not hold. If \( D \) is strong then, by (a), \( F \) has exactly one large cycle \( C \). Since \( p' = p = 0 \) and \( F^{(1)} \) is a reduction of \( F \) it follows from Definition 17 that \( F^{(1)} \) also has a large cycle \( C^{(1)} \). Therefore, as \( q' < q \), \( F^{(1)} \) has fewer small cycles than \( F \) in some \( Q_j \) and, since \( F^{(1)} \) intersects \( Q_j \) in exactly \( pc(Q'_j) \leq p_j \) paths, the restriction of \( F^{(1)} \) to \( Q_j \) is a reduction of \( F_j \), showing that (b) does not hold.

Finally, suppose that \( p = p' > 0 \). As \( q' < q \) and (a) holds for \( F \), \( F^{(1)} \) contains fewer small cycles than \( F \) in some \( Q_j \) and, since \( F^{(1)} \) intersects \( Q_j \) in exactly \( pc(Q'_j) \leq p_i \) paths, the restriction of \( F^{(1)} \) to \( Q_j \) is a reduction of \( F_j \), showing that (b) does not hold.

Note that if \( F \) is a Hamilton path (and hence irreducible) it may not satisfy condition (c) of Lemma 35.

Theorem 36. There is an \( O(n^5) \) algorithm which, given a path-cycle factor \( F \) in a quasi-transitive digraph \( D = R[H_1, H_2, \ldots, H_r] \), either confirms that \( F \) is irreducible or returns a reduction of \( F \). Hence in time \( O(n^6) \) any given path-cycle factor can be converted into an irreducible one.

Proof. We shall use the terminology of Lemma 35. Let \( F \) be a \( p \)-path-\( q \)-cycle factor in \( D = R[H_1, H_2, \ldots, H_r] \). If \( F \) is a Hamilton path of \( D \), it is irreducible and we are done. Otherwise, by Lemma 35, \( F \) is irreducible if and only if (a), (b), and (c) hold. It follows from the proofs of Lemmas 18 and 29 that checking whether (a) holds and, if it does not hold, producing a reduction of \( F \) can be done in time \( O(n^3) \) by applying the algorithm for finding a longest cycle in \( E(D) \) (and hence the numbers \( m_i(D), i = 1, 2, \ldots, r \)) and the algorithms of Lemma 14 and Theorem 6. So assume that (a) holds.

Now, find the canonical total decomposition of \( D \) in \( O(n^3) \). Let \( p_i \) and \( q_i \) be defined as in Lemma 35 and let \( J \) be the set of indices \( i \in \{1, 2, \ldots, r\} \) such that \( q_i > 0 \). Compute, for those \( i \in J \) such that \( p_i > 0 \), \( pc(Q'_j) \) as well as the path covering numbers for all quasi-transitive subdigraphs encountered when constructing the canonical total decomposition of all \( Q_j \)'s in total time \( O(n^4) \). Then define lower and upper bounds on the vertices of \( N_R \) as in Lemma 35.

To check whether (c) holds, we first find a minimum value feasible flow, \( x \), in \( N_R \). This can be done in time \( O(n^3 \) ([1, Section 3.9])). If \( x \) has value \( d < p \), we can construct a reduction \( F' \) from \( x \) as in the proof of Lemma 35 and we are done, so assume that \( d = p \). Now, checking whether \( N_R \) has any flow of value \( p \) which satisfies \( x(v_i) > l(v_i) \), for some \( i \in J \), can be done using a minimum cost flow calculation: form a new network \( N'_R \) by splitting each vertex \( v_i, i \in J \), into two similar vertices \( v_{i,1}, v_{i,2} \) and assign bounds \( l(v_{i,1}) = u(v_{i,1}) = l(v_i) \) and \( l(v_{i,2}) = 0, u(v_{i,2}) = u(v_i) - l(v_i) \). Finally, assign cost zero to each of \( v_{1,1}, \ldots, v_{1,1} \) and cost minus one to each of \( v_{1,2}, \ldots, v_{2,2} \). Now it is easy to check that every feasible flow in \( N_R \) corresponds to a feasible flow of the same value in \( N'_R \) and
vice versa. Furthermore, \( N'_R \) has a flow of value \( p \) and of negative cost if and only if the second part of (c) is violated by some flow, \( x' \), in \( N_R \). Since one iteration of the cycle cancelling algorithm (see [1, section 3.10]) is sufficient to establish a flow of negative cost, starting from the flow \( x \) in \( N_R \), the complexity of this part is bounded by the time to construct \( N'_R \) and check for a negative cycle in the residual network with respect to \( x \) (modified to be a flow in \( N'_R \)). This can be done in time \( O(n^3) \) by standard techniques. If we find a negative cycle, we can use the corresponding flow, \( x'' \), which we obtained from \( x \) by augmenting along this cycle, to modify \( F \) and return the reduction \( F' \). Thus, the total time for checking whether (c) is violated and, if it is, constructing a reduction is bounded by \( O(n^3) \). So suppose that (c) holds.

It remains to check whether (b) holds. We can do this by making recursive calls on each \( H_j, j \in J \). The recursion stops when the current quasi-transitive digraph is transitive or a strong semicomplete digraph. Every strong semicomplete digraph has a Hamiltonian cycle which can be found in time \( O(n^2) \), and we can find a minimum path factor of a transitive oriented graph in time \( O(n^2) \) (see [1, Thm. 5.3.1]). It follows that the total time to make the recursive calls is \( T(n) \leq O(n^4) + \sum_{i=1}^{|J|} T(|V(H_i)|) \in O(n^5) \). □

It is easy to see that an irreducible cycle factor \( F \) with \( c \leq 2 \) cycles in a strong quasi-transitive digraph \( D \) is also minimum: if \( c = 2 \), every Hamilton cycle in \( D \) would be a reduction of \( F \). It is thus natural at this point to ask whether every irreducible cycle factor in a strong quasi-transitive digraph \( D \) is also minimum or, at least, close to being minimum. Unfortunately, this is not always the case. In fact, as the example in Fig. 4 shows, \( F \) may be irreducible and have arbitrarily many cycles even though \( \lambda_{\text{min}}(D) = 2 \).

When considering the minimum cycle factor problem an obvious approach is to start from an arbitrary cycle factor and then try to reduce the number of cycles by some merging process involving two or more cycles. For some classes of digraphs, e.g. extended semicomplete (cf. Theorem 5) and semicomplete bipartite [9] digraphs, this line of attack has indeed proven successful. As we have just seen, however, for quasi-transitive digraphs the notion of irreducibility, as given by Definition 17, does not coincide with that of minimality. One may well ask whether this could be remedied by a different definition of irreducibility that still allows for algorithmic results such as those of Lemma 35 and Theorem 36. It seems that any natural such definition, which is not merely stating that \( F \) is minimum, should include something like the second requirement in (b) of Definition 17 and, obviously, the second requirement of (a). We shall now try to account for the various other requirements of Definition 17. In the following, given a path-cycle factor, \( F \) (or \( F' \)), in a quasi-transitive digraph, \( D = R(Q_1, Q_2, \ldots, Q_r) \), we let \( p_i \) and \( q_i \) (or \( p'_i \) and \( q'_i \)) be defined as in Lemma 35. Our definition is motivated by the recursive structure of \( D = R(Q_1, Q_2, \ldots, Q_r) \), i.e. the canonical total decomposition, which suggests the requirement that if \( F \) is a \( q \)-cycle factor in \( D \) and \( F' \) is a reduction of \( F \) such that \( p'_i \leq p_i, q'_i \leq q_i \) and \( p'_i + q'_i < p + q \) (for some \( i = 1, 2, \ldots, r \)) then \( F'_i \) should be a reduction of \( F_i \). Unless we also introduce the first part of (b) in the definition, this will not always be the case, since a small cycle which is merged into a large cycle of \( F' \) may have its vertices distributed on several paths of \( F'_i \). Furthermore, we must extend the definition to path-cycle factors with arbitrarily many paths (not just zero), since in general \( F_i \) will be a path-cycle factor of \( Q_i \). It is also desirable to require that a reduction, \( F'_i \), of \( F_i \), for some \( i \), implies a reduction of \( F \) obtained by substituting \( F'_i \) for \( F_i \) in \( F \). This necessitates the first part of (a) as well as (c) in the definition, since

![Fig. 4. A strong quasi-transitive digraph D = S(Q1, Q2, Q3) with an irreducible but non-minimum cycle factor. In Q1 there are no arcs between Ck+1 and the other cycles, and V(C1) \( \rightarrow \) V(Cj) if and only if 1 \( \leq i < j \leq k \). Let F be the (k + 1)-cycle factor consisting of the cycles C1, C2, . . . , Ck and a large cycle covering V(Ck+1) \( \cup \) V(Q2) \( \cup \) V(Q3). Then F is irreducible (since V(Ck+1) must remain on a large cycle in any potential reduction) but there is a 2-cycle factor consisting of Ck+1 and a large cycle.](image-url)
otherwise \( F_i' \) would be allowed to merge some vertices from a large cycle, \( C \), of \( F \) into small cycles of \( Q_i \) implying that \( C \) might violate (b).

It seems an interesting research problem to derive further properties of irreducible path-cycle factors in quasi-transitive digraphs and combine these with new ideas showing how to convert some irreducible but not minimum cycle factor into a cycle factor with fewer cycles. A similar approach enabled the authors of [2] to solve the Hamiltonian cycle problem for semicomplete multipartite digraphs starting from results by Yeo on irreducible cycle factors in this class of digraphs (see [1, Section 5.7]).

References