Note

The Ramsey size number of dipaths

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Abstract

Let $H$ be a finite graph. The Ramsey size number of $H$, $\hat{r}(H,H)$, is the minimum number of edges required to construct a graph such that when its edges are 2-colored it contains a monochromatic subgraph, $H$. In this paper we prove that the Ramsey size number of a directed path is quadratic. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $G$, $R$ and $B$ be finite graphs. The notation $G \rightarrow (R,B)$ implies when each of the edges of $G$ are arbitrarily colored red or blue, we can always find a red copy of $R$ or a blue copy of $B$ in graph $G$. It is natural to ask how large does $G$ have to be in order to have this property? The Ramsey size number, $\hat{r}(R,B)$, and the Ramsey number, $r(R,B)$, represent the minimum size $G$ must be, measured in edges and vertices, respectively, if it has the property $G \rightarrow (R,B)$. Specifically,

$$\hat{r}(R,B) \equiv \left\{ \min_G (|E(G)| : G \rightarrow (R,B)) \right\},$$

$$r(R,B) \equiv \left\{ \min_G (|V(G)| : G \rightarrow (R,B)) \right\}.$$

Let $P_n$ and $\tilde{P}_n$ be the path and dipath of length $n$, respectively. It is well known that $O(r(P_n, P_n))$ is linear [2]. This implies that $O(\hat{r}(\tilde{P}_n, \tilde{P}_n)) \leq n^2$. This follows because if
\(|V(G)| \leq cn\), for some constant \(c\), then

\(|E(G)| \leq \left(\frac{cn}{2}\right)\),

which is order \(n^2\). In 1983 Beck [1] proved that \(O(\hat{r}(P_n, P_n))\) is also linear, which in turn implies \(O(\hat{r}(\vec{P}_n, \vec{P}_n)) \geq n\). In this paper we will show that the behavior of Ramsey size numbers for directed paths is radically different than that of undirected graphs even for graphs as simple as \(\vec{P}_n\), the directed path with \(n\) edges. Specifically:

**Theorem 1.** \(\frac{(n-5/2)^2}{2} \leq \hat{r}(\vec{P}_n, \vec{P}_n)\).

2. **Proof of the theorem**

To prove the lower bound we will construct an edge-coloring algorithm, which restricts the size of monochromatic paths.

**Lemma 2.** Any graph \(G\) with \(m\) edges can have its vertices partitioned into \(k \equiv \lceil \sqrt{m/2} \rceil\) stable sets and a remainder set with not more than \(\sqrt{2m}\) vertices.

**Proof.** Let \(G\) be a graph with \(m\) edges. Create disjoint sets \(S_1\) through \(S_k\) by placing as many vertices of \(G\) as possible into the sets while maintaining the stability of each \(S_i\). Let \(R\) be the set of unused vertices. Clearly, for \(R\) to be minimal, each vertex of \(R\) must have an edge to each \(S_i\). So \(G\) has at least \(|R|k\) edges giving

\(m \geq |R|\lceil \sqrt{m/2} \rceil\)

and therefore

\(|R| \leq \frac{m}{\lceil \sqrt{m/2} \rceil} \leq \sqrt{2m}\). \(\square\)

**Lemma 3.** Any directed graph \(G\) with \(m\) edges can be two colored such that its largest monochromatic path has at most \(\sqrt{2m + 5/2}\) vertices.

**Proof.** Let \(G\) be a directed graph and apply the previous lemma. Arbitrarily partition the vertices of \(R\) into two sets \(S_{k+1}\) and \(S_{k+2}\) such that \(|S_{k+1}| = \lfloor |R|/2 \rfloor\) and \(|S_{k+2}| = \lceil |R|/2 \rceil\). Let \((v, w)\) denote the directed edge going from \(v\) to \(w\). Color the edges of \(G\) as follows:

\( (v, w) \in \text{Red} \) if \(\begin{cases} v \in S_i, & w \in S_j \text{ where } i < j, \\ v \in S_i, & w \in S_j \text{ where } i = j = k + 1, \end{cases}\)

\( (v, w) \in \text{Blue} \) if \(\begin{cases} v \in S_i, & w \in S_j \text{ where } i > j, \\ v \in S_i, & w \in S_j \text{ where } i = j = k + 2. \end{cases}\)
Consider red dipath, $\tilde{P}_n$, in $G$ colored as above. As we follow the dipath, the index of the set, $S_i$ containing the current vertex of the dipath is nondecreasing and can only remain constant in $S_{k+1}$. Therefore, $\tilde{P}_n$ can contain only one vertex of each $S_i$ where $i \neq k + 1$. So

$$|V(\tilde{P}_n)| \leq |\{S_i\}_{i \neq k+1}| + |S_{k+1}|.$$ 

Because the length, $L$, of path $\tilde{P}_n$ is one less than the number of vertices we get

$$L \leq k + 1 + \left\lceil \frac{|R|}{2} \right\rceil \leq \left\lceil \frac{m}{2} \right\rceil + \frac{|R|}{2} + \frac{3}{2} \leq \sqrt{2m} + \frac{5}{2}.$$ 

Similarly, blue dipaths are bounded by the same length. □

When the above algorithm is applied to graphs where

$$|E(G)| \leq \frac{(n - 5/2)^2}{2}$$ 

it produces a coloring where the length of monochromatic dipaths is less than $n$, hence proving the main theorem. □

**References**