Almost fully decomposable infinite rank lattices over orders

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Abstract

Let be an order over a Dedekind domain with quotient field . An object of , the category of -projective -modules, is said to be fully decomposable if it admits a decomposition into (finitely generated) -lattices. In a previous article [W. Rump, Large lattices over orders, Proc. London Math. Soc. 91 (2005) 105–128], we give a necessary and sufficient criterion for -orders in a separable -algebra with the property that every -module is fully decomposable. In the present paper, we assume that is separable, but that the -adic completion is not semisimple for at least one . We show that there exists such that admits a decomposition with , where is fully decomposable, but itself is not fully decomposable.

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0. Introduction

Infinite rank lattices over orders form a rather new subject of study. For a cyclic group of prime order , it was shown by Butler, Campbell, and Kovács [3] that every -free -module decomposes into -lattices. The case was settled a little earlier by Butler and Kovács [4], and independently by Benson in a joint paper with Kumjian and Phillips on -theory of -algebras [2].

Let be a Dedekind domain with quotient field , and let be an -order in a finite dimensional -algebra. A -module is said to be a -lattice [11] if is finitely generated and projective as an -module. If the finiteness condition is dropped, i.e. if is just a projective -module, then is said to be a generalized -lattice [3]. The category of generalized -lattices will be denoted by , and its full subcategory of -lattices by . We call an object of a -lattice fully decomposable if it admits a decomposition into -lattices . Butler, Campbell, and Kovács have shown ([3], Theorem 2.1) that in case is lattice-finite, every object of is a direct summand of a fully decomposable one. Thus if is a complete discrete valuation domain, the Crawley–Jønsson–Warfield theorem ([1], Theorem 26.5) then implies that every generalized -lattice is fully decomposable. For such , the converse is also true, i.e. has to be lattice-finite if every is fully decomposable [14]. If is not lattice-finite, however, there even exists an indecomposable object in which is not finitely generated [14].

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In the global case, the situation is much more delicate. In [15] we associate a hypergraph \( H(A) \) to any \( \Lambda \)-order \( A \) in a separable \( K \)-algebra \( A \), which decides whether every generalized \( \Lambda \)-lattice is fully decomposable. For example, the latter property holds for the group ring \( \mathbb{Z}C_{p^2} \) of a cyclic \( p \)-group of order \( p^2 \). However, there are plenty of \( R \)-orders \( \Lambda \), even lattice-finite ones, with generalized \( \Lambda \)-lattices \( L \) which are not fully decomposable. For such \( L \in \Lambda \text{-Lat} \), the decomposability behaviour can be quite different. For example, there are \( R \)-orders \( \Lambda \) (see [15], Example 2) which admit a projective \( L \in \Lambda \text{-Lat} \) without non-zero \( \Lambda \)-lattices as direct summands, such that \( L_p \cong L_{(p)} \) for each maximal ideal \( p \in \text{Spec} \ R \). On the other hand, it may happen that every non-finitely generated generalized \( \Lambda \)-lattice has a fully decomposable direct summand of infinite rank, but need not be fully decomposable itself ([15], Example 1).

In the present article, we deal with \( R \)-orders \( \Lambda \) in a finite dimensional \( K \)-algebra \( A \) which do not have a maximal overorder. By a theorem of Drozd [5], this happens if the algebra \( A_p := K_p \otimes_K A \) over the \( p \)-adically complete field \( K_p := R_p \otimes_R K \) is not semisimple for some \( p \in \text{Spec} \ R \). We define \( L \in \Lambda \text{-Lat} \) to be \textit{almost fully decomposable} if the \( A \)-module \( KL := K \otimes_R L \) admits a decomposition \( KL = M_0 \oplus M_1 \) with \( M_0 \) finitely generated, such that \( L \cap M_1 \) is fully decomposable. Our main result (Theorem 2) states that if \( A/\text{Rad} \ A \) is separable, and \( A_p \) is not semisimple for some \( p \in \text{Spec} \ R \), there exists an almost fully decomposable \( L \in \Lambda \text{-Lat} \) which is not fully decomposable. To prove this, we show first that \( \Lambda \) has an overorder \( \Lambda' = \Lambda_0' \oplus (\Lambda' \cap \text{Rad} \ A) \), such that \( \Lambda_0' \) is a maximal order. For a suitable block \( I \) of \( \Lambda_0' \), we construct a dense functor (Theorem 1)

\[
C : \Lambda' \text{-Lat} \rightarrow \Gamma \text{-Mod}.
\]

More precisely, for any \( \Gamma \)-module \( M \), there is a projective presentation \( L_1 \hookrightarrow L_2 \twoheadrightarrow M \), and the inclusion \( L_1 \hookrightarrow L_2 \) gives rise to a generalized \( T_2(\Gamma) \)-lattice \( L_M \), where \( T_2(\Gamma) \) denotes the triangular matrix order

\[
T_2(\Gamma) := \begin{pmatrix} \Gamma & 0 \\ \Gamma & \Gamma \end{pmatrix}.
\]

We define a functor

\[
C' : T_2(\Gamma) \text{-Lat} \rightarrow \Lambda' \text{-Lat},
\]

such that \( CC'(L_M) \cong M \) for all \( M \in \Gamma \text{-Mod} \). If we choose \( M \) to be an \( R \)-torsion indecomposable injective \( \Gamma \)-module, then \( C'(M) \in \Lambda' \text{-Lat} \subset \Lambda \text{-Lat} \) is almost fully decomposable, but not fully decomposable (Theorem 1). Note that the \( R \)-torsion indecomposable injective modules over a maximal order \( \Gamma \) look rather similar to the Prüfer groups \( \mathbb{Z}(p^\infty) \in \mathbb{Z} \text{-Mod} \) (Corollary of Proposition 8).

In a sense, triangular matrix orders \( \Lambda = T_2(\Gamma) \) over a maximal \( R \)-order \( \Gamma \) are the simplest type of \( R \)-orders in a non-semisimple \( K \)-algebra. They are most suitable to explain the nature of almost fully decomposable \( \Lambda \)-lattices. Here the relationship between \( \Gamma \)-modules and their projective presentations gives rise to an equivalence

\[
\Gamma \text{-Mod} \cong T_2(\Gamma) \text{-Lat}
\]

between the category \( \Gamma \text{-Mod} \) of \( \Gamma \)-modules and the “stable” category of \( T_2(\Gamma) \text{-Lat} \) modulo injective objects (Proposition 6). By the way, for an arbitrary \( R \)-order \( \Lambda \), the above mentioned argument ([3], Theorem 2.1) implies that every injective object of \( \Lambda \text{-Lat} \) is a direct summand of a coproduct \( \bigsqcup_{i \in I} E_i \) of injective \( \Lambda \)-lattices \( E_i \) (see Proposition 1). Now if \( M \in \Gamma \text{-Mod} \) does not decompose into finitely generated \( \Gamma \)-modules, the corresponding object \( L_M \in T_2(\Gamma) \text{-Lat} \) given by a projective presentation \( L_1 \hookrightarrow L_2 \twoheadrightarrow M \) cannot be fully decomposable. If \( M \) is not finitely generated, then \( L_2 \) cannot be finitely generated, and that part of the invariant factor theorem that survives in the infinite rank case, implies that the inclusion \( L_1 \hookrightarrow L_2 \) which represents \( L_M \) splits off an infinitude of (non-zero) finitely generated direct summands \( E_1 \hookrightarrow E_2 \). So far, this phenomenon occurs for all \( M \) which are not finitely generated (Proposition 7). If, in particular, \( M \) is chosen to be indecomposable and injective, the corresponding object \( L_M \in T_2(\Gamma) \text{-Lat} \) will be almost fully decomposable.

1. The category \( \Lambda \text{-Lat} \)

We start with some general terminology for an additive category \( \mathcal{A} \). Recall that a pair of morphisms \( X \xrightarrow{a} Y \xrightarrow{b} Z \) in \( \mathcal{A} \) is said to be a \textit{short exact sequence} if \( a = \ker b \) and \( b = \cok a \). We indicate kernels in \( \mathcal{A} \) by \( \hookrightarrow \) and cokernels by \( \twoheadrightarrow \). An object \( Q \) of \( \mathcal{A} \) is said to be \textit{projective} (injective) if for each short exact sequence \( X \xrightarrow{a} Y \xrightarrow{b} Z \) in \( \mathcal{A} \),
every morphism \( Q \rightarrow Z \) factors through \( b \) (resp. every morphism \( X \rightarrow Q \) factors through \( a \)). The full subcategory of projective (injective) objects will be denoted by \( \text{Proj}(\mathcal{A}) \) (resp. \( \text{Inj}(\mathcal{A}) \)). We will say that \( \mathcal{A} \) has enough projectives (in a strict sense) if for each object \( X \) of \( \mathcal{A} \), there exists a cokernel \( P \rightarrow X \) with \( P \in \text{Proj}(\mathcal{A}) \). Similarly, we say that \( \mathcal{A} \) has enough injectives if for each \( X \in \text{Ob} \mathcal{A} \), there is a kernel \( X \leftarrow I \) with \( I \in \text{Inj}(\mathcal{A}) \). For a full subcategory \( \mathcal{E} \) of \( \mathcal{A} \), the ideal of \( \mathcal{A} \) generated by the identical morphisms \( 1_C \) with \( C \in \text{Ob} \mathcal{E} \) will be denoted by \( [\mathcal{E}] \). By \( \text{add} \mathcal{E} \) we denote the full subcategory of objects \( C \in \text{Ob} \mathcal{A} \) with \( 1_C \in [\mathcal{E}] \).

For a ring \( R \), we write \( R\text{-Mod} \) (resp. \( R\text{-mod} \)) for the category of all (resp. finitely presented) left \( R \)-modules. More generally, the coherent functors from \( \mathcal{A}^{op} \) to the category \( \text{Ab} \) of abelian groups can be regarded as an additive category and will be denoted by \( \text{mod}(\mathcal{A}) \). Thus \( R\text{-Mod} \cong \text{mod}(R\text{-Proj}) \), where \( R\text{-Proj} \coloneqq \text{Proj}(R\text{-Mod}) \). There is an equivalent, more explicit description of \( \text{mod}(\mathcal{A}) \) (see [7,12] or [13], Section 2). Let \( \text{Mor}(\mathcal{A}) \) be the additive category with morphisms \( A_1 \xrightarrow{a} A_0 \) in \( \mathcal{A} \) as objects, and commutative squares

\[
\begin{array}{ccc}
A_1 & \xrightarrow{f_1} & B_1 \\
\downarrow{a} & & \downarrow{b} \\
A_0 & \xrightarrow{f_0} & B_0
\end{array}
\]

(1)
as morphisms \( a \rightarrow b \). If \( \mathcal{E} \) denotes the full subcategory of \( \text{Mor}(\mathcal{A}) \) with split epimorphisms \( A_1 \rightarrow A_0 \) as objects, the ideal \([\mathcal{E}]\) consists of the morphisms (1) in \( \text{Mor}(\mathcal{A}) \) which admit a morphism \( h: A_0 \rightarrow B_1 \) with \( f_0 = bh \). Then \( \text{mod}(\mathcal{A}) \) can be represented as a factor category

\[
\text{mod}(\mathcal{A}) \cong \text{Mor}(\mathcal{A})/[\mathcal{E}].
\]

(2)

From now on, let \( R \) be a Dedekind domain with quotient field \( K \), and let \( \Lambda \) be an \( R \)-order [11] in a finite dimensional \( K \)-algebra \( A \). By \( \Lambda\text{-Lat} \) we denote the additive category of generalized \( \Lambda \)-lattices, that is, \( \Lambda \)-modules \( L \) which are projective over \( R \). The objects \( E \) of \( \Lambda\text{-Lat} \) which are finitely generated over \( R \) form the full subcategory \( \Lambda\text{-lat} \) of \( \Lambda\text{-lattices} \). For a generalized \( \Lambda \)-lattice \( L \), the natural homomorphism \( L \rightarrow K \otimes_R L \) is monic since \( K \) is a direct summand of a free \( R \)-module. Therefore, we have a natural embedding \( L \hookrightarrow KL := K \otimes_R L \). In particular, the \( K \)-algebra \( A \) can be identified with \( KA \). For a generalized \( \Lambda \)-lattice \( L \), the cardinal \( \rho(L) \coloneqq \dim KL \) will be called the rational rank of \( L \). If \( \rho(L) \geq N_0 \), we call \( L \) a large \( \Lambda \)-lattice. A generalized \( \Lambda \)-lattice \( L \) is said to be fully decomposable [15] if \( L \cong \bigsqcup E_i \) with \( E_i \in \Lambda\text{-lat} \).

**Remark.** The category \( \Lambda\text{-Lat} \) has kernels, but not every morphism has a cokernel. For example, consider a free presentation \( \mathbb{Z}^1 \xrightarrow{f} \mathbb{Z}^J \rightarrow \mathbb{Z}^{N_0} \) of the Baer–Specker group. Then \( f \) has no cokernel in \( \Lambda\text{-Lat} \). Otherwise, \( \mathbb{Z}^{N_0} \) would have a free direct summand \( \mathbb{Z}^{(N_0)} \) of countable infinite rank, which is impossible by Sasiada’s theorem ([18], Proposition 94.2).

Notice that \( \Lambda\text{-Lat} \) has coproducts. A sequence of morphisms \( L' \xrightarrow{a} L \xrightarrow{b} L'' \) in \( \Lambda\text{-Lat} \) is short exact if and only if it is short exact in \( \Lambda\text{-Mod} \).

**Proposition 1.** The category \( \Lambda\text{-Lat} \) has enough projectives and enough injectives. An object \( L \) of \( \Lambda\text{-Lat} \) is projective (injective) if and only if \( L \) is a direct summand of a coproduct \( \bigsqcup_i Q_i \) with \( Q_i \in \text{Proj}(\Lambda\text{-lat}) \) (resp. \( Q_i \in \text{Inj}(\Lambda\text{-lat}) \)).

**Proof.** For a given \( L \in \Lambda\text{-Lat} \), there is a surjection of a free \( \Lambda \)-module \( \Lambda^J \) onto \( L \), hence a short exact sequence \( L' \rightarrow \Lambda^J \rightarrow L \). Thus \( \Lambda\text{-Lat} \) has enough projectives. Furthermore, we have an \( R \)-split monomorphism \( \Lambda \cong \text{Hom}_{\Lambda}(\Lambda, L) \hookrightarrow \text{Hom}_{R}(\Lambda, L) \cong \Lambda^* \otimes_R L \), where \( \Lambda^* \coloneqq \text{Hom}_R(\Lambda, R) \). To show that \( \Lambda^* \in \text{Inj}(\Lambda\text{-Lat}) \), let \( c: L_1 \rightarrow L_2 \) be a kernel in \( \Lambda\text{-Lat} \). Then the isomorphism \( \text{Hom}_\Lambda(L_i, \Lambda^*) \cong \text{Hom}_R(L_i, R) \) is natural in \( L_i \). Since \( c \) is a split monomorphism in \( R\text{-Mod} \), this implies that \( \Lambda^* \in \text{Inj}(\Lambda\text{-Lat}) \). Thus if \( R \) is a direct summand of \( R^J \), we get a sequence

\[
L \xrightarrow{c} (\Lambda^*)^J \xrightarrow{f} (\Lambda^*)^J
\]

(3)

with \( c = \ker f \). Hence \( \Lambda\text{-Lat} \) has enough injectives. The remaining assertions follow immediately. \( \square \)
Corollary. The categories $\text{Proj}(A\text{-}\text{Lat})$ and $\text{Inj}(A\text{-}\text{Lat})$ are equivalent.

Proof. The Nakayama functor $P \mapsto \text{Hom}_A(P, A)^*$ gives an equivalence $\text{add}(A) \cong \text{add}(A^*)$. Now the corollary follows by [15], Proposition 1. □

Definition. We call a generalized $A$-lattice $L$ almost fully decomposable if there is a decomposition $KL = M_0 \oplus M_1$ of $A$-modules with $M_0$ finitely generated, such that $L \cap M_1$ is fully decomposable.

We are interested in almost fully decomposable generalized $A$-lattices $L$ which are not fully decomposable. For such $L$, the projection into $M_0$ cannot be finitely generated.

Proposition 2. Let $L$ be a generalized $A$-lattice with a fully decomposable submodule $L' \in A\text{-}\text{Lat}$ such that $L/L'$ is finitely generated. Then $L$ is fully decomposable.

Proof. There is a short exact sequence $L' \hookrightarrow L \twoheadrightarrow E$ with $E$ finitely generated. Hence there exists a finitely generated submodule $F$ of $L$ with $L' + F = L$. This gives a commutative diagram

$$
\begin{array}{ccc}
L' \cap F & \hookrightarrow & F \\
\downarrow & & \downarrow \\
L' & \overset{e}{\hookrightarrow} & L \\
\downarrow & & \downarrow \\
L' & \twoheadrightarrow & E
\end{array}
$$

with short exact rows. Assume that $L' = \bigsqcup_{i \in I} E_i$ with $E_i \in A\text{-}\text{Lat}$. Since $L' \cap F$ is finitely generated, there is a finite subset $J \subset I$ with $L' \cap F \subset \bigsqcup_{i \in J} E_i$. Consequently, the projection $p': L' \to \bigsqcup_{i \in J} E_i$ factors through the inclusion $e$. So we get a split epimorphism $q: L \to \bigsqcup_{i \in J} E_i$ and a short exact sequence $\bigsqcup_{j \in J} E_j \hookrightarrow \text{Ker } q \twoheadrightarrow E$ which shows that $\text{Ker } q$ is finitely generated. Hence $L$ is fully decomposable. □

The following proposition shows that almost full decomposability behaves nicely with respect to $\text{Hom}$-functors.

Proposition 3. Let $E$ be a $A$-lattice with $\Gamma := \text{End}_A(E)^{\text{op}}$, and let $L \in A\text{-}\text{Lat}$ be almost fully decomposable. Then $\text{Hom}_A(E, L) \in \Gamma\text{-}\text{Lat}$ is almost fully decomposable.

Proof. By definition, there exists a decomposition $KL = M_0 \oplus M_1$ with $M_0$ finitely generated and $L \cap M_1 \cong \bigsqcup_{i \in I} E_i$, such that $E_i \in A\text{-}\text{Lat}$ for all $i \in I$. Hence

$$
K \text{Hom}_A(E, L) = \text{Hom}_K(E, KL) = \text{Hom}_K(E, M_0) \oplus \text{Hom}_K(E, M_1),
$$

and $\text{Hom}_A(E, L) \cap \text{Hom}_K(E, M_1) = \text{Hom}_A(E, L \cap M_1) \cong \bigsqcup_{i \in I} \text{Hom}_A(E, E_i)$. □

For a maximal ideal $p$ of $R$, let $R_p$ denote the $p$-adic completion of $R$. Then $A_p := R_p \otimes_R A$ is an $R_p$-order in $A_p := R_p \otimes_R A$. For $L \in A\text{-}\text{Lat}$, we define $L_p := R_p \otimes_R L$.

Lemma 1. Let $A$ be an $R$-order in a finite dimensional $K$-algebra $A$. For an $A$-module $M$ and a maximal ideal $p$ of $R$, let $L' \in A_p\text{-}\text{Lat}$ be a $A_p$-submodule of $M_p := R_p \otimes_R M$ with $KL' = M_p$. Then there exists an $L \in A\text{-}\text{Lat}$ with $KL = M$ and $L_p = L'$.

Proof. By [15], Proposition 4, the intersection $L' \cap M$ is a generalized $(A_p \cap A)$-lattice, and there exists a generalized $A$-lattice $H$ with $KH = M$ by [15], Lemma 1. Choose an automorphism $\alpha$ of $K M$ with $\alpha(L' \cap M) \subset H_p \cap M$. Then $\text{Hom}_A(L', M) \cong \text{Hom}_K(E, KL) \subset \alpha^{-1} H_p$. Thus if we replace $H$ by $\alpha^{-1} H$, we can assume that $L' \subset H_p$. Now we define $L := L' \cap \bigcap_{q \in \mathfrak{p}} H_q$, where $q$ runs through $\text{Spec } R \setminus \{p\}$. Then $L \subset H_p \cap \bigcap_{q \notin \mathfrak{p}} H_q = H$ by [15], Lemma 2. Hence $L \in A\text{-}\text{Lat}$. Moreover, [15], Lemma 2, yields $L_p = R_p L' \cap \bigcap_{q \notin \mathfrak{p}} H_q = L' \cap M_p = L'$. □

Proposition 4. Let $p$ be a maximal ideal of $R$, and let $L' \in A_p\text{-}\text{Lat}$ be almost fully decomposable but not fully decomposable. Assume that $KL'$ is an injective $A_p$-module. Then there exists an almost fully decomposable but not fully decomposable $L \in A\text{-}\text{Lat}$, and a fully decomposable $L'' \in A_p\text{-}\text{Lat}$ with $L_p \cong L' \oplus L''$ and $\rho(L) = \rho(L')$. 

□
Proof. By assumption, $KL' \cong N_0 \oplus N_1$ with $N_0 \in \mathcal{A}_p\text{-mod}$ and $L' \cap N_1 \cong \bigsqcup_{i \in I} F_i$, such that $0 \neq F_i \in \mathcal{A}_p\text{-lat}$ for all $i \in I$. Since $K F_i$ is injective, there is a finitely generated projective right $A$-module $Q_i$ such that $K F_i$ is a direct summand of the $A$-module $\text{Hom}_{K_p}(K_p \otimes_K Q_i, K_p) \cong K_p \otimes_K \text{Hom}_A(Q_i, K)$. So there exist $E_i \in \mathcal{A}_p\text{-lat}$ and $F'_i \in \mathcal{A}_p\text{-lat}$ with $(E_i)_p \cong F_i \oplus F'_i$ for all $i \in I$. Similarly, we find $E \in \mathcal{A}_p\text{-lat}$ and $F, F' \in \mathcal{A}_p\text{-lat}$ with $K F = N_0$ and $E_p \cong F \oplus F'$. With $M_0 := K E$ and $M_i := \bigsqcup_{i \in I} K E_i$, this gives $(M_0 \oplus M_1)_p \cong K F \oplus K F' \oplus \bigsqcup_{i \in I}(K F_i \oplus K F'_i) \cong N_0 \oplus K (L' \cap N_1) \oplus K F' \oplus \bigsqcup_{i \in I} K F'_i \cong K (L' \oplus F' \oplus \bigsqcup_{i \in I} F'_i)$. By Lemma 1, there exists a generalized $A$-lattice $H$ with $K H = M_0 \oplus M_1$ and $H_p \cong L' \oplus F' \oplus \bigsqcup_{i \in I} F'_i$. Hence $(H \cap M_1)_p \cong H_p \cap \bigsqcup_{i \in I}(K F_i \oplus K F'_i) \cong (L' \cap N_1) \oplus \bigsqcup_{i \in I} F'_i = \bigsqcup_{i \in I}(F_i \oplus F'_i) = \bigsqcup_{i \in I}(E_i)_p$. Now we define

$$L := H_p \cap \bigcap_q \left((E \cap H)_q \oplus \bigsqcup_{i \in I}(E_i \cap H)_q\right),$$

where $q$ runs through $\text{Spec } R \setminus \{p\}$. Thus $L \subset H_p \cap \bigcap_q H_q = H$ implies that $L \in \mathcal{A}_p\text{-Lat}$. Moreover, $L_p \cong H_p \cong L' \oplus L''$ for some fully decomposable $L''$ with $\rho(L') \leq \rho(L')$. Hence $\rho(L) = \rho(L')$. Also, we have $L \cap M_1 = L \cap (M_1)_p = \bigcap_q L \cap (M_1)_q = \bigcap_q (H \cap M_1)_q = \bigcap_q \bigcap_{i \in I}(E_i \cap H)_q = \bigcap_{i \in I}(E_i \cap H)$. Hence $L$ is almost fully decomposable. Finally, suppose that $L$ is fully decomposable. Then $L_p \cong L' \oplus L''$ is so, and the Crawley–Jønsson–Warfield theorem ([1], Theorem 26.5) implies that $L'$ is fully decomposable, a contradiction. □

2. A typical example

In this section, we consider an $R$-order $A$ with no large indecomposables, but with an abundance of generalized $A$-lattices which are not fully decomposable. Let $N(A)$ denote the prime radical of $A$, that is, the intersection of all prime ideals of $A$. Since $A$ is noetherian, the prime radical $N(A)$ is nilpotent. Hence

$$N(A) = A \cap \text{Rad } A. \quad (4)$$

We will say that $A$ has a splitting prime radical if there is a suborder $A_0$ of $A$ with $A = A_0 \oplus N(A)$.

Proposition 5. Assume that $A/\text{Rad } A$ is a separable $K$-algebra. Then $A$ admits an overorder $\Gamma'$ with splitting prime radical.

Proof. By the Wedderburn–Mal’cev theorem, there is a subalgebra $A_0$ of $A$ with $A = A_0 \oplus \text{Rad } A$. Let $\pi : A \to A_0$ be the projection with respect to this decomposition. Then $A_0 := \pi(A)$ is an $R$-order in $A_0$, and there is a non-zero $\lambda \in R$ with $A \subset A_0 \oplus \lambda^{-1} N(A)$. Hence $A_0 \subset A + \lambda^{-1} N(A)$. So it follows that

$$\Gamma' := A_0 + \lambda^{-1} N(A) + \lambda^{-2} N(A)^2 + \cdots$$

is an overorder of $A$ with splitting prime radical. □

For a maximal $R$-order $A_0$ in a semisimple $K$-algebra $A_0$, let us consider the triangular $R$-order

$$T_2(A_0) := \begin{pmatrix} A_0 & 0 \\ 0 & A_0 \end{pmatrix} \subset T_2(A_0) = \begin{pmatrix} A_0 & 0 \\ A_0 & A_0 \end{pmatrix}. \quad (5)$$

By [15], Proposition 11, every generalized $A_0$-lattice is fully decomposable, hence projective over $A_0$. So we have an equivalence

$$T_2(A_0)\text{-Lat} \cong \text{Mor}(A_0\text{-Proj}). \quad (6)$$

Therefore, the functor $\text{Mor}(A_0\text{-Proj}) \to A_0\text{-Mod}$ which maps an object $L_1 \xrightarrow{f} L_2$ to $\text{Cok } f \in A_0\text{-Mod}$ induces an additive functor

$$C_0 : T_2(A_0)\text{-Lat} \to A_0\text{-Mod}. \quad (7)$$

Proposition 6. The functor $(7)$ induces an equivalence of additive categories

$$T_2(A_0)\text{-Lat}/[\text{Inj}(T_2(A_0)\text{-Lat})] \cong A_0\text{-Mod}. \quad (8)$$
Proof. Since every $A_0$-module has a projective presentation, the functor (7) is full and dense. As $A_0\text{-}\mathbf{Mod} \cong \text{mod}(A_0\text{-}\mathbf{Proj})$, Eq. (2) implies that $\text{Mor}(A_0\text{-}\mathbf{Proj})/\{\mathcal{E}\} \cong A_0\text{-}\mathbf{Mod}$, where $\mathcal{E}$ consists of the objects $L_1 \xrightarrow{f} L_2$ in $\text{Mor}(A_0\text{-}\mathbf{Proj})$ for which $f$ is a split epimorphism in $A_0\text{-}\mathbf{Proj}$. On the other hand, the injective objects in $T_2(A_0)\text{-}\mathbf{lat}$ correspond to split epimorphisms in $A_0\text{-}\mathbf{lat}$. So the equivalence (8) follows by Proposition 1. \hfill \Box

Lemma 2. Let $A_0$ be a maximal $R$-order, and let $L$ be a generalized $A_0$-lattice with a finitely generated submodule $E$, such that $L/E$ is $R$-torsion-free. Then $L/E \in A_0\text{-}\mathbf{Proj}$.

Proof. By [15], Proposition 11, there is a decomposition $L = \bigoplus_{i \in I} E_i$ with $E_i \in A_0\text{-}\mathbf{lat}$. Hence there is a finite subset $J$ of $I$ with $E \subset \bigoplus_{i \in I \setminus J} E_i$. This gives a commutative diagram

\[
\begin{array}{cccc}
e & \rightarrow & L & \rightarrow & \bigoplus_{i \in I \setminus J} E_i \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{i \in J} E_i & \rightarrow & \bigoplus_{i \in I \setminus J} E_i \\
\end{array}
\]

with exact rows and columns. Hence $L/E \cong F \oplus \bigoplus_{i \in I \setminus J} E_i \in A_0\text{-}\mathbf{Proj}$. \hfill \Box

Proposition 6 shows that a $T_2(A_0)$-lattice $L$ is not fully decomposable unless $C_0(L)$ decomposes into finitely generated $A_0$-modules. Nevertheless, the equivalence (8) does not imply the existence of indecomposable large $T_2(A_0)$-lattices. On the contrary, we have

Proposition 7. For a maximal $R$-order $A_0$, every indecomposable $L \in T_2(A_0)\text{-}\mathbf{Lat}$ is finitely generated.

Proof. Firstly, the block decomposition of $A_0$ carries over to $T_2(A_0)$. Therefore, via Morita equivalence, we can assume that $A_0$ is a maximal order $\Delta$ in a division algebra $D$ over $K$. Thus, by (6) and [15], Proposition 11, a generalized $T_2(\Delta)$-lattice $L$ is given by a morphism $f: L_1 \rightarrow L_2$ in $A_0\text{-}\mathbf{Proj}$. As $\text{Im} f$ is projective, $f$ decomposes into $\text{Ker} f \rightarrow 0$ and $\text{Im} f \hookrightarrow L_2$. Therefore, we can assume that $L$ is given by an embedding $L_1 \hookrightarrow L_2$. It suffices to show that $L_2$ admits a decomposition $L_2 = F + C$ with $0 \neq F \in \Delta\text{-}\mathbf{lat}$, such that $L_1 = (L_1 \cap F) \oplus (L_1 \cap C)$. For any non-zero $x \in K L_2$, consider the ideal $I_x := \{a \in D \mid a(D x \cap L_2) \subset L_1\}$ of $\Delta$. We set $I := \sum_{x \in KL_2} I_x$. If $I = 0$, then $L_1 = 0$, which implies that $L$ is fully decomposable. Thus we assume that $I \neq 0$. Since $\Delta$ is left noetherian, there are $x_1, \ldots, x_n \in L_2$ with $I_{x_i} \neq 0$ and $I = I_{x_1} + \cdots + I_{x_n}$. By Lemma 2, we infer that $E := (D x_1 + \cdots + D x_n) \cap L_2$ is a direct summand of $L_2$. Moreover, there is a non-zero ideal $a$ of $R$ with $aE \subset L_1$. Now we proceed as in the lattice case [10]. Let $p_1, \ldots, p_m$ be the maximal ideals of $R$ which contain $a$. Then there are elements $y_1, \ldots, y_m \in E$, such that $R_p \otimes_R I_{y_i} = R_p \otimes_R I$. By the Strong Approximation Theorem ([11], Theorem 4.11), we find an element $y \in E$ with $R_p \otimes_R I = R_p \otimes_R I$ for $i \in \{1, \ldots, m\}$. Since $a \subset I_y \subset I$, this gives $I_y = I$. On the other hand, every $x \in L_1$ satisfies $x \in D x \cap L_1 = I_x(D x \cap L_2) \subset I(D x \cap L_2)$, whence $L_1 \subset IL_2$. By Lemma 2, there is a decomposition $L_2 = (D y \cap L_2) \oplus C$. Hence $1L_2 = I(D y \cap L_2) \oplus IC$, where $I(D y \cap L_2) = I_y(D y \cap L_2) \subset L_1$. Therefore, we get $L_1 = (I(D y \cap L_2) + IC) \cap L_1 = I(D y \cap L_2) \oplus (IC \cap L_1)$. \hfill \Box

3. Almost fully decomposable large $A$-lattices

Let $\Gamma$ be a maximal $R$-order in $A$. Then $\Gamma_p$ is a maximal $R_p$-order in $A_p$ for any maximal ideal $p$ of $R$. Hence $A_p$ is semisimple. For a simple $A_p$-module $S$, the non-zero $\Gamma_p$-submodules $E$ of $S$ form a chain of isomorphic $\Gamma_p$-lattices. Therefore, the isomorphism class of $S(p^\infty) := S/E \in \Gamma^-\text{-}\mathbf{Mod}$ does not depend on $E$. This definition can be extended to every $p \in \text{Spec } R$. Namely, for $p = 0$ and a simple $A$-module $S$, we set $A_p := A$ and $S(p^\infty) := S$.

Lemma 3. Let $A$ be an $R$-order in $A$. Every simple $A$-module is isomorphic to a factor module $F/E$ with $E, F \in A\text{-}\mathbf{lat}$, such that $K E = K F$ is a simple $A$-module.
Proof. Every simple \( A \)-module is isomorphic to a factor module \( F/E \) with \( E, F \in A\text{-lat} \) and \( KE = KF \). (Choose, e.g., \( F := A \).) Consider a short exact sequence \( N \twoheadrightarrow KF \twoheadrightarrow S \) with \( S \in A\text{-mod} \) simple. If \( p(F) \neq p(E) \), then \( F/E \cong p(F)/p(E) \), and we are done. If \( p(F) = p(E) \), we get a commutative diagram

\[
\begin{array}{ccc}
E' & \rightarrow & E \\
\downarrow & & \downarrow \\
p(E) & \rightarrow & p(F)
\end{array}
\]

with short exact rows. Hence \( F'/E' \cong F/E \) and \( \dim KE' < \dim KF \). Therefore, the lemma follows by induction.

\[ \square \]

Proposition 8. Let \( \Gamma \) be a maximal \( R \)-order in \( A \). Up to isomorphism, there is a one-to-one correspondence between simple \( \Gamma \)-modules \( U \) and simple \( A_\mathfrak{p} \)-modules \( S \), where \( \mathfrak{p} \) runs through the maximal ideals of \( R \). The correspondence is given by \( U \cong \text{Soc} S(p^{\infty}) \).

Proof. Let \( U \) be a simple \( \Gamma \)-module. Then there is a maximal ideal \( \mathfrak{p} \) of \( R \) with \( pU = 0 \). Since \( \Gamma_\mathfrak{p} \cong R_\mathfrak{p} \otimes_R \Gamma \), every \( \Gamma \)-submodule of \( U \) can be regarded as a \( \Gamma_\mathfrak{p} \)-module. Thus \( U \) is simple as a \( \Gamma_\mathfrak{p} \)-module. By Lemma 3, there is a simple \( A_\mathfrak{p} \)-module \( S \), such that \( U \cong F/E \) for some \( E, F \in A_\mathfrak{p}\text{-lat} \) with \( KE = KF = S \). Thus \( U \cong \text{Soc} S(p^{\infty}) \). Conversely, every simple \( \Gamma_\mathfrak{p} \)-module is simple as a \( \Gamma \)-module. This establishes the one-to-one correspondence.

As a consequence, we get the following generalization of the classification of indecomposable injective \( \mathbb{Z} \)-modules, which will be needed for the construction of almost fully decomposables.

Corollary. Let \( \Gamma \) be a maximal \( R \)-order in \( A \). For every simple \( A_\mathfrak{p} \)-module \( S \) with \( \mathfrak{p} \in \text{Spec} R \), the \( \Gamma \)-module \( S(p^{\infty}) \) is indecomposable and injective, and every indecomposable injective \( \Gamma \)-module is of this form.

Proof. Since \( A \) is semisimple, every \( A \)-module is injective in \( \Gamma\text{-Mod} \). This shows that for each \( \mathfrak{p} \in \text{Spec} R \), the simple \( A_\mathfrak{p} \)-modules are injective in \( \Gamma\text{-Mod} \). Since \( \Gamma \) is hereditary, this implies that the factor modules \( S(p^{\infty}) \in \Gamma\text{-Mod} \) are injective. Clearly, \( S(p^{\infty}) \) is indecomposable for \( p = 0 \). To show that \( S(p^{\infty}) \) is indecomposable for \( p \neq 0 \), let \( S(p^{\infty}) = S/E_0 \) with \( E_0 \in \Gamma_\mathfrak{p}\text{-lat} \) and \( KE_0 = S \in A_\mathfrak{p}\text{-mod} \). The proper non-zero \( \Gamma_\mathfrak{p} \)-submodules of \( S \) form a chain \( \{E_i \mid i \in \mathbb{Z}\} \) with \( E_i \subset E_j \) for \( i > j \). We will show that any proper \( \Gamma \)-submodule \( E \) of \( S \) with \( E_0 \subset E \) coincides with some \( E_i \). Since \( E \) is a union of finitely generated \( \Gamma \)-submodules, we can assume that \( E \) is finitely generated. Hence \( p^n E \subset E_0 \) for some \( n \in \mathbb{N} \). Suppose that \( n \) is minimal with respect to \( p^n E \subset E_i \subset E \) for some \( i \in \mathbb{Z} \). If \( n > 0 \), then \( R_\mathfrak{p}(E_i + p^{n-1} E) = (R + p R_\mathfrak{p})(E_i + p^{n-1} E) \subset E_i + p^{n-1} E \). Hence \( E_i + p^{n-1} E = E_j \) for some \( j \in \mathbb{Z} \), and \( p^{n-1} E \subset E_j \subset E \), a contradiction. This proves that the \( \Gamma \)-module \( S(p^{\infty}) \) is indecomposable. Conversely, let \( I \) be an indecomposable injective \( \Gamma \)-module. If \( I \) is \( R \)-torsion-free, then \( KE = KF \) and \( KE \) must be simple. If \( I \) is not \( R \)-torsion-free, the submodule \( [p]I := \{x \in I \mid px = 0\} \) of \( I \) is non-zero for some maximal ideal \( \mathfrak{p} \) of \( R \). Hence \( I \) is the injective envelope of a simple \( \Gamma \)-module \( U \). By Proposition 8, this implies that \( I \) is of the desired form.

Now we turn our attention to the construction of almost fully decomposable generalized \( A \)-lattices which are not fully decomposable. First, we assume that the \( R \)-order \( \Lambda_0 := \Lambda/N(\Lambda) \) in \( A/\text{Rad} A \) is maximal. For \( L \in A\text{-Lat} \), consider the short exact sequence

\[
N(\Lambda)L \cap \text{Soc} KL \hookrightarrow L \cap \text{Soc} KL \rightarrow CL
\]

of \( A_0 \)-modules, where \( \text{Soc} KL \) denotes the socle of \( KL \in A\text{-mod} \). Then \( L \mapsto CL \) defines an additive functor

\[
C : A\text{-Lat} \rightarrow A_0\text{-Mod},
\]

and (9) gives a projective resolution of \( CL \). Note that \( C \) maps \( A \)-lattices to finitely generated \( A_0 \)-modules.

Theorem 1. Let \( R \) be a Dedekind domain with quotient field \( K \), and let \( \Lambda \) be an \( R \)-order in a finite dimensional \( K \)-algebra \( A \) such that \( \Lambda_0 := \Lambda/N(\Lambda) \) is a maximal order. Assume that \( \Lambda \) has a splitting prime radical. Let \( \Lambda_1 \) be the product of the blocks \( \Gamma \) of \( \Lambda_0 \) with \( \Gamma(\text{Rad} A/\text{Rad}^2 A) \neq 0 \).
(a) Every $I \in \Lambda_1$-Mod is isomorphic to $C_L$ for some $L \in A$-Lat.
(b) If $I$ is $R$-torsion and indecomposable injective, then $L$ can be chosen to be almost fully decomposable, not fully decomposable, with $KL$ injective and $\rho(L) = S_0$.

**Proof.** (a) By assumption, $A_0$ can be regarded as a suborder of $A$ with $A = A_0 \oplus N$, where $N := N(A)$. Since $A_1$ is maximal, $I$ has a projective resolution $L_1 \hookrightarrow L_2 \twoheadrightarrow I$ with $L_1, L_2 \in A_1$-Lat. Define
\[
L := \{ f \in \text{Hom}_{A_0}(A, L_2) \mid f(N) \subset L_1 \} \in A$-Lat.
\]
With $J := \text{Rad} A$, $A_0 := K A_0$, and $X := KL_2$, we have
\[
KL = \{ f \in \text{Hom}_{A_0}(A, X) \mid f(J) \subset KL_1 \}.
\]
Therefore, $\text{Soc} KL = \{ f \in \text{Hom}_{A_0}(A, X) \mid f(J) = 0 \} \cong \text{Hom}_{A_0}(A/J, X)$ can be identified with $X$. With this identification,$$
L \cap \text{Soc} KL = \{ f \in \text{Hom}_{A_0}(A, L_2) \mid f(N) = 0 \} = \text{Hom}_{A_0}(A/N, L_2) = L_2 \subset X.
$$
Let us show that
\[
\{ f \in \text{Hom}_{A_0}(A, L_1) \mid f(N) = 0 \} \subset NL.
\]
By assumption, every block of $KA_1$ has a simple direct summand occurring in $J/J^2$. By Harada’s theorem ([9], Theorem 1.1; [11], Theorem 22.7), this implies that there is a surjection $p: N^m \to A_1$ in $A$-Lat for some $m \in \mathbb{N}$. Every $f \in \text{Hom}_{A_0}(A, L_1)$ with $f(N) = 0$ factors through the natural map $q: A \to A/N \to A_1$. Since $q \in A$-Lat, we have $q = pr$ for some $r: A \to N^m$ in $A$-Lat. Hence there are $r_i \in \text{Hom}_A(A, N)$ and $f_i \in \text{Hom}_{A_0}(N, L_1)$ with $f = \sum_{i=1}^m r_i f_i$. Since $A_0 N$ is a direct summand of $A_0 A$, the $f_i$ can be extended to $A$, whence $f = \sum_{i=1}^m r_i(1)f_i \in NL$. Therefore, we get
\[
NL \cap \text{Soc} KL = \{ f \in \text{Hom}_{A_0}(A, L_1) \mid f(N) = 0 \} = \text{Hom}_{A_0}(A/N, L_1) = L_1 \subset X,
\]
which yields $CL \cong I$.

(b) If $I$ is an $R$-torsion module, then $KL_1 = KL_2$. Hence $KL = \text{Hom}_{A_0}(A, X)$ by Eq. (12). Since $A_A$ is flat and $A_0 X$ is injective, this implies that $A KL$ is injective ([6], Theorem 3.2.9). Assume, in addition, that $I$ is indecomposable injective. By Proposition 8 and its Corollary, there is a maximal ideal $p$ of $R$ and a simple $A_1$-module $U$ with injective envelope $I$, such that $p U = 0$. By Lemma 3, $U$ can be represented as a factor module $E_1/E_0$ with $E_0, E_1 \in A_1$-Lat and $S := KL E_0 = K E_1 \in K A_1$-mod simple. Since $S/E_0$ is an injective $A_1$-module, the Corollary of Proposition 8 implies that the inclusion $E_0 \subset E_1$ can be extended to a chain of proper $A_1$-submodules $E_0 \subset E_1 \subset E_2 \subset \cdots$ with $E_{i+1}/E_i \cong E_1/E_0$, such that $\tilde{E} := \bigcup_{i=0}^{\infty} E_i$ satisfies $\tilde{E}/E_0 \cong I$. Define $L_2 := \bigcap_{i=0}^{\infty} E_i$. Then the inclusions $E_i \hookrightarrow \tilde{E}$ induce a natural epimorphism $\rho: L_2 \to \tilde{E}/E_0 \cong I$. We set $L_1 := \text{Ker} \rho$. Then the generalized $A$-lattice (11) satisfies $\rho(L) = S_0$ and is not fully decomposable. It remains to be shown that $L$ is almost fully decomposable. Let $e_i$ denote the inclusion $E_i \hookrightarrow E_{i+1}$. Then
\[
f_i : E_i \overset{1}{\rightarrow} E_i \oplus E_{i+1} \hookrightarrow L_2
\]
is monic with $f_i(E_i) \subset L_1$. Moreover, a straightforward calculation yields
\[
KL_2 = KE_0 \oplus \bigoplus_{i=0}^{\infty} Kf_i(E_i).
\]
With $C := \bigoplus_{i=0}^{\infty} Kf_i(E_i)$, this shows that $C \cap L_2 = C \cap L_1$. Hence $L$ is almost fully decomposable. \[\square\]

**Corollary.** Assume that $A_0 := A/N(A)$ is a maximal order and that $A$ has a splitting prime radical. The following are equivalent.

(a) The functor (10) is dense.

(b) Every simple $A$-module is a direct summand of $\text{Rad} A/\text{Rad}^2 A$. 

Proof. (a) ⇒ (b): Let $S$ be a simple $A$-module which does not occur as a direct summand of $J/J^2$, where $J := \text{Rad } A$. Suppose that $S \cong C L$ for some $L \in \Lambda$-Lat. Then (9) gives rise to a short exact sequence

$$J L \cap \text{Soc } K L \hookrightarrow \text{Soc } K L \xrightarrow{\rho} S$$

with $\rho(L \cap \text{Soc } K L) = S$. We set $L' := L \cap \text{Soc } K L$ and choose $i \in \text{Hom}_A(S, \text{Soc } K L)$ with $pi = 1$. Then $E := i^{-1}(L') \subset S$ is an $A_0$-lattice with $KE = S$. So there is an $A_0$-lattice $F$ with $KF = S$ and $E \subsetneq F$. Hence $(i(F) + L')/L' \cong (i(F)/(i(F) \cap L')) \cong F/E$. This gives a commutative diagram

$$
\begin{array}{c}
F' := (i(F) + L') \cap J L \бург \rightarrow i(F) + L' \hookrightarrow S \\
E' := L' \cap J L \бург \rightarrow L' \hookrightarrow S
\end{array}
$$

(15)

with exact rows. As $F$ is finitely generated, we have $i(F) + L' \in A_0$-Lat, whence $F' \in A_0$-Lat. Moreover, the diagram (15) yields an isomorphism $F/E \cong F'/E'$ of $A_0$-modules. Since $A_0$ is maximal, this isomorphism is induced by a homomorphism $F \rightarrow F'$. Therefore, we get a non-zero morphism $\rho(S) = K F \rightarrow K F' \hookrightarrow J L$. Since $J$ is nilpotent, it follows that $S$ is a direct summand of $J^m L/J^{m+1} L$ for some $m > 0$. Hence we get an epimorphism $J \otimes_A J^{m-1} L \rightarrow J^m L \rightarrow S$, which shows that $\text{Hom}_A(J, S) \neq 0$. Thus $S$ is a direct summand of $J/J^2$, in contrast to the assumption. The reverse implication $(b) \Rightarrow (a)$ follows immediately by Theorem 1. 

Now we are ready to prove our main result. Assume that $A_p$ is a maximal $R_p$-order for almost all $p \neq 0$ in $\text{Spec } R$. By Drozd’s theorem [5], there exists a maximal overorder of $\Lambda$ if and only if $A_p$ is semisimple for every $p \in \text{Spec } R$.

**Theorem 2.** Let $R$ be a Dedekind domain with quotient field $K$, and let $\Lambda$ be an $R$-order in a finite dimensional $K$-algebra $A$ such that $A/\text{Rad } A$ is separable over $K$. Assume that $A_p$ is not semisimple for some $p \in \text{Spec } R$. Then there exists an almost fully decomposable, but not fully decomposable $L \in \Lambda$-Lat with $\rho(L) = \aleph_0$.

**Proof.** With regard to Proposition 4, we assume that $A$ is not semisimple. Then it suffices to construct an almost fully decomposable $L \in \Lambda$-Lat with $K L$ injective and $\rho(L) = \aleph_0$, such that $L$ is not fully decomposable. By Proposition 5, there is an overorder $\Lambda'$ of $\Lambda$ with splitting prime radical. Thus $\Lambda' = \Lambda_0' \oplus N(\Lambda')$. Choose a maximal overorder $\Gamma_0$ of $\Lambda_0'$. Then $\Gamma := \Gamma_0 \oplus \Gamma_0 N(\Lambda)/\Gamma_0$ is an overorder of $\Lambda$. Therefore, Theorem 1 yields an almost fully decomposable $L \in \Gamma$-Lat with $K L$ injective and $\rho(L) = \aleph_0$, such that $L$ is not fully decomposable. Hence $L$ is almost fully decomposable in $\Lambda$-Lat. If $L$ would be fully decomposable, say, $L = \bigoplus_{i \in I} E_i$ with $E_i \in \Lambda$-lat, then $L = \Gamma L = \bigoplus_{i \in I} \Gamma E_i$, a contradiction. Thus $L \in \Lambda$-Lat meets the requirements. \( \square \)

References