ELSEVIER
Discrete Mathematics 257 (2002) 173–175
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

The Ramsey size number of dipaths

David Reimer
Department of Mathematics and Statistics, The College of New Jersey, P.O. Box 7718, Ewing, NJ 08628-0718, USA

Received 13 July 2001; accepted 10 December 2001

Abstract

Let H be a finite graph. The Ramsey size number of H, $\hat{r}(H,H)$, is the minimum number of edges required to construct a graph such that when its edges are 2-colored it contains a monochromatic subgraph, H. In this paper we prove that the Ramsey size number of a directed path is quadratic. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Ramsey size number; Dipath; Graph theory

1. Introduction

Let G, R and B be finite graphs. The notation $G \rightarrow (R,B)$ implies when each of the edges of G are arbitrarily colored red or blue, we can always find a red copy of R or a blue copy of B in graph G. It is natural to ask how large does G have to be in order to have this property? The Ramsey size number, $\hat{r}(R,B)$, and the Ramsey number, $r(R,B)$, represent the minimum size G must be, measured in edges and vertices, respectively, if it has the property $G \rightarrow (R,B)$. Specifically,

$$\hat{r}(R,B) \equiv \left\{ \min_G (|E(G)| : G \rightarrow (R,B)) \right\},$$

$$r(R,B) \equiv \left\{ \min_G (|V(G)| : G \rightarrow (R,B)) \right\}.$$

Let P_n and $\vec{P}_n$ be the path and dipath of length n, respectively. It is well known that $O(r(P_n,P_n))$ is linear [2]. This implies that $O(\hat{r}(\vec{P}_n,\vec{P}_n)) \leqslant n^2$. This follows because if

E-mail address: dreimer@tcnj.edu (D. Reimer).

PII: S0012-365X(02)00396-5

$|V(G)| \leqslant cn$, for some constant c, then

$$|E(G)| \leqslant \binom{cn}{2},$$

which is order n^2. In 1983 Beck [1] proved that $O(\hat{r}(P_n, P_n))$ is also linear, which in turn implies $O(\hat{r}(\vec{P_n}, \vec{P_n})) \geqslant n$. In this paper we will show that the behavior of Ramsey size numbers for directed paths is radically different than that of undirected graphs even for graphs as simple as $\vec{P_n}$, the directed path with n edges. Specifically:

Theorem 1. $\frac{(n-5/2)^2}{2} \leqslant \hat{r}(\vec{P_n}, \vec{P_n})$.

2. Proof of the theorem

To prove the lower bound we will construct an edge-coloring algorithm, which restricts the size of monochromatic paths.

Lemma 2. *Any graph G with m edges can have its vertices partitioned into $k \equiv \lceil \sqrt{m/2} \rceil$ stable sets and a remainder set with not more than $\sqrt{2m}$ vertices.*

Proof. Let G be a graph with m edges. Create disjoint sets S_1 through S_k by placing as many vertices of G as possible into the sets while maintaining the stability of each S_i. Let R be the set of unused vertices. Clearly, for R to be minimal, each vertex of R must have an edge to each S_i. So G has at least $|R|k$ edges giving

$$m \geqslant |R| \lceil \sqrt{m/2} \rceil$$

and therefore

$$|R| \leqslant \frac{m}{\lceil \sqrt{m/2} \rceil} \leqslant \sqrt{2m}. \qquad \square$$

Lemma 3. *Any directed graph G with m edges can be two colored such that its largest monochromatic path has at most $\sqrt{2m} + 5/2$ vertices.*

Proof. Let G be a directed graph and apply the previous lemma. Arbitrarily partition the vertices of R into two sets S_{k+1} and S_{k+2} such that $|S_{k+1}| = \lceil |R|/2 \rceil$ and $|S_{k+2}| = \lfloor |R|/2 \rfloor$. Let (v, w) denote the directed edge going from v to w. Color the edges of G as follows:

$$(v, w) \in \text{Red} \quad \text{if} \begin{cases} v \in S_i, \ w \in S_j \text{ where } i < j, \\ v \in S_i, \ w \in S_j \text{ where } i = j = k+1, \end{cases}$$

$$(v, w) \in \text{Blue} \quad \text{if} \begin{cases} v \in S_i, \ w \in S_j \text{ where } i > j, \\ v \in S_i, \ w \in S_j \text{ where } i = j = k+2. \end{cases}$$

Consider red dipath, $\vec{P_n}$, in G colored as above. As we follow the dipath, the index of the set, S_i containing the current vertex of the dipath is nondecreasing and can only remain constant in S_{k+1}. Therefore, $\vec{P_n}$ can contain only one vertex of each S_i where $i \neq k+1$. So

$$|V(P_n)| \leqslant |\{S_i\}_{i \neq k+1}| + |S_{k+1}|.$$

Because the length, L, of path $\vec{P_n}$ is one less than the number of vertices we get

$$L \leqslant k + 1 + \left\lceil \frac{|R|}{2} \right\rceil \leqslant \left\lceil \sqrt{\frac{m}{2}} \right\rceil + \frac{|R|}{2} + \frac{3}{2} \leqslant \sqrt{2m} + \frac{5}{2}.$$

Similarly, blue dipaths are bounded by the same length. □

When the above algorithm is applied to graphs where

$$|E(G)| < \frac{(n - 5/2)^2}{2}$$

it produces a coloring where the length of monochromatic dipaths is less than n, hence proving the main theorem. □

References

[1] J. Beck, On Ramsey number of paths, trees and circuits, Internat. J. Graph Theory 7 (1983) 115–129.
[2] R. Graham, B. Rothschild, J. Spencer, Ramsey Theory, Wiley, New York, 1980.