Longtime behavior of the Kirchhoff type equation with strong damping on $\mathbb{R}^N$

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Abstract

The paper studies the longtime behavior of the Kirchhoff type equation with strong damping on $\mathbb{R}^N$: $u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_t + u + u_t + g(x, u) = f(x)$. It shows that the related continuous semigroup $S(t)$ possesses a global attractor which is connected and has finite fractal and Hausdorff dimension. © 2007 Elsevier Inc. All rights reserved.

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1. Introduction

In this paper, we are concerned with the longtime behavior of solutions to the Cauchy problem for the Kirchhoff type equation with strong damping

$$u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_t + u + u_t + g(x, u) = f(x) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^+,$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \mathbb{R}^N,$$

where $M(s) = 1 + s^{m/2}, m \geq 1, g(x, u)$ is a nonlinear function specified later, and $f(x)$ is an external force term.

When $N = 1$, Eq. (1.1) without dissipative term $-\Delta u_t$ was introduced by Kirchhoff [9] to describe small vibrations of an elastic stretched string. The bibliography of studies on the Kirchhoff type equations is plentiful (cf. Lions [12], Arosio and Garavaldi [1], Brito [4], etc.). The global existence and decay properties of solutions to the equation with dissipation $-\Delta u_t$ or $u_t$
or more delicate ones have been well discussed for the case \( g = f = 0 \) or \( f = 0 \) (cf. Matsuyama and Ikehata [13], Mizumachi [14], Nakao [15], Nishihara [17], Ono [18,19], Bae and Nakao [3], and Cavalcanti et al. [5], etc.).

Global attractor is a basic concept in the study of the asymptotic behavior of solutions for nonlinear evolution equations with various dissipation (cf. Babin and Vishik [2], Hale [8], Temam [21]). There have been a lot of profound results on the global attractors for the semi-linear wave equations, but there are only very few ones on the quasilinear problem (cf. Chen, Guo and Wang [6], Ghidaglia and Marzocchi [7], Karachalois and Stavrakakis [10], Zhou [23]).

Recently, as the first step of investigation, Nakao and Yang [16] discussed the longtime behavior of solutions to the initial boundary value problem (IBVP) of Eq. (1.1), under the assumptions that, roughly speaking, \( g \in C^1(\mathbb{R}^N, \mathbb{R}) \), \( g(x,s) \) is of polynomial growth order on \( s \), say \( p \), with \( 1 \leq p \leq \frac{4}{(N-4)^+} \), and \( g \) satisfies appropriate Lipschitz condition, the authors established the existence of global attractor in the phase space \((H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)\) by proving the existence of absorbing sets and the asymptotic compactness of the related continuous semigroup. But does the global attractor have finite Hausdorff dimension? Does the dynamical system associated with Cauchy problem (1.1), (1.2) still possess a global attractor provided that the similar conditions hold? What happens to its Hausdorff dimension? All these questions are still open.

In general, the existence of global attractor depends on some kind compactness. For Cauchy problem, the main question is how to overcome the difficulty of lacking the compactness of Sobolev embedding in \( \mathbb{R}^N \). Recently, Wang [22] introduced a new method to study the existence of global attractor for reaction–diffusion equations in unbounded domains, and he proved the asymptotic compactness of the solutions and then established the global attractor in \( L^2(\mathbb{R}^N) \) by approaching \( \mathbb{R}^N \) by a bounded domain \( \Omega_k \), and combining the tail estimates in spatial variables with the compactness of Sobolev embedding in bounded domain \( \Omega_k \). This method has been also used in second-order evolution equations and systems (see [11,20]). But the method ceases to be effective for further discussion of the Hausdorff dimension of the global attractor.

In the present paper, by combining the decomposition idea (see Temam [21] and Hale [8]) (rather than proving the asymptotic compactness of the related continuous semigroup as done in [16,22]) with the tail estimates in the phase spaces \( H^2 \times L^2 \) and \( H^2 \times H^1 \), both in the time and spatial variables, which are different from those used in [16,22], the author, motivated by the idea in [22], further discusses the longtime behavior of solutions to Cauchy problem (1.1), (1.2), and proves that under the conditions similar to those in [16] the related dynamical system possesses a global attractor in phase space \( H^2 \times H^1 \), which is connected and has finite fractal and Hausdorff dimension. And the results hold true for the IBVP of Eq. (1.1).

The paper is organized as follows. In Section 2, some notations and the main results are stated. The global existence and global estimate of solution to Cauchy problem (1.1), (1.2) are established in Section 3. In Section 4, the existence and finite fractal and Hausdorff dimension of global attractor to the dynamical system associated with Cauchy problem (1.1), (1.2) are proved.

2. Statement of main results

We first introduce the following abbreviations:

\[
L^p = L^p(\mathbb{R}^N), \quad H^k = H^k(\mathbb{R}^N), \quad \| \cdot \|_p = \| \cdot \|_{L^p}, \\
\| \cdot \| = \| \cdot \|_{L^2}, \quad X = H^2 \times H^1, \quad X^* = H^3 \times H^2, \quad (2.1)
\]
with \( p \geq 1, k = 1, 2, 3 \). The notation \((\cdot,\cdot)\) denotes the \( L^2 \)-inner product, the same letter \( C \) denotes different positive constants, \( C(\cdot \cdot) \) denotes positive constants depending on the quantities appearing in the parenthesis, \( B(R) \) denotes the ball in \( \mathbb{R}^N \) centered at 0 with the radius \( R \), and \( B(R) \) denotes that in \( X \). \( B(R)^C = \mathbb{R}^N - B(R) \), and \( a^+ = \max\{a, 0\} \).

Now, we state the main results.

**Theorem 2.1.** Assume that

(i) \( g \in C(\mathbb{R}^N \times \mathbb{R}), |g(x,s)| \leq \alpha(|s|^p + L_1(x)), \)

\[
|g(x,v_1) - g(x,v_2)| \leq \beta(|v_1|^{p-1} + |v_2|^{p-1} + L_1(x)|v_1 - v_2|,
\]

\[
g(x,u)u + L_0(x) \geq \nu \left( \int_0^u g(x,s) \, ds + L(x) \right) \geq 0 \tag{2.2}
\]

for a.e. \( x \in \mathbb{R}^N \) and all \( u \in \mathbb{R} \), where \( \alpha, \beta, \nu \) are positive real numbers, \( L_0, L \in L^1 \), \( L_1 \in L^2 \cap L^{p+1} \), with \( 1 \leq p < \frac{N+2}{N-2} \).

(ii) \( f \in L^2, (u_0, u_1) \in X \).

Then problem (1.1), (1.2) admits a unique solution \( u \in C([0, \infty); \mathbb{H}^2) \cap C^1([0, \infty); \mathbb{H}_1) \cap \mathbb{H}^1([0, \infty); \mathbb{H}^2) \cap \mathbb{H}^2([0, \infty); \mathbb{L}^2) \), \( u \) is continuous on initial data in \( X \) and

\[
\|u(t)\|_{\mathbb{H}^2}^2 + \|u_t(t)\|_{\mathbb{H}_1}^2 \leq \tilde{C} e^{-\kappa t} + C(M_0, L^*, I_0), \quad t > 0, \tag{2.3}
\]

where (and in the sequel) \( \tilde{C} = C(\|u_0\|_{\mathbb{H}^2},\|u_1\|_{\mathbb{H}_1}), \kappa > 0 \) is a small constant, \( M_0 = \|f\|, L^* = \|L_0\|_{\mathbb{L}^1}, I_0 = \|L_1\| + \|L_1\|_{\mathbb{L}^{p+1}/p-1} \).

We denote the solution in Theorem 2.1 with \( S(t)(u_0, u_1) = (u(t), u_t(t)) \). Then \( S(t) \) composes a continuous semigroup on \( X \). We have the following theorem.

**Theorem 2.2.** In addition to the assumptions of Theorem 2.1, if also \( f \in H^1, g \in C^1(\mathbb{R}^N \times \mathbb{R}) \) and

\[
\left| \frac{\partial g}{\partial s} \right| \leq \beta(|s|^{p-1} + L_1(x)), \quad |\nabla_x g| \leq \gamma(|s|^p + L_2(x)), \tag{2.4}
\]

where \( \beta, \gamma > 0 \) are constants, \( L_1 \in L^{2(p+1)/(p-1)}, L_2 \in L^2, \) and \( 1 \leq p \leq \frac{4}{(N-4)^+} \), then the semigroup \( S(t) \) possesses a global attractor \( A \) which is connected and included in a ball \( B(R) \) in \( X \) centered at 0 with a radius \( R \), and for any bounded set \( B_0 \subset X \),

\[
\text{dist}(S(t)B_0, B(R)) \leq C(B_0)e^{-\kappa t}. \tag{2.5}
\]

Moreover, \( A \) has finite fractal and Hausdorff dimension.

**Remark 1.** From the proof process of Theorems 2.1–2.2 it can be known that for the Dirichlet problem of Eq. (1.1), with or without \( u \) and \( u_t \), the same results as stated in Theorems 2.1–2.2 hold true.
3. Global existence and global estimates of solutions

Proof of Theorem 2.1. Because the proof is similar to that in [16], we only state the outline here. The proof of existence and uniqueness of local in time solution in the space \( Z = C([0, T]; H^2) \cap C^1([0, T]; H^1) \cap H^1([0, T]; H^2) \cap H^2([0, T]; L^2) \) with small \( T > 0 \) is given by using the Banach fixed point theorem in the metric space

\[
X_{R,T} = \{ v(t) \in Z \mid \rho(v(t)) \leq R^2, \ t \in [0, T], \ v(0) = u_0, \ v_t(0) = u_1 \} \tag{3.1}
\]

for \( R > 0 \) and \( T > 0 \) equipped with the distance \( \sqrt{\rho(u, v)} \), where

\[
\rho(u, v) = \sup_{0 \leq t \leq T} \left\{ \| u(t) - v(t) \|_{H^2}^2 + \| u_t(t) - v_t(t) \|_{H^1}^2 \right. \\
\left. + \int_0^T \left( \| u_{tt}(\tau) - v_{tt}(\tau) \|_{H^1}^2 + \| u_t(\tau) - v_t(\tau) \|_{H^2}^2 \right) d\tau \right\}, \tag{3.2}
\]

and where the mapping \( u = Fv \) in \( X_{R,T} \) is defined by

\[
u_{tt} - \left(1 + \| \nabla v(t) \|^m \right) \Delta u - \Delta u_t + u + u_t = -g(x, v) + f(x) \equiv G_0 \tag{3.3}
\]

with initial conditions. Let \([0, T^0)\) be the maximal interval of existence of \( u \). According to the standard method the global existence, i.e. \( T^0 = \infty \), comes from the global estimate (2.3), which can be obtained by using the similar arguments as in [16], and the continuity of solution on initial data in \( X \) can also be easily obtained, so we omit the process. Theorem 2.1 is proved.

4. Global attractor

The general theory (cf. [21]) indicates that the continuous semigroup \( S(t) \) defined on a Banach space \( X \) has a global attractor which is connected when the following conditions are satisfied:

(i) There exists a bounded absorbing set \( B \subset X \) such that for any bounded set \( B_0 \subset X \), \( \text{dist}(S(t)B_0, B) \to 0 \) as \( t \to +\infty \).

(ii) \( S(t) \) can be decomposed as \( S(t) = P(t) + U(t) \), where \( P(t) \) is a continuous map from \( X \) to itself with the property that, for any bounded set \( B_0 \subset X \),

\[
\sup_{\phi \in B_0} \| P(t)\phi \|_X \to 0 \quad \text{as} \ t \to \infty, \tag{4.1}
\]

and \( U(t) \) is precompact for \( t > T_0 \), for some \( T_0 \).

Proof of Theorem 2.2. Existence of an absorbing set. The global estimate (2.3) shows that the set

\[
B = \{ (u, v) \in X \mid \| (u, v) \|_X^2 \leq R \}, \tag{4.2}
\]
Lemma 4.1. Assume that the conditions of Theorem 2.2 hold, and \((u_0, u_1) \in \mathcal{B}(R_0) \subset X (R_0 > 0)\), \(u\) is the solution of problem (1.1), (1.2) as shown in Theorem 2.1. Then \(u_t\) can be written as \(u_t = \bar{v} + \hat{v}\), where \(\bar{v} \in L^\infty([0, \infty); H^1) \cap L^2_{\text{loc}}([0, \infty); H^2)\) is the unique solution of the problem

\[
\bar{v}_t - \Delta \bar{v} + \bar{v} = 0, \quad t > 0, \quad \bar{v}(x, 0) = u_1.
\]

and \(\hat{v} \in L^\infty([0, \infty); H^2)\) is the unique solution of the problem

\[
\hat{v}_t - \Delta \hat{v} + \hat{v} = M(\|\nabla u\|^2) \Delta u - g(x, u) + f \equiv G_1, \quad t > 0, \quad \hat{v}(x, 0) = 0.
\]

Moreover,

\[
\|\bar{v}(t)\|_{H^1}^2 \leq \|u_1\|_{H^1}^2 e^{-\lambda t}, \quad t > 0,
\]

\[
\|\hat{v}(t)\|_{H^2}^2 \leq C(M_0, L^*, I_0), \quad t \geq T_0,
\]

where \(T_0 = T(R_0)\).

**Proof.** The existence and uniqueness of \(\bar{v}\) and \(\hat{v}\) are obvious because Theorem 2.1 implies \(G_1 \in L^\infty([0, \infty); L^2)\), and by (2.3),

\[
\|G_1(t)\| \leq M(\|\nabla u\|^2) \|\Delta u(t)\| + \|u(t)\| + \alpha(\|u(t)\|_{2p} + \|L_1\|) + \|f\| \leq \tilde{C}e^{-\lambda t} + C(M_0, L^*, I_0), \quad t > 0.
\]

The uniqueness of solutions of problems (4.4) and (4.5) implies \(u_t = \bar{v} + \hat{v}\). Taking \(L^2\)-inner product with \(\bar{v} + \hat{v}_t\) in the equation in (4.5), we obtain

\[
\frac{d}{dt} \left[ 2\|\bar{v}(t)\|^2 + \|\nabla \bar{v}(t)\|^2 \right] + 2\|\nabla \hat{v}(t)\|^2 + \|\hat{v}(t)\|^2 + \|\hat{v}_t(t)\|^2 \leq 4G_1(t) \leq 4\|G_1(t)\|^2.
\]

Differentiating the equation in (4.5) with respect to \(t\) and taking \(L^2\)-inner product with \(\hat{v}_t\) in the resulting expression, we have

\[
\frac{1}{2} \frac{d}{dt} \|\hat{v}_t(t)\|^2 + \|\hat{v}_t(t)\|_{H^1}^2 \leq -M(\|\nabla u\|^2)(\nabla u_t, \nabla \hat{v}_t) + (g_u u_t, \hat{v}_t) + \frac{1}{4} \|\hat{v}_t(t)\|^2 + C[M(\|\nabla u\|^2) \|\nabla u\| \|\nabla u_t\| \|\Delta u\|^2 + \|u_t\|^2 \|u_t\|^2] \leq \frac{1}{2} \|\hat{v}_t(t)\|_{H^1}^2 + \tilde{C}e^{-\lambda t} + C(M_0, L^*, I_0), \quad t > 0,
\]
where (2.3) and the fact
\[
\left| (g_u u_t, \hat{v}_t) \right| \leq \beta \left\| \hat{v}_t (t) \right\|_{L^1} \left( \left\| u(t) \right\|_{L^p}^{p-1} + \left\| L_1 \right\|_{L^p}^{1} \right)
\leq \frac{1}{8} \left\| \hat{v}_t (t) \right\|_{H^1}^2 + \bar{C} e^{-\kappa t} + C(M_0, L^*, I_0), \quad t > 0,
\]
(4.11)
have been used. The combination of (4.9) with (4.10) yields
\[
\frac{d}{dt} \left[ \left\| \hat{v}_t (t) \right\|_{H^1}^2 + 2 \left\| \hat{v} (t) \right\|_{H^1}^2 + \left\| \nabla \hat{v} (t) \right\|_{H^1}^2 \right] + \left\| \hat{v}_t (t) \right\|_{H^1}^2 + \left\| \hat{v} (t) \right\|_{H^1}^2 \leq \bar{C} e^{-\kappa t} + C(M_0, L^*, I_0), \quad t > 0.
\]
(4.12)
It follows from (4.5), (4.8) and (4.13) that
\[
\left\| \Delta \hat{v} (t) \right\|_{H^1} \leq 3 \left( \left\| \hat{v}_t (t) \right\|_{H^1}^2 + \left\| \hat{v} (t) \right\|_{H^1}^2 + \left\| G_1 (t) \right\|_{H^1}^2 \right) \leq \bar{C} e^{-\kappa t} + C(M_0, L^*, I_0), \quad t > 0.
\]
(4.14)
(4.13) and (4.14) imply (4.7). (4.6) is immediate. Lemma 4.1 is proved. \(\Box\)

**Step 2 (The decomposition of \( u \)).**

**Lemma 4.2.** Under the assumptions of Lemma 4.1, the above mentioned solution \( u \) can be decomposed as \( u = \bar{u} + \hat{u} \), where \( \bar{u} \in L^\infty ([0, \infty); H^2) \) is the unique solution of the problem
\[
(\bar{u}_t - \Delta \bar{u} + \bar{u})_t + M \left( \left\| \nabla u \right\|_{L^2}^2 \right) (\bar{u}_t - \Delta \bar{u} + \bar{u}) = M \left( \left\| \nabla u \right\|_{L^2}^2 \right) \bar{v}, \quad t > 0,
\]
(4.15)
\[
\bar{u} (x, 0) = u_0, \quad \bar{u}_t (x, 0) = u_1,
\]
(4.16)
and \( \hat{u} \in L^\infty ([0, \infty); H^3) \) is the unique solution of the problem
\[
(\hat{u}_t - \Delta \hat{u} + \hat{u})_t + M \left( \left\| \nabla u \right\|_{L^2}^2 \right) (\hat{u}_t - \Delta \hat{u} + \hat{u}) = G_2, \quad t > 0,
\]
(4.17)
\[
\hat{u} (x, 0) = \hat{u}_t (x, 0) = 0,
\]
(4.18)
where \( G_2 = M \left( \left\| \nabla u \right\|_{L^2}^2 \right) (\hat{v} + u) - g(x, u) + f(x) \). Moreover,
\[
\left\| \bar{u} (t) \right\|_{H^2}^2 \leq C \left( \left\| u_0 \right\|_{L^2} \left\| u_1 \right\|_{L^2} \right)^2 e^{-\kappa t}, \quad t > 0,
\]
(4.19)
\[
\left\| \hat{u} (t) \right\|_{H^3}^2 \leq C (M_1, I_0, I_1, L^*), \quad t \geq T_0,
\]
(4.20)
where \( \bar{v} \) and \( \hat{v} \) are as shown in Lemma 4.1, \( M_1 = \left\| f \right\|_{H^1} \) and \( I_1 = \left\| L_2 \right\| + \left\| L_1 \right\|_{L^2}^{(p+1) \left( p-1 \right)} \).
Proof. The uniqueness of solutions \( \bar{u} \) and \( \hat{u} \) of problems (4.15), (4.16) and (4.17), (4.18) is obvious. For the existence of \( \bar{u} \), first, let \( \bar{y} \) be the unique solution of the linear problem

\[
\bar{y}_t + M(\|\nabla u\|^2)\bar{y} = M(\|\nabla u\|^2)\bar{v}, \quad t > 0, \\
\bar{y}(0) = u_1 - \Delta u_0 + u_0. 
\]  

(4.21)

By virtue of (4.6) and the standard estimate, we have

\[
\|\bar{y}(t)\|^2 + \|\bar{y}_t(t)\|^2 \leq C\|(u_0, u_1)\|_X^2 e^{-\kappa t}, \quad t > 0, 
\]  

(4.22)

where the standard estimate indicates that the exponential decay estimate of \( \bar{y} \) in \( L^2 \) is gotten by taking \( L^2 \)-inner product with \( \bar{y} \) in the equation in (4.21), and the exponential decay estimate of \( \bar{y}_t \) in \( L^2 \) is obtained by using (4.21). Next, we consider the linear problem

\[
\bar{u}_t - \Delta \bar{u} + \bar{u} = \bar{y}, \quad t > 0, \\
\bar{u}(0) = u_0. 
\]  

(4.23)

Since \( \bar{y} \in W^{1, \infty}([0, \infty); L^2) \), problem (4.23) admits a unique solution \( \bar{u} \), with \( \bar{u} \in L^\infty([0, \infty); H^2) \), \( \bar{u}_t \in L^\infty([0, \infty); L^2) \). \( \bar{u} \) is the solution of problem (4.15), (4.16). With the standard method, estimate (4.19) is obtained, where the standard method indicates that the exponential decay estimate of \( \bar{u} \) in \( H^1 \) is gotten by taking \( L^2 \)-inner product with \( \bar{u} + \bar{u}_t \) in the equation in (4.23), the exponential decay estimate of \( \bar{u}_t \) in \( L^2 \) is obtained by differentiating the equation in (4.23) with respect to \( t \) and taking \( L^2 \)-inner product with \( \bar{u}_t \) in the resulting expression, and the exponential decay estimate of \( \Delta \bar{u} \) in \( L^2 \) is received by using (4.23) and (4.22).

Similarly, let \( \hat{y} \) be the unique solution of the linear problem

\[
\hat{y}_t + M(\|\nabla u\|^2)\hat{y} = G_2, \quad t > 0, \\
\hat{y}(0) = 0. 
\]  

(4.24)

Since \( G_2 \in L^\infty([0, \infty); H^1) \) from (2.3)–(2.4) and (4.13), problem (4.24) admits a unique solution \( \hat{y} \in L^\infty([0, \infty); H^1) \). Note that when \( N = 4 \),

\[
H^2 \hookrightarrow H^s \hookrightarrow L^q, \quad \text{with} \ 1 < s < 2 \ \text{and} \ 2 \leq q \leq \frac{4}{2-s},
\]

by the arbitrariness of \( s \): \( 1 < s < 2 \) we get that of \( q \): \( 2 \leq q < +\infty \), and hence, by the interpolation theorem,

\[
\|u\|_{2(p+1)} \leq C\|u\|_{H^2}, \quad p \leq \frac{4}{(N-4)^+} (p < \infty).
\]

Similarly,

\[
\|u\|_{2p} \leq C\|u\|_{H^2}, \quad p \leq \frac{N}{(N-4)^+} (p < \infty).
\]

Therefore,

\[
\|G_2(t)\|_{H^1}^2 \leq \tilde{C} e^{-\kappa t} + C_2, \quad t > 0,
\]  

(4.25)
where (and in the sequel) $C_2 = C(M_1, I_0, I_1, L^*)$, (2.3) and the fact
\[
\|\nabla x g + g_u \nabla u\| \leq C\left[\|u\|_{L^p}^p + \|L_2\| + \|u\|^{p-1} \nabla u\| + \|L_1 \nabla u\|\right]
\]
\[
\leq C\left[\|u\|_{H^2}^p + \|L_2\| + \|\nabla u\|_{p+1} (\|u\|_{L^2}^{p-1} + \|L_1\|^{2(p+1)})\right]
\]
\[
\leq C\left[\|u\|_{H^2}^p + \|L_2\| + \|L_1\|^{2(p+1)}\right]
\]
have been used. By virtue of (4.25) we get easily from (4.24) that
\[
\|\hat{\gamma}(t)\|^2 + \|\hat{y}(t)\|^2 \leq \tilde{C} e^{-\kappa t} + C_2, \quad t > 0.
\] (4.26)
Taking $L^2$-inner product with $-\Delta \hat{y}$ in (4.24), we have
\[
\frac{1}{2} \int_{\mathbb{R}^N} M(\|\nabla u\|^2) \nabla \hat{y}\|^2 \, dx = (\nabla G_2, \nabla \hat{y}),
\]
\[
\frac{d}{dt} \|\nabla \hat{y}(t)\|^2 + \frac{1}{2} \|\hat{y}(t)\|^2 \leq \tilde{C} e^{-\kappa t} + C_2,
\]
\[
\|\nabla \hat{y}(t)\|^2 \leq \tilde{C} e^{-\kappa t} + C_2, \quad t > 0.
\] (4.27)
Next, we consider the linear problem
\[
\hat{u}_t - \Delta \hat{u} + \hat{u} = \hat{y}, \quad t > 0, \quad \hat{u}(0) = 0.
\] (4.28)
Since $\hat{y} \in L^\infty((0, \infty); H^1)$ and $\hat{y}_t \in L^\infty((0, \infty); L^2)$, problem (4.28) admits a unique solution $\hat{u} \in L^\infty((0, \infty); H^3)$. Taking $L^2$-inner product with $\hat{u} - \Delta \hat{u}$ in the equation in (4.28) and exploiting (4.26), we arrive at
\[
\frac{1}{2} \frac{d}{dt} \left(\|\hat{u}(t)\|^2 + \|\nabla \hat{u}(t)\|^2\right) + 2 \|\nabla \hat{u}(t)\|^2 + \frac{1}{2} \left(\|\hat{u}(t)\|^2 + \|\Delta \hat{u}(t)\|^2\right) \leq \|\hat{y}(t)\|^2,
\]
\[
\|\hat{u}(t)\|_{H^1}^2 \leq \tilde{C} e^{-\kappa t} + C_2, \quad t > 0.
\] (4.29)
Differentiating (4.28) with respect to $t$, taking $L^2$-inner product with $-\Delta \hat{u}_t$ in the resulting expression and making use of (4.26), we get
\[
\frac{1}{2} \frac{d}{dt} \left(\|\nabla \hat{u}_t(t)\|^2 + \|\nabla \hat{u}_t(t)\|^2 + \frac{1}{2} \|\Delta \hat{u}_t(t)\|^2\right) \leq \frac{1}{2} \|\hat{y}_t(t)\|^2,
\]
\[
\|\nabla \hat{u}_t(t)\|^2 \leq \tilde{C} e^{-\kappa t} + C_2, \quad t > 0.
\] (4.30)
Therefore, it follows from (4.27)–(4.30) that
\[
\|\nabla \Delta \hat{u}(t)\|^2 \leq \|\nabla \hat{y}(t)\|^2 + \|\nabla \hat{u}(t)\|^2 + \|\nabla \hat{u}_t(t)\|^2
\]
\[
\leq C(M_1, L^*, I_0, I_1), \quad t \geq T_0.
\] (4.31)
The combination of (4.29) with (4.31) yields (4.20). Lemma 4.2 is proved. \qed
Step 3 (The decomposition of $S(t)$).

Define

$$P(t)(u_0, u_1) = \left( \bar{u}(t), \bar{v}(t) \right), \quad U(t)(u_0, u_1) = \left( \hat{u}(t), \hat{v}(t) \right).$$

(4.32)

Lemmas 4.1–4.2 imply $S(t) = P(t) + U(t)$. The combination of (4.19) with (4.6) shows that for any $(u_0, u_1) \in B_0 \subset X$,

$$\| \left( \bar{u}(t), \bar{v}(t) \right) \|_X \leq C \| (u_0, u_1) \|_X e^{-\kappa t} \to 0 \quad \text{as} \quad t \to \infty,$$

(4.33)

i.e. the map $P(t) : X \to X$ is continuous and satisfies (4.1). The combination of (4.20) with (4.7) yields

$$\| \left( \hat{u}(t), \hat{v}(t) \right) \|_{X^*} \leq C(M_1, I_0, I_1, L^*), \quad t \geq T_0.$$

(4.34)

In order to get the precompactness of $U(t)$, besides (4.34), we also need a tail estimate for $(\hat{u}, \hat{v})$ in $X$. For this purpose, we first discuss the tail estimates of $(u, u_t)$ in space $X_1 = H^2 \times L^2$ by using a directly cut-off technique both in $x$ and $t$.

Step 4 (The tail estimates of $(u, u_t)$ in $X_1 = H^2 \times L^2$).

Let

$$K_0(s) =\begin{cases} 0, & 0 \leq s \leq 1, \\ s - 1, & 1 < s \leq 2, \\ 1, & s > 2, \end{cases}$$

$$K_\delta(s) = (\rho_\delta \ast K_0)(s) = \int_R \rho_\delta(s - y) K_0(y) \, dy,$$

(4.35)

where $\rho_\delta(s)$ is a standard mollifier on $\mathbb{R}$ with supp $\rho_\delta \subset [-\delta, \delta]$. Obviously, $K_\delta \in C^\infty(\mathbb{R})$, $0 \leq K_\delta(s) \leq 1$, $K_\delta(s) = 0$ when $0 \leq s < 1$; $K_\delta(s) = 1$ when $s > 2$, with $\delta \ll 1$, and

$$0 \leq \left[ \frac{d}{ds} K_\delta(s) \right]^{1+\theta} \leq C(\theta) K_\delta(s), \quad 0 < \theta \leq 1.$$

(4.36)

Let $\varphi(x) = K_\delta(\frac{|x|}{R})$, $\psi(t) = K_\delta(\frac{t}{T})$, then $0 \leq \varphi(x), \psi(t) \leq 1$, especially,

$$\psi(T) = K_\delta(1) < \delta, \quad \left| \nabla \varphi(x) \right|^2 \leq \frac{C}{R^2} \varphi(x), \quad x \in \mathbb{R}^N,$$

$$\left| \psi'(t) \right|^2 \leq \frac{C}{T^2} \psi(t), \quad t \in \mathbb{R}^+.$$

(4.37)

Lemma 4.3. Under the assumptions of Lemma 4.1, we have

$$\| \varphi \psi u(t) \|^2 + \| \varphi \psi \nabla u(t) \|^2 + \| \varphi \psi \Delta u(t) \|^2 + \| \varphi \psi u_t(t) \|^2 \leq C \delta + A_1(R, T), \quad t > 0,$$

(4.38)
where

\[ A_1(R, T) = A(R, T) + A^{p(1-\theta_1)}(R/2, T/2) + \|L_1\|^2_{L^2(B(R)C)}, \]

\[ A(R, T) = \bar{C} \left( \frac{1}{R^2} + \frac{1}{T^2} \right) + C \left( \|f\|^2_{L^2(B(R)C)} + \|L_0\|_{L^1(B(R)C)} \right), \tag{4.39} \]

\[ \theta_1 = \frac{(p-1)N}{p[4(p+1)-(p-1)N]} \quad (0 < \theta_1 < 1) \quad \text{and} \quad \bar{C} = C(R_0, M_0, L^*, I_0). \]

**Proof.** Obviously, \( \|\varphi \psi \nabla u(t)\|^2 \equiv 0, t \in [0, T), \delta \ll 1. \) We discuss the following two cases.

1. If \( \frac{d}{dt}\|\varphi \psi \nabla u(t)\|^2 \geq 0, t \in [0, T + t_0] \) \((t_0 > 0)\), by taking \( L^2 \)-inner product with \( \varphi^2\psi^2u_t \) in (1.1) and by noticing

\[ M(\|\nabla u\|^2) \frac{d}{dt} \|\varphi \psi \nabla u\|^2 \geq M(\|\varphi \psi \nabla u\|^2) \frac{d}{dt} \|\varphi \psi \nabla u\|^2, \quad t \in [0, T + t_0], \tag{4.40} \]

we obtain

\[ \frac{d}{dt} e_1^*(t) + \|\varphi \psi u_t(t)\|^2 + \|\varphi \psi \nabla u(t)\|^2 \]

\[ \leq \psi \psi' \left( \|\varphi u_t(t)\|^2 + \|\varphi u(t)\|^2 + M(\|\nabla u\|^2) \|\varphi \nabla u\|^2 \right) \]

\[ + 2\psi \psi' \int_{\mathbb{R}^N} \varphi^2 (G(x, u) + L(x) - fu) \, dx \]

\[ - 2 \int_{\mathbb{R}^N} \varphi \psi^2 \nabla \varphi u_t (\nabla u_t + M(\|\nabla u\|^2) \nabla u) \, dx \equiv h(t), \quad t \in [0, T + t_0]. \tag{4.41} \]

where

\[ e_1^*(t) = \frac{1}{2} \left( \|\varphi u_t(t)\|^2 + \|\varphi u(t)\|^2 + \|\varphi \nabla u(t)\|^2 + \frac{2}{m+2} \|\varphi \nabla u(t)\|^{m+2} \right) \]

\[ + \int_{\mathbb{R}^N} \varphi^2 \psi^2 (G(x, u) + L(x)) \, dx - \int_{\mathbb{R}^N} \varphi^2 \psi^2 fu \, dx, \quad t \in [0, T + t_0]. \tag{4.42} \]

Let \( \tilde{e}_1^*(t) = e_1^*(t) + \|\varphi f\|^2, \) then \( \frac{d}{dt} \tilde{e}_1^*(t) = \frac{d}{dt} e_1^*(t), \) and

\[ \tilde{e}_1^*(t) \geq \frac{1}{4} \|\varphi u_t(t)\|^2 + \frac{1}{2} \left( \|\varphi u_t(t)\|^2 + \|\varphi \nabla u(t)\|^2 + \frac{2}{m+2} \|\varphi \nabla u(t)\|^{m+2} \right) \]

\[ + \int_{\mathbb{R}^N} \varphi^2 \psi^2 (G(x, u) + L(x)) \, dx, \quad t \geq 0. \tag{4.43} \]

Taking \( L^2 \)-inner product with \( \varphi^2\psi^2u \) in (1.1), we get
\[
\frac{d}{dt} \left[ \frac{1}{2} \left( \| \phi \psi u(t) \|^2 + \| \phi \psi \nabla u(t) \|^2 \right) + (\phi \psi u, \phi \psi u_t) \right] + \| \phi \psi u(t) \|^2 \\
+ M(\| \nabla u \|^2) \| \phi \psi \nabla u(t) \|^2 + \int_{\mathbb{R}^N} \phi^2 \psi^2 [g(x, u) + L_0(x)] \, dx \\
= \| \phi \psi u_t(t) \|^2 + \psi \psi' \left( \| \phi u(t) \|^2 + \| \phi \nabla u(t) \|^2 \right) + 2 \int_{\mathbb{R}^N} \psi \psi' \phi \psi u \, dx \\
- 2 \int_{\mathbb{R}^N} \phi \psi^2 \nabla \phi \cdot \left( M(\| \nabla u \|^2) \nabla u + \nabla u_t \right) \, dx \\
+ \int_{\mathbb{R}^N} \phi^2 \psi^2 (L_0(x) + fu) \, dx, \quad t > 0.
(4.44)
\]

The combination of (4.41) with (4.44) yields

\[
\frac{d}{dt} e^*_2(t) + h^*_1(t) \leq 5 |\psi \psi'|(\| \phi u_t(t) \|^2 + \| \phi u(t) \|^2 + M(\| \nabla u \|^2) \| \phi \nabla u(t) \|^2) \\
+ \psi \psi' \int_{\mathbb{R}^N} \phi^2 [8 (G(x, u) + L(x) - fu) + 2 u_t u] \, dx \\
- \int_{\mathbb{R}^N} \phi \psi^2 \nabla \phi (8 u_t + 2 u) \cdot (\nabla u_t + M(\| \nabla u \|^2) \nabla u) \, dx \\
+ \| \phi L_0 \|_{L^1} + \frac{1}{8} \| \phi \psi u(t) \|^2 + 2 \| \phi f \|^2, \quad t \in [0, T + t_0].
(4.45)
\]

where

\[
e^*_2(t) = 4 \bar{e}^*_1(t) + \frac{1}{2} \left( \| \phi \psi u(t) \|^2 + \| \phi \psi \nabla u(t) \|^2 \right) + (\phi \psi u, \phi \psi u_t) \\
\geq \| \phi \psi u_t(t) \|^2 + \| \phi \psi u(t) \|^2 + \| \phi \psi \nabla u(t) \|^2 \\
+ \frac{4}{m+2} \| \phi \psi \nabla u(t) \|^{m+2} + 4 \int_{\mathbb{R}^N} \phi^2 \psi^2 (G(x, u) + L(x)) \, dx, 
(4.46)
\]

\[
h^*_1(t) = 3 \| \phi \psi u_t(t) \|^2 + 4 \| \phi \psi \nabla u_t(t) \|^2 + \| \phi \psi u(t) \|^2 + M(\| \nabla u \|^2) \| \phi \psi \nabla u(t) \|^2 \\
+ \int_{\mathbb{R}^N} \phi^2 \psi^2 (g(x, u)u + L_0(x)) \, dx, \quad t \in [0, T + t_0].
(4.47)
\]

By the Cauchy inequality, (4.37) and (2.3),
\[5 \psi'(\|\psi u_t(t)\|^2 + \|\varphi u(t)\|^2 + M(\|\nabla u\|^2)\|\varphi \nabla u(t)\|^2)\]
\[\leq \frac{1}{8}[\|\psi u_t(t)\|^2 + \|\varphi u(t)\|^2 + M(\|\nabla u\|^2)\|\varphi \nabla u(t)\|^2] + \frac{\bar{C}}{T^2}, \quad (4.48)\]
\[\psi' \int_{\mathbb{R}^N} \varphi^2 [8(G(x, u) + L(x) - fu) + 2u_t u] \, dx\]
\[\leq \frac{1}{8} \left( \int_{\mathbb{R}^N} \varphi^2 \psi^2 (G(x, u) + L(x)) \, dx + \|\varphi u(t)\|^2 \right) + \frac{\bar{C}}{T^2}, \quad (4.49)\]
\[\int_{\mathbb{R}^N} \varphi \psi \nabla \varphi (8u_t + 2u) \cdot (\nabla u_t + M(\|\nabla u\|^2)\nabla u) \, dx\]
\[\leq \frac{1}{8}[\|\varphi \nabla u_t(t)\|^2 + M(\|\nabla u\|^2)\|\varphi \nabla u(t)\|^2] + \frac{\bar{C}}{R^2}, \quad t > 0. \quad (4.50)\]

Substituting (4.48)–(4.50) into (4.45), integrating the resulting expression over \((0, t)\), noticing \(e^*_2(0) = 4\|\varphi f\|^2\),
\[h^*_1(t) - \kappa e^*_2(t) \geq -C\|\varphi f\|^2, \quad (4.51)\]

and (4.46), we have
\[\|\varphi \psi u_t(t)\|^2 + \|\varphi \psi u(t)\|^2 + \|\varphi \psi \nabla u(t)\|^2 + \int_{\mathbb{R}^N} \varphi^2 \psi^2 (G(x, u) + L(x)) \, dx\]
\[\leq \tilde{C} \left( \frac{1}{R^2} + \frac{1}{T^2} \right) + C(\|f\|^2_{L^2(B(R)c)} + \|L_0\|_{L^1(B(R)c)})\]
\[\equiv A(R, T), \quad t \in [0, T + t_0]. \quad (4.52)\]

If \(\frac{d}{dt} \|\varphi \psi \nabla u(t)\|^2 \leq 0, t \in [T + t_0, T + t_1], (t_1 > t_0)\), then by (4.52),
\[\|\varphi \psi \nabla u(t)\|^2 \leq \|\varphi \psi \nabla u(T + t_0)\|^2 \leq A(R, T), \quad t \in [T + t_0, T + t_1]. \quad (4.53)\]

Since
\[\frac{1}{2}M(\|\nabla u\|^2) \frac{d}{dt} \|\varphi \psi \nabla u(t)\|^2\]
\[= \frac{1}{2} \frac{d}{dt} [M(\|\nabla u\|^2)\|\varphi \psi \nabla u(t)\|^2] - M'(\|\nabla u\|^2)(\nabla u, \nabla u_t) \|\varphi \psi \nabla u(t)\|^2, \quad (4.54)\]

replacing (4.41) we have
\[\frac{d}{dt} e^*_1(t) + \|\varphi \psi u_t(t)\|^2 + \|\varphi \psi \nabla u(t)\|^2\]
\[\leq h(t) + M'(\|\nabla u\|^2)(\nabla u, \nabla u_t) \|\varphi \psi \nabla u(t)\|^2, \quad t \in [T + t_0, T + t_1], \quad (4.55)\]
where \( h(t) \) is as shown in (4.41), and

\[
\bar{e}_1^*(t) = \frac{1}{2} \left( \| \varphi \psi u_t(t) \|^2 + \| \varphi \psi u(t) \|^2 + M(\| \nabla u \|^2) \| \varphi \psi \nabla u(t) \|^2 \right) \\
\quad + \int_{\mathbb{R}^N} \varphi^2 \psi^2 (G(x, u) + L(x)) \, dx \\
\quad - \int_{\mathbb{R}^N} \varphi^2 \psi^2 f u \, dx + \| \varphi f \|^2, \quad t \in [T + t_0, T + t_1]. \tag{4.56}
\]

By (2.3) and (4.53),

\[
\left| M' (\| \nabla u \|^2) (\nabla u, \nabla u_t) \| \varphi \psi \nabla u(t) \|^2 \right| \leq A(R, T), \quad t \in [T + t_0, T + t_1]. \tag{4.57}
\]

Repeating the same arguments as above, we still get

\[
\frac{d}{dt} e_2^*(t) + \kappa e_2^*(t) \leq A(R, T), \quad t \in [T + t_0, T + t_1], \tag{4.58}
\]

where

\[
e_2^*(t) = 4 \bar{e}_1^*(t) + \frac{1}{2} \left( \| \varphi \psi u(t) \|^2 + \| \varphi \psi \nabla u(t) \|^2 \right) + (\varphi \psi u, \varphi \psi u_t).
\]

Hence,

\[
\| \varphi \psi u_t(t) \|^2 + \| \varphi \psi u(t) \|^2 + \| \varphi \psi \nabla u(t) \|^2 + \int_{\mathbb{R}^N} \varphi^2 \psi^2 (G(x, u) + L(x)) \, dx \\
\leq e_2^*(T + t_0) + A(R, T) \leq A(R, T), \quad t \in [0, T + t_1], \tag{4.59}
\]

and where (4.46) and (4.52) have been used.

2. If \( \frac{d}{dt} \| \varphi \psi \nabla u(t) \|^2 \leq 0, \) \( t \in [0, T + t_0] \) \( (t_0 > 0) \), then by (2.3) and (4.37),

\[
\| \varphi \psi \nabla u(t) \|^2 \leq \psi^2(T) \| \varphi \nabla u(T) \|^2 \leq C \delta, \quad t \in [0, T + t_0]. \tag{4.60}
\]

Repeating the similar arguments as used in the second part of case 1, we have

\[
\| \varphi \psi u_t(t) \|^2 + \| \varphi \psi u(t) \|^2 + \| \varphi \psi \nabla u(t) \|^2 + \int_{\mathbb{R}^N} \varphi^2 \psi^2 (G(x, u) + L(x)) \, dx \\
\leq C \delta + A(R, T), \quad t \in [0, T + t_0]. \tag{4.61}
\]

If \( \frac{d}{dt} \| \varphi \psi \nabla u(t) \|^2 \geq 0, \) \( t \in [T + t_0, T + t_1] \) \( (t_1 > t_0) \), we repeat the similar arguments as used in the first part of case 1 and know that (4.61) holds true on \( [0, T + t_1] \).

Therefore, repeating above process we see that (4.61) holds true for \( t \geq 0 \), and hence

\[
\| u_t(t) \|^2_{L^2(B(2R)^c)} + \| u(t) \|^2_{H^1(B(2R)^c)} \leq C \delta + A(R, T), \quad t > 2T. \tag{4.62}
\]
Taking $L^2$-inner product with $-\varphi^2\psi^2\Delta u$ in (1.1), we have
\[
\frac{d}{dt} e_3(t) + M \left( \| \nabla u \|^2 \right) \| \varphi \psi \Delta u(t) \|^2 \\
\leq |\psi'\| \| \varphi \Delta u(t) \|^2 - 2 \int_{\mathbb{R}^N} \psi' \varphi^2 u_t \Delta u \, dx + \frac{1}{4} \| \varphi \psi \Delta u(t) \|^2 \\
+ C \left( \| \varphi \psi u_t(t) \|^2 + \| \varphi \psi u(t) \|^2 + \| \varphi \psi g(t) \|^2 + \| \varphi f \|^2 \right),
\]
(4.63)
where
\[
e_3(t) = \frac{1}{2} \| \varphi \psi \Delta u(t) \|^2 - (\varphi \psi u_t, \varphi \psi \Delta u) \\
\geq \frac{1}{4} \| \varphi \psi \Delta u(t) \|^2 - \| \varphi \psi u_t(t) \|^2, \quad t > 0.
\]
(4.64)

By the Gagliardo–Nirenberg inequality, (4.62), (2.3) and (4.37),
\[
\| \varphi \psi g(t) \|^2 \leq C \psi^2 \left( \| u(t) \|^2_{L^p(B(R)c)} + \| \varphi L_1 \|^2 \right) \\
\leq C \psi^2 \left( \| u(t) \|^2_{L^{p(1-\theta_1)}(B(R)c)} \| \Delta u(t) \|^2_{L^2(B(R)c)} + \| L_1 \|^2_{L^2(B(R)c)} \right) \\
\leq C \psi^2 \left( \| u(t) \|^2_{L^{2(p-1)}(B(R)c)} + \| L_1 \|^2_{L^{2}(B(R)c)} \right) \\
\leq C \delta + A \rho(1-\theta_1)(R/2, T/2) + \| L_1 \|^2_{L^{2}(B(R)c)},
\]
(4.65)
\[
|\psi'| \| \psi \Delta u(t) \|^2 \leq \frac{1}{8} \| \psi \psi u(t) \|^2 + \frac{C}{T^2},
\]
(4.66)
\[
\left| 2 \int_{\mathbb{R}^N} \psi' \varphi^2 u_t \Delta u \, dx \right| \leq \frac{1}{8} \| \psi \psi \Delta u(t) \|^2 + \frac{C}{T^2}, \quad t > 0,
\]
(4.67)
where $\theta_1$ is as shown in (4.39). Substituting (4.65)–(4.67) into (4.63), noticing
\[
\frac{1}{2} M \left( \| \nabla u \|^2 \right) \| \varphi \psi \Delta u(t) \|^2 - \kappa e_3(t) \geq -C \kappa \| \varphi \psi u_t(t) \|^2, \quad t > 0,
\]
(4.68)
and exploiting (4.61) and (4.64), we get
\[
\| \varphi \psi \Delta u(t) \|^2 \leq C \delta + A_1(R, T), \quad t > 0,
\]
(4.69)
where $A_1(R, T)$ is as shown in (4.39). The combination of (4.61) with (4.69) yields (4.38). Lemma 4.3 is proved. □

**Step 5.** (The tail estimates of $(\hat{u}, \hat{v})$ in $X$).
Lemma 4.4. Under the assumptions of Lemma 4.1,

\[ \| (\hat{u}(t), \hat{v}(t)) \|_{X(B(2R)^C)}^2 \leq A_1(R, T), \quad t > 2T, \]  

(4.70)

where \( A_1(R, T) \) is as shown in (4.39).

Remark 2. It can be seen from (4.70) that for any \( \epsilon > 0 \), there exist \( R_1 = R_1(R_0) > 0 \) and \( T_0 = T_0(R_0) > 0 \) such that

\[ \| (\hat{u}(t), \hat{v}(t)) \|_{X(B(2R)^C)}^2 < \epsilon \]  

(4.71)

as long as \( R > R_1 \) and \( t > T_0 \).

Proof. Taking \( L^2 \)-inner product with \( \varphi^2 \psi^2(\hat{v} + \hat{v}_t) \) in the equation in (4.5), we obtain

\[ \frac{d}{dt} \left[ \| \varphi \psi \hat{v}(t) \|^2 + \frac{1}{2} \| \varphi \psi \nabla \hat{v}(t) \|^2 \right] + \| \varphi \psi \hat{v}(t) \|^2 + \| \varphi \psi \nabla \hat{v}(t) \|^2 + \| \varphi \psi \hat{v}_t(t) \|^2 \\
= \psi \psi' \left( 2 \| \varphi \hat{v}(t) \|^2 + \| \varphi \nabla \hat{v}(t) \|^2 \right) - 2 \int \mathbb{R}^N \varphi^2 (\hat{v} + \hat{v}_t) \nabla \varphi \cdot \nabla \hat{v} \, dx + (\varphi \psi G_1, \varphi \psi (\hat{v} + \hat{v}_t)) \\
\leq \frac{1}{4} \left( \| \varphi \psi \hat{v}(t) \|^2 + \| \varphi \psi \nabla \hat{v}(t) \|^2 + \| \varphi \psi \hat{v}_t(t) \|^2 \right) + \tilde{C} \left( \frac{1}{R^2} + \frac{1}{T^2} \right) \\
+ C \| \varphi \psi G_1(t) \|^2, \quad t > 0. \]  

(4.72)

Since

\[ \| \varphi \psi G_1(t) \|^2 \leq 4(M^2(\| \nabla u \|^2) \| \varphi \psi \Delta u(t) \|^2 + \| \varphi \psi u(t) \|^2 + \| \varphi \psi g(t) \|^2 + \| \varphi f \|^2) \\
\leq C \delta + A_1(R, T), \quad t > 0, \]  

(4.73)

where estimates (4.38) and (4.65) have been used, we get from (4.72) that

\[ \| \varphi \psi \hat{v}(t) \|^2 + \| \varphi \psi \nabla \hat{v}(t) \|^2 \leq C \delta + A_1(R, T), \quad t > 0. \]  

(4.74)

Taking \( L^2 \)-inner product with \( \varphi^2 \psi^2 \hat{y} \) in the equation in (4.24), we arrive at

\[ \frac{1}{2} \frac{d}{dt} \| \varphi \psi \hat{y}(t) \|^2 + M^2(\| \nabla u \|^2) \| \varphi \psi \hat{y}(t) \|^2 \\
= (\varphi \psi G_2, \varphi \psi \hat{y}) + \psi \psi' \| \varphi \hat{y}(t) \|^2 \\
\leq \frac{1}{2} \| \varphi \psi \hat{y}(t) \|^2 + \| \varphi \psi G_2(t) \|^2 + \tilde{C} T^2 \\
\leq \frac{1}{2} \| \varphi \psi \hat{y}(t) \|^2 + C \delta + A_1(R, T), \quad t > 0, \]  

(4.75)

where (4.26) and the fact
\[ \| \varphi \psi G_2(t) \| ^2 \leq 4 \left[ M^2(\| \nabla u \| ^2)(\| \varphi \psi \hat{u}(t) \| ^2 + \| \varphi \psi u(t) \| ^2) + \| \varphi \psi g(t) \| ^2 + \| \varphi \psi f \| ^2 \right] \leq C\delta + A_1(R, T), \quad t > 0 \]  

(4.76)

have been used. Hence, it follows from (4.75) and Eq. (4.24) that

\[ \| \varphi \psi \hat{y}(t) \| ^2 + \| \varphi \psi \hat{y}_t(t) \| ^2 \leq C\delta + A_1(R, T), \quad t > 0. \]  

(4.77)

Taking \( L^2 \)-inner product with \( \varphi^2 \psi^2 \hat{u} \) in (4.28), we arrive at

\[
\frac{1}{2} \frac{d}{dt} \| \varphi \psi \hat{u}(t) \| ^2 + \| \varphi \psi \hat{u}_t(t) \| ^2 + \| \varphi \psi \nabla \hat{u}(t) \| ^2 = \psi \varphi' \| \varphi \hat{u}(t) \| ^2 - 2 \int_{\mathbb{R}^N} \varphi^2 \psi \nabla \varphi \cdot \nabla \hat{u} \, dx + (\varphi \psi \hat{y}, \varphi \psi \hat{u}) \\
\leq \frac{1}{2} \| \varphi \psi \hat{u}(t) \| ^2 + \tilde{C} \left( \frac{1}{R^2} + \frac{1}{T^2} \right) + \| \varphi \psi \hat{y}(t) \| ^2, \\
\| \varphi \psi \hat{u}(t) \| ^2 \leq C\delta + A_1(R, T), \quad t > 0. \]  

(4.78)

Differentiating (4.28) with respect to \( t \) and taking \( L^2 \)-inner product by \( \varphi^2 \psi^2 \hat{u}_t \) in the resulting expression, we have

\[
\frac{1}{2} \frac{d}{dt} \| \varphi \psi \hat{u}_t(t) \| ^2 + \| \varphi \psi \hat{u}_t(t) \| ^2 + \| \varphi \psi \nabla \hat{u}_t(t) \| ^2 \\
= \psi \varphi' \| \varphi \hat{u}_t(t) \| ^2 - 2 \int_{\mathbb{R}^N} \varphi^2 \psi \nabla \varphi \cdot \nabla \hat{u}_t \, dx + (\varphi \psi \hat{y}_t, \varphi \psi \hat{u}_t) \\
\leq \frac{1}{2} \left( \| \varphi \psi \hat{u}_t(t) \| ^2 + \| \varphi \psi \nabla \hat{u}_t(t) \| ^2 \right) + \tilde{C} \left( \frac{1}{R^2} + \frac{1}{T^2} \right) + C \| \varphi \psi \hat{y}_t(t) \| ^2, \\
\| \varphi \psi \hat{u}_t(t) \| ^2 \leq C\delta + A_1(R, T), \quad t > 0, \]  

(4.79)

where (4.37) and (4.77) have been used. Therefore, by (4.28), (4.77)–(4.79),

\[ \| \varphi \psi \Delta \hat{u}(t) \| ^2 \leq 3\left( \| \varphi \psi \hat{u}_t(t) \| ^2 + \| \varphi \psi \hat{u}(t) \| ^2 + \| \varphi \psi \hat{y}(t) \| ^2 \right) \leq C\delta + A_1(R, T), \quad t > 0. \]  

(4.80)

The combination of (4.80), (4.78) with (4.74) yields

\[ \| (\hat{u}(t), \hat{v}(t)) \| ^2_{L^2(\mathbb{R}^N \times (2R)^C)} \leq C\delta + A_1(R, T), \quad t > 2T. \]  

(4.81)

By the arbitrariness of \( \delta \), we get (4.70) from (4.81). Lemma 4.4 is proved. \( \square \)
Lemma 4.4 shows that for any $\epsilon > 0$ and any bounded sequence $\{(u^m_0, u^m_1)\} \subset B(R_0) \subset X$, there exist $R_1 = R_1(R_0) > 0$ and $T_0 = T_0(R_0) > 0$ such that

$$ \| (\hat{u}^m(t), \hat{v}^m(t)) \|_{X(B(2R)C)} ^2 < \epsilon $$

as long as $R > R_1$, $t > T_0$, where $\hat{v}^m(t)$ are the solutions of problem (4.5) (substituting $u$ there by $u^m$, where $u^m$ are the solutions of problem (1.1), (1.2) corresponding to initial data $(u^m_0, u^m_1)$), and $\hat{u}^m(t)$ are the solutions of problem (4.17), (4.18) (substituting $u$ and $\hat{v}$ there by $u^m$ and $\hat{v}^m$, respectively). Since $\{(\hat{u}^m(t), \hat{v}^m(t))\}$ is uniformly bounded in $X^* = H^3 \times H^2$ as $t \geq T_0$ (see (4.34)), and $X^*(B(2R)) \hookrightarrow \hookrightarrow X(B(2R))$ for any $R > 0$, we can select a subsequence $\{(\hat{u}^{m'}(t), \hat{v}^{m'}(t))\}$ such that

$$ \| (\hat{u}^{m'}(t), \hat{v}^{m'}(t)) - (\hat{u}^{n'}(t), \hat{v}^{n'}(t)) \|_X \leq \| (\hat{u}^{m'}(t), \hat{v}^{m'}(t)) - (\hat{u}^{n'}(t), \hat{v}^{n'}(t)) \|_{X(B(2R))} + 2\epsilon $$

as long as $m', n' > N(\epsilon)$, $R > R_1$ and $t > T_0$, i.e. $(\hat{u}^{m'}(t), \hat{v}^{m'}(t))$ is convergent in $X$, that is that $U(t)$ (see (4.32)) is precompact as $t > T_0$.

Then the continuous semigroup $S(t)$ possesses a global attractor $\mathcal{A}$.

**The finite fractal and Hausdorff dimension of $\mathcal{A}$.

**Lemma 4.5.** (See [8].) Let $X$ be a Banach space and $S : X \to X$ be a continuous map. If $S$ is bounded (that is, it takes bounded sets to bounded sets), and $S = P + U$, where $P$ is global Lipschitz continuous with a Lipschitz constant $k \in [0, 1)$, and $U$ is conditionally completely continuous, then $S$ is a $\beta$-contraction, and hence, an $\alpha$-contraction (see [8] for the definition of both $\beta$-contraction and $\alpha$-contraction).

The global estimate (2.3) shows that the continuous semigroup $S(t)$ is bounded for $t \geq 0$. Since $U(t)$ (see (4.32)) is completely continuous as $t \geq T_0$, by Lemma 4.5, $S(t) = P(t) + U(t)$ is a $\alpha$-contraction as $t > T_0$ if for any $(u_0, u_1), (\tilde{u}_0, \tilde{u}_1) \in X$,

$$ \| P(t)(u_0, u_1) - P(t)(\tilde{u}_0, \tilde{u}_1) \|_X \leq k \| (w_0, w_1) \|_X \quad (4.83) $$

as $t > T_0$, where $k \in [0, 1)$, $w_0 = u_0 - \tilde{u}_0$, $w_1 = u_1 - \tilde{u}_1$.

Indeed, let $w(t) = \tilde{u}(t) - \tilde{u}(t)$, $v(t) = \tilde{v}(t) - \tilde{v}(t)$, where $(\tilde{u}(t), \tilde{v}(t)) = P(t)(\tilde{u}_0, \tilde{u}_1)$, and $(\tilde{u}(t), \tilde{v}(t))$ as shown in (4.32). By (4.4) and (4.15)–(4.16) we see that $v(t)$ and $w(t)$ satisfy, respectively,

$$ v_t - \Delta v + v = 0, \quad t > 0, \quad v(\cdot, 0) = w_1, \quad (4.84) $$

$$ (w_t - \Delta w + w, t + M \| \nabla u \|^2 (w_t - \Delta w + w) = M \| \nabla u \|^2 v, \quad t > 0, \quad (4.85) $$

$$ w(\cdot, 0) = w_0, \quad w_t(\cdot, 0) = w_1 \quad (4.86) $$

By Lemmas 4.1 and 4.2 (see (4.6) and (4.19)),

$$ \| w(t) \|^2_{H^2} + \| v(t) \|^2_{H^1} \leq C \| (w_0, w_1) \|^2_X e^{-\Delta t}, \quad t > 0. \quad (4.87) $$
(4.87) implies
\[
\| (w(t), v(t)) \|_X \leq k \| (w_0, w_1) \|_X, \quad t > T_0,
\]
with \( k \in [0, 1) \), i.e. (4.83) holds. Therefore, \( S(t) \) is an \( \alpha \)-contraction as \( t > T_0 \). As a direct consequence of Theorem 2.8.1 of Hale [8] we know that \( \mathcal{A} \) has finite fractal and Hausdorff dimension. Theorem 2.2 is proved. \( \Box \)

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**References**