A NEW PROOF OF THE FREYD’S THEOREM

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In [1], P.J. Freyd solved the concreteness problem of categories proving that the Isbell’s necessary condition [2] is also sufficient. The new proof we are presenting here is based on an explicite construction of the faithful functor. The construction has, moreover, the following property: for countable categories with finite sets of morphisms between fixed objects, the functor has finite values. Thus we obtain that the Freyd’s theorem holds also in the finite set theory (which did not follow by [1]).

I am indebted to L. Kucera and A. Pultr for valuable advice. The impulse to consider the problem in finite set theory came from Z. Hedrlin and L. Kucera.

Throughout the paper, \( C \) is a category satisfying the Isbell’s condition (see [2]). Using the axiom of choice for classes, we can assume without loss of generality that objects of \( C \) are ordinals.

The objects will be indicated by letters \( A, B, C, ..., X, Y, Z, ... \), the morphisms by letters \( a, b, ..., f, g, ..., a', b', ... \), the domain (range resp.) of \( f \) by \( d(f) (r(f) \text{ resp.}) \), \( \exp X \) designates the power-set of \( X \). Disjoint unions are indicated by \( \cup \).

Definition 1. Put \( L_{A,X} = \{(a,x) ; d(a) = d(x), r(a) = A, r(x) = X\} \). Define an equivalence \( I_{A,X} \) on \( L_{A,X} \) putting \((a,x)I_{A,X}(a',x')\) whenever for every couple \((f:A \to C, g:X \to C)fa = gx \text{ iff } fa' = gx'\).

Remark. The Isbell’s condition (see [2]) says, roughly speaking, that every \( L_{A,X}/I_{A,X} \) is a set. More exactly, there is a set \( R \) of elements of \( L_{A,X} \) such that every \((a,x) \in L_{A,X}\) is equivalent to some member of \( R \).

Notation. For every couple \( A, X \) choose a subset
\[ R_{A,X} \subset L_{A,X} \]
such that no two different elements of \( R_{A,X} \) are equivalent and that every \((a, x) \in L_{A,X}\) is equivalent to an element of \( R_{A,X} \), which will be denoted by 
\( \langle a, x \rangle \).

**Definition 2.** Define a mapping \([\ , \ ]_{A,X}: L_{A,X} \rightarrow R_{A,X} \times \Pi_{Y<X} \exp R_{A,Y}\) (< is the natural ordering of ordinals) putting \([a, x]_{A,X} = (\langle a, x \rangle, (a_Y)_{Y<X})\), where \(a_Y = \{(a, y); y: d(a) \rightarrow Y\}\). Denote by \([L_{A,X}]\) the image of \(L_{A,X}\) under \([\ , \ ]_{A,X}\). We shall often write \([a, x]\) instead of \([a, x]_{A,X}\).

**Lemma 1.** Let \((a, x), (a', x')\) be elements of \(L_{A,X}, f:A \rightarrow B\). Then \([a, x] = [a', x']\) implies \([fa, x] = [fa', x']\).

**Proof.** Evidently, \(\langle a, x \rangle = \langle a', x' \rangle\) implies \(\langle fa, x \rangle = \langle fa', x' \rangle\). Similarly, \(a_Y = a'_Y\) implies \((fa)_Y = (fa')_Y\).

**Definition 3.** Put \(M_{A,X} = \{(a, x) \in L_{A,X}; a \text{ factors through } no Y < X\}\).

**Lemma 2.** If \([a, x] = [a', x']\) and \((a, x) \in M_{A,X}\), then \((a', x') \in M_{A,X}\).

**Proof.** Suppose that \(a' = B \rightarrow Y \rightarrow A, Y < X\). Then \(\langle a', y \rangle \in a'_Y\). Because \(a_Y = a'_Y\), there exists a \(u: d(a) \rightarrow Y\) such that \(\langle a, u \rangle = \langle a', y \rangle\). Since

\[
\begin{array}{ccc}
A & \xrightarrow{1_A} & A \\
\downarrow{a'} \quad \downarrow{1_A} \quad \downarrow{a} \\
Y & \xrightarrow{y} & Y \\
\end{array}
\]

commutes, the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{1_A} & A \\
\downarrow{u} \quad \downarrow{1_A} \quad \downarrow{u} \\
Y & \xrightarrow{z} & Y \\
\end{array}
\]
also commutes. This contradicts the assumption of \((a, x) \in M_{A, X}\).

A new proof of the Freyd's theorem

(a) For an object \(A\) of \(C\) put

\[
F(A) = \bigvee_{X \in \text{C}} [M_{A, X}] \cup \{O_A\}.
\]

Since \(M_{A, X} = \emptyset\) for \(X > A\), \(F(A)\) is a set.

For a morphism \(f: A \to B\) define a mapping \(F(f): F(A) \to F(B)\) as follows:

\[
F(f)(O_A) = O_B;
\]

\[
F(f) [a, x] = [fa, x], \quad \text{if} \quad (fa, x) \in M_{B, r(x)};
\]

\[
F(f) [a, x] = O_B \quad \text{otherwise}.
\]

According to Lemma 1 and Lemma 2, \(F(f)\) is correctly defined.

(b) \(F\) is a functor:

Obviously, \(F(1_A) = 1_{F(A)}\). Let \(f: A \to B, g: B \to C\) be morphisms, \([a, x] \in F(A)\).

Consider three cases:

(aa) \(gfa\) factors through no \(Y < r(x)\);

(bb) \(gfa\) factors through \(Y < r(x)\), but \(fa\) factors through no \(Z < r(x)\);

(cc) \(fa\) factors through \(Z < r(x)\).

In the case (aa) obviously \(fa\) factors through no \(Y < r(x)\). Hence \(F(gf)[a, x] = [gfa, x] = F(g)F(f)[a, x]\). In the case (bb) \((gfa, x) \notin M_{C, r(x)}\) and \(F(gf)[a, x] = O_C = F(g)[fa, x] = F(g)F(f)[a, x]\). In the case (cc) obviously \(gfa\) factors through \(Z < r(x)\). Hence \(F(gf)[a, x] = O_C = F(g)(O_B) = F(g)F(f)[a, x]\).

(c) Finally, \(F\) is faithful. Let \(f, g: A \to B\) be morphisms, \(F(f) = F(g)\). Suppose that \(f = [A \to X \to B]\) and that \(f\) factors through no \(Y < X\). Obviously,

\((1_A, x) \in M_{A, X}, [M_{A, X}] \subset F(A), (f, x) \in M_{B, X}, [M_{B, X}] \subset F(B)\) and

\(F(f)[1_A, x] = [f, x]\). Hence \([f, x] = F(g)[1_A, x] = [g, x]\), which implies \(\langle f, x \rangle = \langle g, x \rangle\). Since

\[
\begin{array}{c}
\text{f} \\
\Downarrow \quad \Downarrow \\
\text{x} \quad \text{y}
\end{array}
\]

commutes, the diagram

\[
\begin{array}{c}
\text{f} \\
\Downarrow \quad \Downarrow \\
1_B \\
\quad \Downarrow \\
\text{y}
\end{array}
\]
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\[
\begin{array}{c}
\text{commutes, too. Therefore } f = yx = g. \text{ Q.E.D.}
\end{array}
\]

Remark. The construction in the presented proof enables us to estimate the necessary cardinality of sets representing objects of a given category:

Let the cardinality of \( R_{A,X} \) be \( c(A, X) \). Then

\[
\text{card } F(A) = \left( \sum_{X \in A} \text{card } [M_{A,X}] \right) + 1 \leq \left( \sum_{X \in A} \text{card } [L_{A,X}] \right) + 1
\]

\[
\leq \sum_{X \in A} \left( c(A, X) \cdot \prod_{Y \in X} 2^{c(A,Y)} \right) + 1.
\]

Consequently:

1. If the class \( |C| \) of all the objects of the given category \( C \) is countable and \( c(A, X) \) is finite for every \( A, X \), then all the sets \( F(A) \) of the given representation are finite. Thus, the Freyd's theorem holds also in the finite set theory.

2. If \( |C| \) is a set, cardinality of which is a strongly inaccessible cardinal \( K \) and \( c(A, X) < K \) for every \( A, X \), then \( \text{card } F(A) < K \) for every object \( A \).

References