Discrete Applied Mathematics 157 (2009) 685–690

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Minimally 3-restricted edge connected graphs☆

Qinghai Liu, Yanmei Hong, Zhao Zhang*

College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, PR China

ARTICLE INFO

Article history:
Received 12 December 2007
Received in revised form 13 June 2008
Accepted 31 July 2008
Available online 27 August 2008

Keywords:
Fault tolerance
Restricted edge connectivity

ABSTRACT

For a connected graph $G = (V, E)$, an edge set $S \subset E$ is a 3-restricted edge cut if $G - S$ is disconnected and every component of $G - S$ has order at least three. The cardinality of a minimum 3-restricted edge cut of G is the 3-restricted edge connectivity of G, denoted by $\lambda_3(G)$. A graph G is called minimally 3-restricted edge connected if $\lambda_3(G - e) < \lambda_3(G)$ for each edge $e \in E$. A graph G is λ_3-optimal if $\lambda_3(G) = \xi_3(G)$, where $\xi_3(G) = \max\{\omega(U) : U \subset V(G),\ G[U]$ is connected, $|U| = 3\}$, $\omega(U)$ is the number of edges between U and $V \setminus U$, and $G[U]$ is the subgraph of G induced by vertex set U. We show in this paper that a minimally 3-restricted edge connected graph is always λ_3-optimal except the 3-cube.

1. Introduction

A network can be conveniently modeled as a graph $G = (V, E)$. A classic measure of the fault tolerance of a network is the edge connectivity $\lambda(G)$. In general, the larger $\lambda(G)$ is, the more reliable the network is [2]. A more refined measure known as restricted edge connectivity was proposed by Esfahanian and Hakimi [5], which was further generalized to k-restricted edge connectivity by Fábrega and Fiol [6] (called k-extra edge connectivity in their paper).

Let G be a connected graph. An edge set $S \subset E(G)$ is said to be a *k-restricted edge cut* of G if $G - S$ is disconnected and each component of $G - S$ has at least k vertices. The minimum cardinality of a k-restricted edge cut is called the *k-restricted edge connectivity of* G, denoting by $\lambda_k(G)$. A k-restricted edge cut S with $|S| = \lambda_k(G)$ is called a *λ_k-cut*. Not all graphs have λ_k-cuts [4,5,12,21]. Those which do have λ_k-cuts are called *λ_k-connected* graphs. According to current studies on k-restricted edge connectivity [10,11,13,17], it seems that the larger $\lambda_k(G)$ is, the more reliable the network is. In [21], Zhang and Yuan proved that $\lambda_k(G) \leqslant \xi_k(G)$ holds for any integer $k \leqslant \delta(G) + 1$ except for a class of graphs (such a graph is constructed from a set of complete subgraphs K_δ by adding a new vertex u and connect u to every other vertex), where $\delta(G)$ is the minimum degree of G and $\xi_k(G) = \min\{\omega(U) : U \subset V(G)$, G[U] is connected,$|U| = k\}$, $\omega(U)$ is the number of edges between U and $V \setminus U$, and $G[U]$ is the subgraph of G induced by U. A graph G is called *λ_k-optimal* if $\lambda_k(G) = \xi_k(G)$. There is much research on sufficient conditions for a graph to be λ_k-optimal, such as symmetric conditions [11,13,17,18], degree conditions [14,15,19], and girth-diameter conditions [1,6,16,20]. For more information on this topic, we refer the readers to the nice survey paper by Hellwig and Volkmann [7].

In this paper, we give another type of sufficient condition called a minimally restricted edge connected condition. A graph G is a *minimally k-restricted edge connected graph* (*minimally λ_k-graph* for short) if $\lambda_k(G - e) < \lambda_k(G)$ (and thus $\lambda_k(G - e) = \lambda_k(G) - 1$) for each edge $e \in E(G)$. It is implied in the definition that $\lambda_k(G - e)$ exists for each edge e. If e is a pending edge, then $G - e$ does not have λ_k-cut for $k \geqslant 2$. So, we always assume $\delta(G) \geqslant 2$ when G is a minimally λ_k-graph for some $k \geqslant 2$. A minimally λ_1-graph is exactly a minimally edge connected graph, which has been shown to be λ-optimal ([9] Exercise 49). In [8], the authors have proved that every minimally λ_2-graph is λ_2-optimal. In this paper, we show that every minimally λ_3-graph is always λ_3-optimal except the 3-cube.

☆ This research is supported by NSFC (60603003), the Key Project of Chinese Ministry of Education (208161) and XJEDU.

* Corresponding author. Tel.: +86 13899960204; fax: +86 991 8585505.
E-mail address: zhzhao@xju.edu.cn (Z. Zhang).

0166-218X/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2008.07.009

2. Preliminaries and terminologies

Let $G = (V, E)$ be a graph. For two disjoint vertex sets $U_1, U_2 \subset V(G)$, denote by $[U_1, U_2]_G$ the set of edges of G with one end in U_1 and the other end in U_2, $G[U]$ is the subgraph of G induced by vertex set $U \subset V(G)$, $\overline{U} = V(G) \setminus U$ is the complement of U, $\omega_G(U) = |[U, \overline{U}]_G|$ is the number of edges between U and $\overline{U}$. When the graph under consideration is obvious, we omit the subscript G. Write $d_A(U) = |[U, A \setminus U]|$, $d_A(u) = d_A(\{u\})$. Sometimes, we use a graph itself to represent its vertex set. For example, $\omega(C)$ is used instead of $\omega(V(C))$, where C is a subgraph of G; for an edge $e = uv$, $d_A(e)$ is used instead of $d_A(\{u, v\})$, etc.

A λ_3-*fragment* is a subset U of $V(G)$ with $[U, \overline{U}]$ being a λ_3-cut. If U is a λ_3-fragment, then so is $\overline{U}$, and both $G[U]$ and $G[\overline{U}]$ are connected. A λ_3-fragment with minimum order is called a λ_3-*atom*. The order of a λ_3-atom is denoted by $\alpha_3(G)$. Clearly, $\alpha_3(G) \leqslant |V(G)|/2$.

A graph H is λ_3-*independent* if each component of H has at most two vertices. A connected graph of order at most two is λ_3-*trivial*. A graph is called λ_3-*non-trivial* if it has a component which contains at least three vertices.

The following two observations will be used frequently without mentioning them explicitly. The first is that if two connected subgraphs G_1 and G_2 have nonempty intersection, then $G_1 \cup G_2$ is also connected. The second is that for a vertex set F of a connected graph G and a component C of $G - F$, if $G[F]$ is connected, then so is $G - C$.

For terminologies not given here, we refer to [3] for reference.

3. Main result

First, it should be noted that if $\alpha_3(G) = 3$, then G is λ_3-optimal. In fact, Bonsma et al. [4] have proved that $\lambda_3(G) \leqslant \xi_3(G)$ holds for any λ_3-connected graph G. On the other hand, considering a λ_3-atom A of G, we have $\lambda_3(G) = \omega(A) \geqslant \xi_3(G)$. So, $\lambda_3(G) = \xi_3(G)$. In view of this observation, to derive our main theorem, it suffices to show that $\alpha_3(G) = 3$. In the following, we prove that if G is a minimally 3-restricted edge connected graph with $\alpha_3(G) \geqslant 4$, then G is isomorphic to 3-cube.

Lemma 1. *Let G be a λ_3-connected graph with $\delta(G) \geqslant 2$, F be a subset of G with $G[\overline{F}]$ being connected. If one of the following conditions is satisfied:*

(a) $\omega(F) < \lambda_3(G)$, *or*
(b) $\omega(F) = \lambda_3(G)$ *and* $|F| < \alpha_3(G)$,

then $G[F]$ is λ_3-independent.

Proof. Suppose F has a λ_3-non-trivial component C. Since $G[\overline{C}]$ is connected, we have $\omega(F) \geqslant \omega(C) \geqslant \lambda_3(G)$, which is clearly a contradiction to condition (a). If condition (b) occurs, then $\omega(F) = \omega(C) = \lambda_3(G)$. Hence $V(C)$ is a λ_3-fragment. But then $|F| \geqslant |C| \geqslant \alpha_3(G)$, contradicting $|F| < \alpha_3(G)$. □

Lemma 2. *Let G be a λ_3-connected graph with $\delta(G) \geqslant 2$ and $\alpha_3(G) \geqslant 4$, A be a λ_3-atom of G, and B be a λ_3-fragment of G.*

(a) *If a subset U of A is such that $G[U]$ is connected and $G[A \setminus U]$ has a λ_3-non-trivial component, then $d_A(U) > d_{\overline{A}}(U)$.*
(b) *If a subset U of B is such that $G[U]$ is connected and $G[B \setminus U]$ has a λ_3-non-trivial component, then $d_B(U) \geqslant d_{\overline{B}}(U)$.*
(c) $\delta(G[A]) \geqslant 2$.

Proof. (a). Suppose $d_A(U) \leqslant d_{\overline{A}}(U)$. Then $\omega(A \setminus U) = \omega(A) + d_A(U) - d_{\overline{A}}(U) \leqslant \omega(A) = \lambda_3(G)$. By noting that $|A \setminus U| < |A| = \alpha_3(G)$, it follows from Lemma 1 that $G[A \setminus U]$ is λ_3-independent, a contradiction.
(b). The proof of (b) is similar to that of (a); note that under the assumption $d_B(U) < d_{\overline{B}}(U)$, it can be deduced that $\omega(B \setminus U) < \lambda_3(G)$.
(c) is a consequence of (a). In fact, for each vertex $x \in A$, if $A - x$ is λ_3-independent, then $d_A(x) \geqslant 2$ since $|A| \geqslant 4$. If $A - x$ contains a λ_3-non-trivial component, taking $U = \{x\}$ in (a), we have $d_A(x) > d_{\overline{A}}(x)$. Then it follows from $d_A(x) > \frac{1}{2} \cdot (d_A(x) + d_{\overline{A}}(x)) = \frac{1}{2} \cdot d_G(x) \geqslant 1$ that $d_A(x) \geqslant 2$. □

Similar to Lemma 2, we can prove the following lemma. The key observation to the proof, as well as some proofs after it, is that for any edge $e \in E(G)$, if $\lambda_3(G - e) < \lambda_3(G)$, then any λ_3-fragment of $G - e$ contains exactly one end of e, and is a λ_3-fragment of G. Note that the observation is true when G is minimally 3-restricted edge connected.

Lemma 3. *Let G be a λ_3-connected graph with $\delta(G) \geqslant 2$ and $\alpha_3(G) \geqslant 4$, $e = uv$ be an edge of G, $\lambda_3(G - e) < \lambda_3(G)$, A be a λ_3-atom of $G - e$ with $u \in A$ and $v \notin A$.*

(a) *If a subset U of A is such that $G[U]$ is connected, $G[A] - U$ has a λ_3-non-trivial component, and e is not incident with U, then $d_A(U) > d_{\overline{A}}(U)$.*
(b) $d_{G[A]}(x) \geqslant 2$ *for each $x (\neq u) \in A$.*

Lemma 4. *Let G be a λ_3-connected graph with $\delta(G) \geqslant 2$ and $\alpha_3(G) \geqslant 4$, A be a λ_3-atom of G, B be a λ_3-fragment of G, $A \cap B \neq \emptyset$. Then for any component C of $G[A \cap B]$, either $G[A] - C$ or $G[B] - C$ is λ_3-independent.*

Proof. Suppose both $G[A]-C$ and $G[B]-C$ have λ_3-non-trivial components. Then by taking $U=V(C)$ in Lemma 2, we have

$$d_A(U) > d_{\overline{A}}(U) \geqslant d_{B\setminus A}(U) = d_B(U)$$

and

$$d_B(U) \geqslant d_{\overline{B}}(U) \geqslant d_{A\setminus B}(U) = d_A(U),$$

a contradiction. □

Similar to Lemma 4, by using Lemma 3 instead of Lemma 2, it can be proved that

Lemma 5. *Let G be a λ_3-connected graph with $\delta(G)\geqslant 2$ and $\alpha_3(G)\geqslant 4$, $e=uv$ be an edge of G, $\lambda_3(G-e)<\lambda_e(G)$, A be a λ_3-atom of $G-e$ with $u\in A$ and $v\notin A$, B be a λ_3-fragment of G such that $A\cap B\neq\emptyset$ and e is not incident with B. For any component C of $G[A\cap B]$, either $G[A]-C$ or $G[B]-C$ is λ_3-independent.*

Lemma 6. *Let G be a minimally λ_3-graph with $\alpha_3(G)\geqslant 4$, A be a λ_3-atom of G, $e=uv$ be an edge of $G[A]$, B be a λ_3-atom of $G-e$. Then,*

(a) *$G[A\cap B]$, $G[A\setminus B]$ and $G[B\setminus A]$ are all connected;*
(b) *$|A\cap B|=2$, $|A\setminus B|=2$;*
(c) *$G[A]$ is C_4 (4-cycle) or K_4 (complete graph on 4 vertices) or K_4^- (K_4 minus one edge);*
(d) *$G[\overline{A\cup B}]$ is connected;*
(e) *$|[A\cap B,\overline{A\cup B}]|=|[A\setminus B,B\setminus A]|=0$, and $d_{\overline{A}}(x)=d_A(x)-1$ for each $x\in A$.*

Proof. By the observation before Lemma 3, we may suppose, without loss of generality, that $u\in B$ and $v\in\overline{B}$. Then $A\cap B\neq\emptyset$ and $A\setminus B\neq\emptyset$. Because of $|A|\leqslant\frac{1}{2}|V(G)|$ and $|B|\leqslant\frac{1}{2}|V(G)|$, we have $|\overline{A\cup B}|=|V(G)|-|A|-|B|+|A\cap B|\geqslant|A\cap B|$.

(a). Suppose $G[A\cap B]$ has two components C_1, C_2. If one of C_1 and C_2, say C_2, has at least two vertices, then by the connectedness of $G[A]$ and $G[B]$, both $G[A]-C_1$ and $G[B]-C_1$ have λ_3-non-trivial components containing C_2, contradicting Lemma 4. Next, suppose $V(C_1)=\{x\}$, $V(C_2)=\{y\}$, and $y\neq u$. By Lemma 2(c), we have $d_A(y)\geqslant 2$. By Lemma 3(b), we have $d_B(y)\geqslant 2$. So both $G[A]-C_1$ and $G[B]-C_1$ have λ_3-non-trivial components, again a contradiction. So, $G[A\cap B]$ is connected.

Suppose $G[A\setminus B]$ is not connected. Then a contradiction can be obtained by using Lemma 4 to A and $\overline{B}$ (note that $\overline{B}$ is also a λ_3-fragment). In fact, the case that $G[A\setminus B]$ has a component with at least two vertices can be analyzed similar to the above paragraph. In the case that $A\setminus B$ is an independent set with at least two vertices, we have $|A\cap B|\geqslant 2$ since $\delta(G[A])\geqslant 2$. By the connectedness of $G[A\cap B]$, we see that for any vertex $y\in A\setminus B$, $G[A]-y$ is λ_3-non-trivial. If there is a vertex $y\in A\setminus B$ with degree one in $G[\overline{B}]$, then $G[\overline{B}]-y$ is λ_3-non-trivial. If there exists a vertex $x\in A\setminus B$ with $d_{G[\overline{B}]}(x)\geqslant 2$, let y be another vertex in $A\setminus B$. Then $G[\overline{B}]-y$ has a λ_3-non-trivial component containing x. Taking $C=\{y\}$ in Lemma 4 and using $\overline{B}$ to take the place of B, we arrive at a contradiction.

Similar to the above deduction, by using Lemmas 3 and 5 instead of Lemmas 2 and 4, it can be proved that $G[B\setminus A]$ is also connected.

(b). First suppose $|A\cap B|=1$. By $|B|\geqslant|A|\geqslant 4$, we have $|B\setminus A|\geqslant|A\setminus B|\geqslant 3$. By (a), $G[B\setminus A]$ and $G[A\setminus B]$ are both λ_3-non-trivial, which contradicts Lemma 4(taking $U=A\cap B$ there).

Next suppose $|A\cap B|\geqslant 3$. By (a), $G[A\cap B]$ is λ_3-non-trivial. By the connectedness of $G[\overline{A}]$ and $G[\overline{B}]$, we see that $G[\overline{A\cap B}]=G[\overline{A}\cup\overline{B}]$ is also connected. Taking $F=A\cap B$ in Lemma 1, by noting that $|F|=|A\cap B|<|A|=\alpha_3(G)$, we have $\omega(A\cap B)>\lambda_3(G)$. Combining this with $\omega(A)=\omega(B)=\lambda_3(G)$ and the following well known submodular inequality,

$$\omega(A\cap B)+\omega(A\cup B)\leqslant\omega(A)+\omega(B),$$

we have $\omega(\overline{A\cup B})=\omega(A\cup B)<\lambda_3(G)$. Taking $F=\overline{A\cup B}$ in Lemma 1, we see that $G[\overline{A\cup B}]$ is λ_3-independent, that is, $G[\overline{A\cup B}]$ is composed of some singletons and complete graphs of order two.

We claim that for each component C of $G[\overline{A\cup B}]$, $d_A(C)=d_B(C)$ and $d_{A\cap B}(C)=0$. It should be noted that C is connected to both $A\setminus B$ and $B\setminus A$, since $G[\overline{A}]$ and $G[\overline{B}]$ are connected. First suppose C is a singleton x. Since $|\overline{A\cup B}|\geqslant|A\cap B|\geqslant 3$ and $G[B\setminus A]$ is connected, we see that $G[\overline{A}]-x$ has a connected subgraph of order at least 3. Applying Lemma 2(b) to the λ_3-fragment $\overline{A}$ and the vertex set $\{x\}$, we have $d_{\overline{A}}(x)\geqslant d_A(x)$. Hence

$$d_A(x)\leqslant d_{\overline{A}}(x)=d_{B\setminus A}(x)=d_B(x)-d_{A\cap B}(x).$$

Similarly,

$$d_B(x)\leqslant d_{\overline{B}}(x)=d_{A\setminus B}(x)=d_A(x)-d_{B\cap A}(x).$$

It follows that $d_A(x)=d_B(x)$ and $d_{A\cap B}(x)=0$. Next suppose C is an edge $e=v_1v_2$. If $|\overline{A\cup B}|=3$ and $|A\setminus B|=1$, then $G[\overline{A\cup B}]$ is composed of e and a singleton y. Set $A'=(A\setminus B)\cup\{v_1,v_2\}$. Then both $G[A']$ and $G[\overline{A'}]$ are connected subgraphs of order at least 3. Since $d_{A\setminus B}(y)=d_{B\setminus A}(y)$, we have $\omega(A')=\omega(\overline{A'})=\omega(B\cup\{y\})=\omega(B)+d_{A\setminus B}(y)-d_{B\setminus A}(y)=\omega(B)=\lambda_3(G)$.

So A' is a λ_3-fragment. But then $\alpha_3(G) \leqslant |A'| = 3$, contradicting the assumption that $\alpha_3(G) \geqslant 4$. So, $|\overline{A \cup B}| \geqslant 4$ or $|A \setminus B| \geqslant 2$ (hence $|B \setminus A| \geqslant 2$). Taking the place of x by e in the proof for C being a singleton, we have $d_A(e) = d_B(e)$ and $d_{A\cap B}(e) = 0$. The claim is proved.

As a consequence of the claim, for any vertex set U which is the union of the vertex sets of some components in $G[\overline{A \cup B}]$, we have $d_{A\setminus B}(U) = d_{B\setminus A}(U)$ and $d_{A\cap B}(U) = 0$.

Let U be the vertex set of one or two components of $G[\overline{A \cup B}]$ such that $|U| = 2$. Set $A' = (A \setminus B) \cup U$. Then $3 \leqslant |A'| < |A|$ and both $G[A']$ and $G[\overline{A'}]$ are connected. Since $\omega(A') = \omega(\overline{A'}) = \omega(B) + d_{A\setminus B}(\overline{A \cup B} \setminus U) - d_{B\setminus A}(\overline{A \cup B} \setminus U) = \omega(B) = \lambda_3(G)$, we see that A' is a λ_3-fragment of G with less vertices than A, a contradiction.

Hence $|A \cap B| = 2$. It follows that $|A \setminus B| = |A| - |A \cap B| \geqslant 2$. If $|A \setminus B| \geqslant 3$, then $|B \setminus A| \geqslant 3$. By (a), both $G[A \setminus B]$ and $G[B \setminus A]$ are λ_3-non-trivial, contradicting Lemma 4. So, $|A \setminus B| = 2$, and (b) is proved.

(c) follows from (a), (b) and Lemma 2(c).

Suppose $A \setminus B = \{a, d\}$ and $A \cap B = \{b, c\}$. Then $ad, bc \in E(G)$.

Claim 1. *For each edge $e = xy \in E(G[A])$, $d_A(e) \geqslant d_{\overline{A}}(e)$, equality holds if and only if $d_{\overline{A}}(x) = d_A(x) - 1$ and $d_{\overline{A}}(y) = d_A(y) - 1$.*

Taking $U = \{x\}$ in Lemma 2(a), we have $d_A(x) > d_{\overline{A}}(x)$. Hence $d_{\overline{A}}(x) \leqslant d_A(x) - 1 = d_{A\setminus\{x,y\}}(x)$. Similarly, $d_{\overline{A}}(y) \leqslant d_A(y) - 1 = d_{A\setminus\{x,y\}}(y)$. Then the claim follows from

$$d_A(xy) = d_{A\setminus\{x,y\}}(x) + d_{A\setminus\{x,y\}}(y) \geqslant d_{\overline{A}}(x) + d_{\overline{A}}(y) = d_{\overline{A}}(xy).$$

Claim 2. *$d_{A\setminus B}(bc) = d_{B\setminus A}(bc)$, $d_{\overline{A\cup B}}(bc) = 0$, $d_{\overline{A}}(b) = d_A(b) - 1$ and $d_{\overline{A}}(c) = d_A(c) - 1$.*

By Claim 1, we have

$$d_{A\setminus B}(bc) = d_A(bc) \geqslant d_{\overline{A}}(bc) = d_{B\setminus A}(bc) + d_{\overline{A\cup B}}(bc). \tag{1}$$

If $|B \setminus A| = 2$, then B is also a λ_3-atom of G, and similar to (1), we have

$$d_{B\setminus A}(bc) = d_B(bc) \geqslant d_{\overline{B}}(bc) = d_{A\setminus B}(bc) + d_{\overline{A\cup B}}(bc). \tag{2}$$

If $|B \setminus A| \geqslant 3$, then by (a), $G[B \setminus A]$ is λ_3-non-trivial. By Lemma 2(b), inequality (2) is still valid. Combining (1) and (2), we have the first and the second equality of the claim. As a consequence, equality holds for (1), and thus the third and the fourth equality of the claim follow from Claim 1.

(d). Suppose C_1 and C_2 are two components of $G[\overline{A \cup B}]$. Since $d_A(ad) \geqslant d_{\overline{A}}(ad)$ by Claim 1, we have

$$\begin{aligned}\omega(C_1) < \omega(\overline{A \cup B}) &= \omega(A \cup B) = \omega(B) + d_{\overline{A\cup B}}(ad) - d_{A\cup B}(ad)\\ &= \omega(B) + (d_{\overline{A}}(ad) - d_{B\setminus A}(ad)) - (d_{B\setminus A}(ad) + d_A(ad))\\ &= \omega(B) - d_A(ad) + d_{\overline{A}}(ad) - 2d_{B\setminus A}(ad)\\ &\leqslant \omega(B) = \lambda_3(G).\end{aligned}$$

By Lemma 1, C_1 is λ_3-independent, and thus $|V(C_1)| \leqslant 2$. Since $G[B \setminus A]$ is connected, $|B \setminus A| \geqslant 2$, and C_2 is connected to $B \setminus A$, we see that $G[\overline{A}] - C_1$ has a λ_3-non-trivial component. By Lemma 2(b) and Claim 2, we have

$$d_A(C_1) \leqslant d_{\overline{A}}(C_1) = d_{B\setminus A}(C_1) = d_B(C_1).$$

Similarly,

$$d_B(C_1) \leqslant d_{\overline{B}}(C_1) = d_{A\setminus B}(C_1) = d_A(C_1).$$

It follows that $d_B(C_1) = d_A(C_1)$. Set $A' = (A \setminus B) \cup C_1$. Then both $G[A']$ and $G[\overline{A'}]$ are connected. By Claim 2, we have $\omega(A') = \omega(A\setminus B) + d_B(C_1) - d_{A\setminus B}(C_1) = \omega(A\setminus B) + d_B(C_1) - d_A(C_1) = \omega(A\setminus B) = \omega(A) + d_{A\setminus B}(bc) - d_{B\setminus A}(bc) = \omega(A) = \lambda_3(G)$. Hence A' is a λ_3-fragment of G. since $|A'| \leqslant |A|$, we see that A' is also a λ_3-atom of G. Applying Claim 1 to A' and A respectively, we have

$$d_{A'}(ad) \geqslant d_{\overline{A'}}(ad) \geqslant d_A(ad) + d_{C_2}(ad) > d_A(ad),$$

and

$$d_A(ad) \geqslant d_{\overline{A}}(ad) \geqslant d_{C_1}(ad) = d_{A'}(ad),$$

a contradiction. So, $G[\overline{A \cup B}]$ is connected.

(e). Noting that in proving Claim 2, it suffices for B to be a λ_3-fragment (it needs not to be a λ_3-atom of $G - e$). Hence, by taking the place of B by $\overline{B}$, we have $d_{B\setminus A}(ad) = 0$, and $d_{\overline{A}}(a) = d_A(a) - 1$, $d_{\overline{A}}(d) = d_A(d) - 1$. Together with Claim 2, the results follow immediately. □

Theorem 1. *The only minimally λ_3-connected graph which is not λ_3-optimal is the 3-cube.*

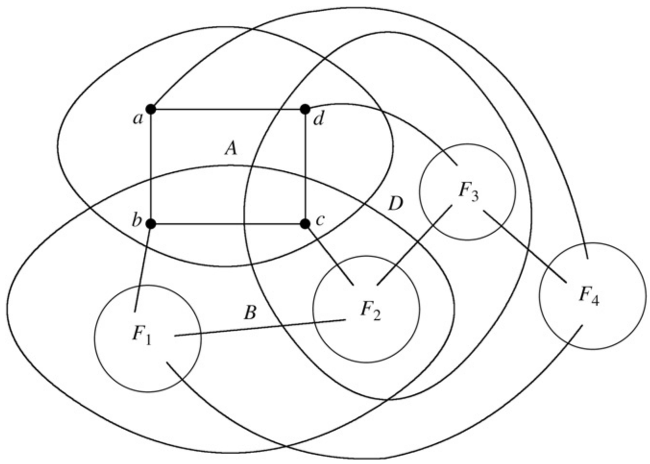

Fig. 1. An illustration for the proof of Theorem 1.

Proof. Suppose G is a minimally λ_3-connected graph which is not λ_3-optimal. Then $\alpha_3(G) \geqslant 4$. Let A be a λ_3-atom of G, $e = uv$ be an edge of $G[A]$, and B be a λ_3-atom of $G - e$. Then Lemma 6 is applicable to A and B. Suppose $A \cap B = \{b, c\}$, $A \setminus B = \{a, d\}$. Then $ad, bc \in E(G)$. By Lemma 6(c), we may assume, without loss of generality, that $ab, cd \in E(G)$.

By Lemma 6(e), $d_{\overline{A}}(x) = d_A(x) - 1$ for each $x \in \{a, b, c, d\}$. Hence

$$\begin{aligned}\lambda_3(G) = \omega(A) &= d_{\overline{A}}(a) + d_{\overline{A}}(b) + d_{\overline{A}}(c) + d_{\overline{A}}(d)\\ &= d_A(a) + d_A(d) - 2 + d_A(b) + d_A(c) - 2\\ &= d_A(ad) + d_A(bc)\\ &= 2|[ad, bc]|.\end{aligned}$$

Let D be a λ_3-atom of $G - bc$. Then Lemma 6 is also applicable to A and D. Suppose, without loss of generality, that $A \cap D = \{c, d\}$. Set $F_1 = (B \setminus A) \setminus D$, $F_2 = (B \setminus A) \cap D$, $F_3 = \overline{(A \cup B)} \cap D$, $F_4 = \overline{(A \cup B)} \setminus D$ (see Fig. 1).

By Lemma 6(e), we have

$$|[\{a, d\}, F_1 \cup F_2]| = |[\{b, c\}, F_3 \cup F_4]| = |[\{a, b\}, F_2 \cup F_3]| = |[\{c, d\}, F_1 \cup F_4]| = 0. \tag{3}$$

It follows that $|[a, F_1 \cup F_2 \cup F_3]| = 0$, and thus $|[a, F_4]| = |[a, \overline{A}]| = d_{\overline{A}}(a) = d_A(a) - 1 \geqslant \delta(G[A]) - 1 \geqslant 1$. By the same argument, we have

$$\begin{aligned}&|[a, F_4]| = d_{\overline{A}}(a) \geqslant 1, \qquad |[b, F_1]| = d_{\overline{A}}(b) \geqslant 1,\\ &|[d, F_3]| = d_{\overline{A}}(d) \geqslant 1, \qquad |[c, F_2]| = d_{\overline{A}}(c) \geqslant 1.\end{aligned} \tag{4}$$

As a consequence,

$$[A, \overline{A}] = [a, F_4] \cup [b, F_1] \cup [c, F_2] \cup [d, F_3], \tag{5}$$

and none of F_1, F_2, F_3, F_4 is empty. Since the four subgraphs $G[F_1 \cup F_2] = G[B \setminus A]$, $G[F_2 \cup F_3] = G[D \setminus A]$, $G[F_1 \cup F_4] = G[\overline{A \cup D}]$ and $G[F_3 \cup F_4] = G[\overline{A \cup B}]$ are all connected, we have

$$|[F_1, F_2]| \geqslant 1, \qquad |[F_2, F_3]| \geqslant 1, \qquad |[F_1, F_4]| \geqslant 1, \qquad |[F_3, F_4]| \geqslant 1. \tag{6}$$

By equality (3), we have

$$\begin{aligned}\omega(B) &= |[bc, ad]| + |[F_1 \cup F_2, F_3 \cup F_4]|,\\ \omega(D) &= |[dc, ab]| + |[F_2 \cup F_3, F_1 \cup F_4]|.\end{aligned} \tag{7}$$

Because of Lemma 6(c), we consider the following three cases.

Case 1. $G[A] \cong C_4$.

In this case, $\lambda_3(G) = 2|[ad, bc]| = 4$. Then equalities hold in (4). By the equalities (7) and the inequalities (6), we have

$$\omega(B) \geqslant 4 \quad \text{and} \quad \omega(D) \geqslant 4. \tag{8}$$

Since $\omega(B) = \omega(D) = \lambda_3(G) = 4$, all inequalities in (8) become equalities. In particular, $|[F_1, F_2]| = |[F_1, F_4]| = 1$ and $|[F_1, F_3]| = 0$. Combining these with $|[F_1, A]| = |[b, F_1]| = 1$, we have $\omega(F_1) = 3 < \lambda_3(G)$. Then by Lemma 1, $G[F_1]$ is λ_3-independent. Since $G[F_1 \cup F_2]$ is connected and $|[F_1, F_2]| = 1$, we see that $G[F_1]$ is connected. Thus $|F_1| \leqslant 2$.

If $|F_1| = 2$, then $G[F]$ is λ_3-non-trivial, where $F = F_1 \cup \{b\}$. On the other hand, by noting that $|F| = 3 < \alpha_3(G)$ and $\omega(F) = \omega(F_1) - |[b, F_1]| + d_A(b) = 3 - 1 + 2 = 4 = \lambda_3(G)$, it follows from Lemma 1 that $G[F]$ is λ_3-independent, a contradiction. So $|F_1| = 1$. Similarly, it can be deduced that $|F_2| = |F_3| = |F_4| = 1$. Then G is a 3-cube.

Case 2. $G[A] \cong K_4^-$.

In this case, $\lambda_3(G) = 2|[ad, bc]| = 6$. We assume, without loss of generality, that $bd \in E(G)$. By equality (7) and $\omega(B) = \omega(D) = \lambda_3(G) = 6$, we have

$$|[F_1, F_3 \cup F_4]| + |[F_2, F_3 \cup F_4]| = |[F_1 \cup F_2, F_3 \cup F_4]| = 3, \tag{9}$$

and

$$|[F_2 \cup F_3, F_1]| + |[F_2 \cup F_3, F_4]| = |[F_2 \cup F_3, F_1 \cup F_4]| = 3. \tag{10}$$

By (6) and (9), one of $|[F_1, F_3 \cup F_4]|$ and $|[F_2, F_3 \cup F_4]|$ is 1. Suppose, without loss of generality, that $|[F_1, F_3 \cup F_4]| = 1$. Then $|[F_1, F_4]| \leq 1$. By (6) and (10), $|[F_1, F_2 \cup F_3]| \leqslant 2$. By equality (5) and Lemma 6(e), $|[b, F_1]| = d_{\overline{A}}^-(b) = d_A(b) - 1 = 2$. Hence $\omega(F_1) = |[b, F_1]| + |[F_1, F_2 \cup F_3]| + |[F_1, F_4]| \leqslant 5 < \lambda_3(G)$. By Lemma 1, $G[F_1]$ is λ_3-independent. Since $G[F_1 \cup F_4]$ is connected and $|[F_1, F_4]| = 1$, we see that $G[F_1]$ is connected and thus $|F_1| \leqslant 2$. Combining this with $|[b, F_1]| = 2$, we have $|F_1| = 2$. Set $F = F_1 \cup \{b\}$. Then both $G[F]$ and $G[\overline{F}]$ are connected. By noting that $|F| = 3 < \alpha_3(G)$ and

$$\omega(F) = \omega(F_1) + d_A(b) - |[b, F_1]| \leqslant 5 + 3 - 2 = 6 = \lambda_3(G),$$

it follows from Lemma 1 that $G[F]$ is λ_3-independent, a contradiction.

Case 3. $G[A] \cong K_4$.

In this case, $\lambda_3(G) = 2|[ad, bc]| = 8$. For each vertex $x \in A$, we have $d_{\overline{A}}^-(x) = d_A(x) - 1 = 2$ by Lemma 6(e). It follows from (5) that $|[a, F_4]| = |[b, F_1]| = |[c, F_2]| = |[d, F_3]| = 2$. Similar to the deduction of (9) and (10), we have

$$|[F_1, F_3 \cup F_4]| + |[F_2, F_3 \cup F_4]| = |[F_2, F_1 \cup F_4]| + |[F_3, F_1 \cup F_4]| = 4.$$

Then $|[F_1, F_3 \cup F_4]| \leqslant 3$. Suppose, without loss of generality, that $|[F_2, F_1 \cup F_4]| \leqslant 2$. Then $|[F_2, F_1]| \leqslant 2$. If $|[F_1, F_3 \cup F_4]| = 3$, then $|[F_2, F_3 \cup F_4]| = 1$. Set $F = F_2 \cup \{c\}$. Then

$$\omega(F) = d_A(c) + |[F_2, F_1]| + |[F_2, F_3 \cup F_4]| \leqslant 3 + 2 + 1 = 6 < \lambda_3(G).$$

By Lemma 1, F is λ_3-independent. On the other hand, since $|[c, F_2]| = 2$, we see that $G[F]$ contains a λ_3-non-trivial component, a contradiction. If $|[F_1, F_3 \cup F_4]| \leqslant 2$, set $F = F_1 \cup \{b\}$. Then

$$\omega(F) = d_A(b) + |[F_1, F_2]| + |[F_1, F_3 \cup F_4]| \leqslant 3 + 2 + 2 = 7 < \lambda_3(G).$$

Similar to the above, we arrive at a contradiction. □

References

[1] C. Balbuena, P. Garcia-Vázquez, X. Marcote, Sufficient conditions for λ'-optimality in graphs with girth g, J. Graph Theory 52 (2006) 73–86.
[2] D. Bauer, F. Boesch, C. Suffel, R. Van Syke, On the validity of a reduction of reliable network design to a graph extremal problem, IEEE Trans. Circuits Syst. 34 (1989) 1579–1581.
[3] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan, London, 1976.
[4] P. Bonsma, N. Ueffing, L. Volkmann, Edge-cuts leaving components of order at least three, Discrete Math. 256 (2002) 431–439.
[5] A.H. Esfahanian, S.L. Hakini, On computing a conditional edge connectivity of a graph, Inform. Process. Lett. 27 (1988) 195–199.
[6] J. Fábrega, M.A. Fiol, Extraconnectivity of graphs with large girth, Discrete Math. 127 (1994) 163–170.
[7] A. Hellwig, L. Volkmann, Maximally edge-connected and vertex-connected graphs and digraphs: A survey, Discrete Math. 308 (2008) 3265–3296.
[8] Y. Hong, Q. Liu, Z. Zhang, Minimally restricted edge connected graphs, Appl. Math. Lett. (2007) doi:10.1016/j.aml.2007.09.004.
[9] L. Lovász, Combinatorial Problems and Exercises, North-Holland Publishing Company, 1979.
[10] J.X. Meng, Y.H. Ji, On a kind of restricted edge connectivity of graphs, Discrete Appl. Math. 117 (2002) 183–193.
[11] J.X. Meng, Optimally super-edge-connected transitive graphs, Discrete Math. 260 (2003) 239–248.
[12] J.P. Ou, Edge cuts leaving components of order at least m, Discrete Math. 305 (2005) 365–371.
[13] M. Wang, Q. Li, Conditional edge conectivity properties, reliability comparisons and transitivity of graphs, Discrete Math. 258 (2002) 205–214.
[14] Ying-Qian Wang, Qiao Li, Super-edge-connectivity properties of graphs with diameter 2, J. Shanghai Jiaotong Univ. 33 (6) (1999) 646–649 (in Chinese).
[15] Ying-Qian Wang, Qiao Li, Sufficient conditions for a graph to be maximally restricted edge-connected, J. Shanghai Jiaotong Univ. 35 (8) (2001) 1253–1255 (in Chinese).
[16] Ying-Qian Wang, Li Qiao, A sufficient condition for the equality between the restricted edge-connectivity and minimum edge-degree of graphs, Appl. Math. J. Chinese Univ. 16A (3) (2001) 269–275 (in Chinese).
[17] J.M. Xu, K.L. Xu, On restricted edge-connectivity of graphs, Discrete Math. 243 (2002) 291–298.
[18] Z. Zhang, J.X. Meng, Restricted edge connectivity of edge transitive graphs, Ars Combinatoria LXXVIII (2006) 297–308.
[19] Z. Zhang, J.J. Yuan, Degree conditions for restricted-edge-connectivity and isoperimetric-edge-connectivity to be optimal, Discrete Math. 307 (2007) 293–298.
[20] Z. Zhang, Q.H. liu, Sufficient conditions for a graph to be λ_k-optimal with given girth and diameter (submitted for publication).
[21] Z. Zhang, J.J. Yuan, A proof of an inequality concerning k-restricted edge connectivity, Discrete Math. 304 (2005) 128–134.