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The Principal Block of Finite Groups with Dihedral Sylow 2-Subgroups

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In order to describe the structure of some given finite group, it is often of great help to have some information on the irreducible characters and the blocks of a corresponding group algebra. Now, the properties of a block are very closely related to the structure of its defect group, and one of the main problems in representation theory of finite groups, therefore, is to give a general description of blocks with given defect group. The most successful results of that kind so far are the work by Dade on blocks with cyclic defect group, for any prime, and Fong's description of p-blocks in p-solvable groups together with the Fong–Swan theorem. However, to go further with this type of general description appears to be extremely difficult. But very useful results have been obtained if the problem is further reduced, for instance, by dealing with the prime 2 only, a natural assumption when working with classification problems. Brauer has obtained strong results in this case in a series of papers (see [1, I–IV]), which seem to be very suitable for applications.

Recently, Brauer has developed a new method, the theory of double chains and subsections, that is very effective in certain cases, and for instance, makes it possible to determine the number of ordinary and modular characters, and furthermore, to give strong relations between the values of ordinary characters and generalized decomposition numbers in 2-blocks with defect groups of normal 2-rank 1, in particular dihedral defect groups [2], using Dade's work. This problem was first attacked in [1, III, Chaps. 5, 7], where the defect group was assumed to be dihedral, and the block was assumed to be principal. Still, it remains to describe the modular characters, and in particular, to determine the decomposition numbers. The first major step would naturally be to do this under the additional assumption that the

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block is principal, and this is the purpose of the present paper, which essentially is a nontrivial generalization of a beautiful trick due to Glauberman, used in an unpublished work of his to give a proof of a problem stated by Brauer in [1, III, p. 238].

One of the most important steps is to find an efficient upper bound for the decomposition numbers. If the conjecture: "The Cartan invariants are bounded above by p^d, where d is the defect of the block" were true, general information on the generalized decomposition numbers would determine the ordinary decomposition numbers almost completely. However, [9, Satz (8)] provides a counterexample to this conjecture.

Section 1 lists the preliminary results, mainly due to Brauer. In Section 2, we restrict the irreducible ordinary representations in the principal block of the group to the centralizer of an involution and compute the dimension of their components in the principal block of the centralizer. This is done by generalizing the idea of Glauberman mentioned above, and relies very heavily on the results referred to in Section 1. In a special case of ours, namely, when the Sylow 2-subgroup is elementary abelian of order 4, this idea gives us a complete solution to the problem almost free, as is demonstrated in Section 7. In Section 3, the decompositions of the corresponding modular representations are described, and information about the vertices of the indecomposable components is obtained. This is based on The Green correspondence and is needed in Section 4, where the components in the nonprincipal blocks of the centralizer are described. Section 5 deals with the corresponding dimensions of the irreducible modular representations. Here, suggestions by Fong have been very useful. Finally, Section 6 uses the results of the previous sections to find constraints on and upper bounds for the decomposition numbers.

1. Notation and Assumptions

Let G be a finite group with a dihedral Sylow 2-subgroup $P = \langle \sigma, \tau \rangle$, $\tau^2 = 1$, $\sigma^\tau = \sigma^{-1}$. Let $j = \sigma^{2^{n-2}}$ be the central involution of P, and set $C = C_G(j)$. Furthermore, let B_0 and b_0 denote the principal blocks of G and C, respectively, and let $\{b_\alpha\}_\alpha$ be the nonprincipal blocks of C.

We need the following results:

Lemma 1.1. *Let $\pi \in P$. Then, $C_G(\pi)$ has a normal 2-complement.*

Proof. See [1, III, Lemma 7A].

Let K denote the normal 2-complement of C.

THEOREM 1.2. *G is of one of the following types:*

1. *G has one conjugacy class of involutions and no normal subgroup of index* 2.

2. *G has two conjugacy classes of involutions and a normal subgroup of index* 2, *but none of index* 4. *If H is a subgroup of index* 2, *H is of type* 1.

3. *G has three conjugacy classes of involutions and a normal* 2*-complement.*

Proof. By transfer. See [6, Theorem 7.7.3].

THEOREM 1.3. *B_0 contains $2^{n-2}+3$ irreducible characters*

$$\chi_0 = 1, \qquad \chi_1, \chi_2, \chi_3, \quad \chi^{(1)}, \dots, \chi^{(2^{n-2}-1)}. \tag{1}$$

If G is of type 1 *or* 2, *then, for $\sigma^\lambda \neq 1$,*

$$\chi_1(\sigma^\lambda) = \delta_1, \qquad \chi_i(\sigma^\lambda) = \delta_i(-1)^{\lambda+1}, \qquad i = 2, 3, \tag{2}$$

where $\delta_j = \pm 1$ for $j = 1, 2, 3$, and

$$\chi^{(k)}(\sigma^\lambda) = \delta_1(u^{k\lambda} + u^{-k\lambda}), \tag{3}$$

u is a primitive 2^{n-1}th root of unity.

Proof. This is one of the main results in [1, III, Chap. 7].

THEOREM 1.4. *For 2-regular elements ρ and all k,*

$$1 + \delta_1\chi_1(\rho) = -\delta_2\chi_2(\rho) - \delta_3\chi_3(\rho) = \delta_1\chi^{(k)}(\rho). \tag{4}$$

Moreover, if G is of type 1 *or* 2, *B_0 contains* 3, *respectively,* 2 *irreducible characters.*

Proof. See [1, III, Chap. 7].

Because of Theorem 1.2 and various theorems in representation theory, we only have to deal with groups of type 1 in order to solve our problem.

Let X_i and $X^{(k)}$ be representations affording χ_i and $\chi^{(k)}$, respectively, $i = 1, 2, 3$, $k = 1, 2, \dots, 2^{n-2} - 1$. Let F_i be the irreducible modular representations in B_0, $i = 1, 2, 3$, with corresponding Brauer characters ϕ_i, and let F_0 be the trivial representation. We will use the same notation for the corresponding modules.

For every $x \in G$, let $x = x_2x'$ be the 2-decomposition, x' of odd order.

LEMMA 1.5. *For every character $\chi \in B_0$, and for every element $x \in G$, $\chi(x) = \chi(x_2)$.*

Proof. This is just a special case of [1, I, p. 159, Corollary 5 of Theorem 3].

Let as usual $\epsilon(\chi)$ denote the sum $(1/|G|)\sum_{x\in G}\chi(x^2)$ for any character χ of G (the so-called Frobenius–Schur value):

$$\begin{aligned}\epsilon(\chi) &= 0, && \text{if } \chi \neq \bar{\chi}\\ &= -1, && \text{if } \chi = \bar{\chi}, \text{ but } X \neq \bar{X}\\ &= 1, && \text{if } X = \bar{X}.\end{aligned} \tag{5}$$

THEOREM 1.6. *χ_1 is rational. Either χ_2 and χ_3 are real and exactly one δ_i, $i = 1, 2, 3$ is 1, and the others are -1, or $\bar{\chi}_2 = \chi_3$, and $\delta_1 = -1$, $\delta_2 = \delta_3 = 1$.*

Proof. See [1, III, p. 241].

Hence, we get essentially three cases to consider:

I. $\bar{\chi}_2 = \chi_3$, $\delta_1 = -1$, $\delta_2 = 1$, $\delta_3 = 1$.

II. χ_2, χ_3 real, $\delta_1 = 1$, $\delta_2 = -1$, $\delta_3 = 1$.

III. χ_2, χ_3 real, $\delta_1 = -1$, $\delta_2 = 1$, $\delta_3 = -1$.

Also, as is easy to see,

$$\deg \chi_1 \equiv \delta_1, \qquad \mathrm{mod}(2^n)), \tag{6}$$

$$\deg \chi_i \equiv \delta_i + 2^{n-1}, \qquad (\mathrm{mod}(2^n)), \qquad i = 2, 3, \tag{7}$$

$$\deg \chi^{(k)} \equiv 2\delta_1, \qquad (\mathrm{mod}\ 2^n)), \tag{8}$$

for all k, using that $(\chi\mid_P, 1_P)_P$ is a rational integer.

Finally, we remark that if $n = 2$, then G only can be of type 1 or 3. Furthermore,

THEOREM 1.7. *If $n = 2$ and G is of type 1, then B contains four ordinary irreducible characters. For 2-singular elements δ,*

$$\chi_i(\delta) = \epsilon_i = \pm 1, \qquad i = 1, 2, 3. \tag{9}$$

Moreover,

$$1 + \sum_{i=1}^{3} \chi_i(\rho) = 0, \tag{10}$$

for all 2-regular elements ρ.

Proof. Most of this theorem is just a special case of Theorem 1.3. See [1, III, Chap. 5].

Here, we get $\chi_i(1) \equiv \epsilon_i$, (mod(4)), $i = 1, 2, 3$. From the equation above, it follows that at least one ϵ_i is negative and one is positive. However, by [1, III], only Case I and II can occur.

In the following, we will assume that G is of type 1.

2. The Dimension of $(\chi_i|_C)_{b_0}$

LEMMA 2.1. *G has*

(i) $N_i = (2^{i-1}/2^n) \mid G \mid$ *elements* x *with* x_2 *of order* 2^i,

(ii) $N_0 = ((2^{n-1}+1)/2^n) \mid G \mid$ *elements of odd order, where* $1 \leqslant i \leqslant 2^{n-1}$.

Proof. The second statement follows from the first. It is easy to see that G has 2^{i-2} conjugacy classes of elements of order 2^i, namely,

$$\{\sigma^{2^{n-1-i}d}\}, \qquad d = 1, 3, \ldots, 2^{i-1}-1, \quad i \geqslant 2. \tag{11}$$

Since G is of type 1, G has only one conjugacy class of involutions. Let N_{d_i} denote the number of elements x with x_2 conjugate to $\sigma^{2^{n-1-i}d}$, and set $C_{d_i} = C_G(\sigma^{2^{n-1-i}d})$. Then, as C_{d_i} has a normal 2-complement K_{d_i} ($C_{d_1} = C$, $K_{d_1} = K$), by Lemma 1.1, it follows that

$$N_1 = \mid G : C \mid \mid K \mid = \mid G \mid/2^n, \qquad N_{d_i} = \mid G : C_{d_i} \mid \mid K_{d_i} \mid = \mid G \mid/2^{n-1}, \tag{12}$$

for $i \geqslant 2$. Hence, $N_i = (\mid G \mid/2^{n-1})\, 2^{i-2} = (2^{i-1}/2^n) \mid G \mid$.

Next, define

$$O = \{x \in G \mid x \text{ is of odd order}\},$$
$$I = \{x \in G \mid x \text{ is of even order and } x^2 \text{ is of odd order}\},$$
$$R = \{x \in G \mid x^2 \text{ is of even order}\} = G \backslash (I \cup O).$$

LEMMA 2.2.

$$\frac{1}{\mid G \mid} \sum_{x \in I} \chi_i(x^2) = \frac{1}{\mid C \mid} \sum_{x \in K} \chi_i(x) = \frac{1}{2^n} (\chi_i|_K, 1_K)_K \tag{13}$$

Proof. The last equality is obvious. As for the first,

$$\frac{1}{\mid G \mid} \sum_{x \in I} \chi_i(x^2) = \frac{1}{\mid G \mid} \frac{\mid G \mid}{\mid C \mid} \sum_{x \in I \cap C} \chi_i(x^2) = \frac{1}{\mid C \mid} \sum_{x \in K} \chi_i(x'^2), \tag{14}$$

and K is of odd order.

We now define

$$E_1 = \{x \in G \backslash O \mid x_2 \text{ is of maximal order}\}$$
$$E_2 = \{x \in G \backslash O \mid x_2 \text{ is not of maximal order}\}$$

Let $\psi = 1 + \delta_1\chi_1 + \delta_2\chi_2 + \delta_3\chi_3$. Then

$$\begin{aligned} \psi(x) &= 0, && \text{for} \quad x \in O \\ &= 0, && \text{for } x \in E_2 \\ &= 4, && \text{for } x \in E_1, \end{aligned} \tag{15}$$

by Theorem 1.3, 4 and 5. Hence,

$$\frac{1}{|G|} \sum_{x \in E_1} \chi_i(x) = \frac{1}{4}(\chi_i, \psi)_G = \frac{1}{4}\delta_i, \tag{16}$$

and $|E_1| = |G|/4$, since for $x \in E_1$

$$\chi_i(x) = \chi_i(x_2) = \delta_i(-1)^{(1-\delta_{1i})(1+\lambda)} = \delta_i, \tag{17}$$

λ being odd. However, we already know $|E_1|$ from Theorem 2.1.

We are now able to prove

THEOREM 2.3.

$$(\chi_i|_K, 1_K)_K = 2^n\epsilon(\chi_i) + 2^{n-2}\delta_i - (2^{n-2} - 1)\,\delta_i(-1)^{1-\delta_{1i}} \tag{18}$$

Proof. By definition,

$$\epsilon(\chi_i) = \frac{1}{|G|}\left[\sum_{x \in I} \chi_i(x^2) + \sum_{x \in O} \chi_i(x^2) + \sum_{x \in R} \chi_i(x^2)\right]. \tag{19}$$

Now, as we have seen,

$$\chi_i(y) = \chi_i(y_2) = \delta_i(-1)^{(1-\delta_{1i})(1+\lambda)} = \delta_i(-1)^{1-\delta_{1i}}, \tag{20}$$

for all y, provided $y_2 \neq 1$ is not of maximal order, in particular, if y is a square or lies in E_2. Thus,

$$\frac{1}{|G|} \sum_{x \in R} \chi_i(x^2) = \frac{|R|}{|G|}\,\delta_i(-1)^{1-\delta_{1i}}, \tag{21}$$

$$\frac{1}{|G|} \sum_{x \in E_2} \chi_i(x) = \frac{|E_2|}{|G|}\,\delta_i(-1)^{1-\delta_{1i}}. \tag{22}$$

Using this in (19), we obtain

$$\epsilon(\chi_i) = \frac{1}{2^n}(\chi_i|_K, 1_K)_K - \frac{1}{4}\delta_i + \frac{|R| - |E_2|}{|G|}\,\delta_i(-1)^{1-\delta_{1i}} \tag{23}$$

from Lemma 2.2. By Theorem 2.1,

$$|R| = |G \backslash O| - |I| = |G| \left(\frac{2^{n-1}-1}{2^n} - \frac{1}{2^n}\right) = |G| \frac{2^{n-1}-2}{2^n}, \tag{24}$$

$$|E_2| = |G \backslash O| - |E_1| = |G| \left(\frac{2^{n-1}-1}{2^n} - \frac{2^{n-2}}{2^n}\right) = |G| \frac{2^{n-2}-1}{2^n}, \tag{25}$$

and the theorem follows.

This result is the key to our problem because of the following elementary fact:

THEOREM 2.4.

$$(\chi_i|_K, 1_K)_K = \dim(\bar{X}_i|_C)_{b_0}. \tag{26}$$

Proof. As C has a normal 2-complement K, every block b_α of C contains exactly one irreducible modular character, which is just an ordinary character of K. In particular, 1_K is the Brauer character of b_0.

3. THE DECOMPOSITION OF $F_{i|C}$

We start with some remarks on a general fact in representation theory of finite groups, based on a lemma due to Nagao and several of the major results in representation theory, all of which may be found in works by Brauer and Green. We will also use without reference some results in [7, 8] and Brauer's main theorems, which may be found in [3, 4].

Let R be a complete commutative integral domain satisfying the ACC with principal radical $J(R)$, such that $\bar{R}$ $(= R/J(R))$ is a field of characteristic p. Furthermore, let G be a finite group whose order is divisible by p. Then, if F is an indecomposable $R[G]$- or $\bar{R}[G]$-module with vertex V, and H is a subgroup of G containing V,

$$F|_H = f_H^G(F) \oplus \sum_\alpha M_\alpha, \tag{27}$$

where $f_H^G(F)$ is some indecomposable H-module with vertex V such that $F \mid f_H^G(F)^G$. We remark that $f_H^G(F)$ is not uniquely determined unless H contains $N_G(V)$, in which case F and $f_H^G(F)$ are said to be in Green correspondence to each other. This is a very important property, as we shall see.

Now, let Q be a p-subgroup of G and C a subgroup such that

$$QC_G(Q) \leqslant C \leqslant N_G(Q). \tag{28}$$

Then, we can define the Brauer homomorphism

$$\mu\colon Z(\bar{R}[G]) \to Z(\bar{R}[C]), \tag{29}$$

by

$$\mu\colon a_K[K] \to a_K[K \cap C], \tag{30}$$

where, as a basis for $Z(\bar{R}[G])$, we have taken the class sums.

Let for a block B of G, respectively, b of C, the corresponding central primitive idempotent be denoted by e_B, respectively, e_B.

PROPOSITION 3.1. *Let F belong to the block B of G. Then*

$$F\,|_C = F\,|_C\, \mu(e_B) \oplus \sum_\alpha M_\alpha = F\,|_C \sum_{b^G=B} e_b \oplus \sum_\alpha M_\alpha\,, \tag{31}$$

where M is indecomposable with vertex Q_α not containing Q and lies in a block b_α such that $e_{b_\alpha}\mu(e_B) = 0$.

Proof. See [10; or [4, Lemma 56.5]. As proved by Brauer, $\mu(e_B) = \sum_{b^G=B} e_b$.

Moreover, by Brauer's third main theorem, we have

PROPOSITION 3.2. *Suppose F belongs to the principal block B_0 of G. Then*

$$F\,|_C \mu(e_{B_0}) = H \in b_0(C), \tag{32}$$

where $b_0(C)$ is the principal block of C.

Finally, by applying the Green correspondence, we get

PROPOSITION 3.3. *If F has vertex V containing Q and $C \geqslant N_G(V)$, then*

$$f_G^C(F) \mid F\,|_C \mu(e_B). \tag{33}$$

Proof. $f_C^G(F)$ has vertex V. Thus, $f_C^G(F)$ does not occur among the M_α's.

Applying this to the two irreducible nontrivial representations F_1 and F_2 in the previous section, we get

THEOREM 3.4.

$$F_i\,|_C = H_i \oplus \sum_{\alpha_i} M_{i\alpha_i} \tag{34}$$

$i = 1, 2$, where $0 \neq H_i \in b_0$ and $M_{i\alpha_i}$ is indecomposable, with vertex $\langle 1\rangle$ or $\langle\tau\rangle$, τ a noncentral involution of P. Furthermore, if $n > 2$, $f_C^G(F_i) \mid H_i$.

Proof. Set $Q = \langle j\rangle$ above and use Proposition 3.1, 2 and 3.

COROLLARY. *Let* $\chi_i = d_{i0}\phi_0 + d_{i1}\phi_1 + d_{i2}\phi_2$. *Then*

$$(\chi_i|_K, 1_K)_K = d_{i0} + d_{i1} \dim H_1 + d_{i2} \dim H_2 \tag{35}$$

Proof. Since irreducible Brauer characters belonging to different blocks are orthogonal to each other, and the Brauer character of H_i is just $\dim H_i \cdot 1_K$, the statement follows from the theorem.

LEMMA 3.5. *Let*

$$F_i|_C = f_C^G(F_i) \oplus \sum_{\alpha_i} W_{i\alpha_i}, \tag{36}$$

where $W_{i\alpha_i}$ is indecomposable. Then, $W_{i\alpha_i}$ has vertex $\langle 1\rangle$, $\langle\tau\rangle$, or $\langle\tau, j\rangle$, where τ is a noncentral involution of P. This is of interest if H_i is not indecomposable.

Proof. By [7, Theorem 3], in general,

$$F|_C = f_C^G(F) \oplus \sum_{\alpha} Y_\alpha, \tag{37}$$

where Y_α is $P \cap C^g$-projective for some $g \notin C$. Suppose $P \cap C^g$ has order larger than 2. Then $\langle j, j^g\rangle \leqslant Z(P \cap C^g)$. Since, on the other hand, $P \cap C^g$ is dihedral, it must have order 4.

4. THE DECOMPOSITION OF $\chi_{i|C}$ (FOR $n > 3$)

As proved by Brauer in general,

$$\begin{aligned} 2^n\phi(x) &= 0 && \text{on} \quad \text{2-singular elements } x \\ &= 2^n\phi(x) && \text{on} \quad \text{2-regular elements } x, \end{aligned} \tag{38}$$

is a generalized character for any Brauer character. In our case, it easily follows that $2^n\phi_i$, $i = 2, 3$, decomposes into

$$n_{i0}\chi_0 + n_{i1}\chi_1 + n_{i2}\chi_2 + n_{i3}\chi_3 + (n_{i0} + n_{i1}) \sum_{k=1}^{2^{n-2}-1} \chi^{(k)}, \tag{39}$$

where $n_{i0} + \delta_1 n_{i1} = -\delta_2 n_{i2} - \delta_3 n_{i3}$. Furthermore,

$$2^n \dim H_i = \sum_{j=0}^{3} n_{ij}(\chi_j|_K, 1_K) + (n_{i0} + n_{i1}) \sum_{k=1}^{2^{n-2}-1} (\chi^{(k)}|_K, 1_K), \tag{40}$$

and thus, $\dim H_i$ may be computed directly from the Brauer characters.

Next, some remarks on the 2-blocks of C. Since C has a normal 2-complement, every block b_α contains exactly one modular irreducible representation with corresponding Brauer character ϕ_α. If its defect group D_α is cyclic, b_α contains 2^{d_α} irreducible characters, and all decomposition numbers are 1. If it is dihedral, b_α contains $2^{d_\alpha-2}+3$ ordinary irreducible characters. Four decomposition numbers are 1, the rest being 2. Furthermore, for every block b_α, there exists an irreducible character ζ_α of K such that

$$\mathrm{Inv}_C(\zeta_\alpha) = D_\alpha K, \tag{41}$$

and

$$\phi_\alpha = \sum_{\gamma=1}^{2^{n-d}} \zeta_\alpha{}^{c_\gamma}, \tag{42}$$

where $\{c_1, ..., c_{2^{n-d_\alpha}}\}$ is a transversal of D_α in P.

As we have seen in Section 3,

$$F_i\,|_C = H_i \oplus \sum_{\alpha_i} M_{i\alpha_i}, \tag{43}$$

where $M_{i\alpha_i}$ is $\langle 1\rangle$- or $\langle\tau\rangle$-projective, τ a noncentral involution of P. Let $V_{i\alpha_i}$ be the vertex of $M_{i\alpha_i}$, and set $W_{i\alpha_i} = K\langle V_{i\alpha_i}, j\rangle$. Then, $N_C(V_{i\alpha_i}) \leqslant W_{i\alpha_i}$, so $f^C_{W_{i\alpha_i}}(M_{i\alpha_i})$ is uniquely determined. Furthermore, $M_{i\alpha_i} = f^C_{W_{i\alpha_i}}(M_{i\alpha_i})^C$, as $f^C_{W_{i\alpha_i}}(M_{i\alpha_i})^C$ is indecomposable by Green's theorem. Now, since $|\,V_{i\alpha_i}\,| \leqslant 2$, the character of $f^C_{W_{i\alpha_i}}(M_{i\alpha_i})$ is complex, so $f^C_{W_{i\alpha_i}}(M_{i\alpha_i})^* \neq f^C_{W_{i\alpha_i}}(M_{i\alpha_i})$ by our remarks above. Hence, $M^*_{i\alpha_i} \neq M_{i\alpha_i}$ by the Green correspondence. In particular, we have proved

LEMMA 4.1. *Either $F_1{}^* = F_2$, or*

$$\dim F_i \equiv \dim H_i, \qquad (\mathrm{mod}(2^n)), \qquad i = 1, 2. \tag{44}$$

Finally, let

$$\xi_0, \xi_1, \xi_2, \xi_3, \xi^{(1)}, ..., \xi^{(2^{n-2}-1)}, \tag{45}$$

be the ordinary irreducible characters of b_0, where ξ_i, $i = 0, 1, 2, 3$ are just characters of $\langle\sigma, \tau\rangle/\langle\sigma^2\rangle$, and $\xi_1(\sigma)$ is equal to 1, say.

THEOREM 4.2. *Let b_α be a nonprincipal block of C. Then*

$$(\chi_i\,|_C)_{b_\alpha} = (1/2^{d_\alpha})(\chi_i\,|_K, \zeta_\alpha)\,\Phi_\alpha, \tag{46}$$

where Φ_α is the character of the principal indecomposable associated with ϕ_α, for $i = 1, 2, 3$. Similarly,

$$(\chi\,|_C)_{b_\alpha} = (1/2^{d_\alpha})(\chi\,|_K, \zeta_\alpha)\,\Phi_\alpha, \tag{47}$$

where $\chi = \sum_{k=1}^{2^{n-2}-1} \chi^{(k)}$. *In particular,*

$$\delta_1(\chi_1 |_C)_{b_\alpha} = -\delta_2(\chi_2 |_C)_{b_\alpha} - \delta_3(\chi_3 |_C)_{b_\alpha} \tag{48}$$

Proof. Let $\eta \in b_\alpha$ be an ordinary irreducible character with decomposition number d. Then

$$(\chi_i |_C, \eta) = \frac{1}{|C|} \left(\sum_{x \in K} + \sum_{x \in E_1 \cap C} + \sum_{x \in E_2 \cap C} \right) \chi_i(x)\, \eta(x^{-1}), \tag{49}$$

and

$$\left| \frac{1}{|C|} \sum_{x \in E_r \cap C} \chi_i(x)\, \eta(x^{-1}) \right| = \left| \frac{1}{|C|} \sum_{x \in E_r \cap C} \eta(x^{-1}) \right|, \tag{50}$$

where $r = 1, 2$, by Theorem 1.3. Now

$$\frac{1}{|C|} \sum_{x \in E_2 \cap C} \eta(x) = \sum_{x \in E_2 \cap P^*} \sum_{y \in K} \eta(xy) = 0, \tag{51}$$

since b_α is nonprincipal. Similarly,

$$\frac{1}{|C|} \sum_{x \in E_1 \cap C} \eta(x^{-1}) = 0, \tag{52}$$

even if $\sigma \in D_\alpha$. Hence,

$$(\chi_i |_C, \eta)_C = \frac{1}{2^n} (\chi_i |_K, d\phi_\alpha)_K = \frac{2^{n-d_\alpha}}{2^n} d(\chi_i |_K, \zeta_\alpha)_K, \tag{53}$$

and the first part of the lemma follows.

Now,

$$\begin{aligned} \sum_{k=1}^{2^{n-2}-1} (u^{k\lambda} + u^{-k\lambda}) &= 0, \qquad \lambda \text{ odd} \\ &= -2, \qquad \lambda \text{ even}, \end{aligned} \tag{54}$$

where u is a primitive 2^{n-1}th root of unity, so the last part is easily proved in the same way.

Remark. As expected by Brauer's second main theorem, $(\chi_i |_C)_{b_\alpha}(\delta) = 0$ for 2-singular elements δ.

5. The Dimension of $(F_{i|C})_{b_0}$

Let F, respectively, H, denote F_i, respectively, H_i, and let V be the vertex of F. Let $\Gamma = \{p_1, ..., p_r\}$ be a transversal of V in P and $\Delta = \{p_1, ..., p_r, c_1, ..., c_s\}$ a transversal of KV in G such that $c_i \notin C$ for any i.

THEOREM 5.1. *Suppose $j \in V$, and furthermore, assume that j lies in the kernel of $f_C{}^G(F)$. Then, V is dihedral.*

Proof. Suppose V is cyclic. Let $H_f = L^C$ for some indecomposable KV-module L with vertex V, where, in our previous notation, $H_f = f_C^G(F) \mid H$. We remark that $\dim L$ is odd, since V is cyclic. Consider L^G as a $\langle j \rangle$-module:

$$L^G = \sum_{p_\alpha \in \Gamma} L \otimes p_\alpha \oplus \sum_{c_\beta \in \Delta} L \otimes c_\beta\,, \tag{55}$$

$c_\beta^{-1} j c_\beta \notin KV$ for any $c_\beta \in \Delta$, so $\sum_{\alpha_\beta \in \Delta} L \oplus c_\beta$ is a free $\langle j \rangle$-module and splits into blocks

$$\{L \otimes c_\beta\,, L \otimes c_\beta j\}, \tag{56}$$

under the action of j.

For $p_\alpha \in \Gamma$, j acts trivially on $L \otimes p_\alpha$, since $p_\alpha \in C$ and j lies in the kernel of the representation. Hence, j acts trivially on a vector space of dimension $\dim L \mid P : V \mid = \dim H_f$, exactly as we might expect.

Since K also lies in the kernel of the representation, the action of C on H_f is just the action of P on $L^C = H_f$. Hence, some vector $v_0 \neq 0$ is left invariant under the action of C. Now define

$$w = \sum_{c_\alpha \in \Delta} v_0 \otimes c_\beta\,, \tag{57}$$

and let W be the one-dimensional vector space spanned by w. W is clearly a G-module, so we may consider the projection

$$\pi : L^G \to L^G/W. \tag{58}$$

Since F is irreducible, $\pi(F) \simeq F$. Now, consider $\pi(L^G)$ as a $\langle j \rangle$-module. $H_f \mid \pi(L^G)$, as $\pi(L) \simeq F$, and it follows that j acts trivially on a vectorspace containing H_f properly, a contradiction.

LEMMA 5.2. *Suppose $\sigma^{2^{n-3}} \in V$, where $n > 3$. Then, H_f is a faithful $P/\langle j \rangle$-module.*

Proof. We use the same notation as above. Let $u = \sigma^{2^{n-3}}$ and consider L^G as a $\langle u \rangle$-module. We may assume that $\tau \in V$ if u lies in the kernel by Theorem 5.1. Now, $\sum_{c_\beta \in \Delta} L \otimes c_\beta$ splits into blocks

$$\{L \otimes c_\beta\,, L \otimes c_\beta u, L \otimes c_\beta u^2, L \otimes c_\beta u^3\}, \tag{59}$$

whenever $c_\beta^{-1}jc_\beta \notin V$, and into blocks

$$\{L \otimes c_\beta, L \otimes c_\beta u\}, \tag{60}$$

if $c_\beta^{-1}jc_\beta \in V$. Since p_α may be chosen as a power of σ, u acts trivially on $\sum_{p_\alpha \in P} L \otimes p_\alpha$, again, a vector space of dimension $\dim H_f$, if u lies in the kernel of H_f. As in the proof of Theorem 5.1., C centralizes some vector $v_0 \neq 0$, and the argument above takes over and leads to a contradiction once more if u lies in the kernel. Finally, suppose τ is in the kernel. Then, $\tau\sigma\tau = \sigma^{-1}$, and σ have the same action, which implies that $\sigma^2 \in \langle j \rangle$, i.e., $n \leqslant 3$.

COROLLARY. *Either F_1 and F_2 are algebraic conjugate or* $\dim H_i > 2^{n-3}$, $i = 1, 2$.

Proof. If $u \notin V$, $2^{n-2} \mid \dim H_i$.

6. The Decomposition Numbers

We need the following elementary fact from linear algebra;

LEMMA. *Let* $\mathbf{M}$ *be an $R_{m\times n}$-matrix. Then*

$$\det(\operatorname{ad}\mathbf{M} \cdot \mathbf{M}) = \sum_i (\det \mathbf{\Delta}_i)^2, \tag{61}$$

where $\mathbf{\Delta}_i$ is a maximal minor of $\mathbf{M}$ *and $\mathbf{\Delta}_i$ runs through such minors.*

Case I. Let $\chi_2 = e\phi_0 + f\phi_1 + g\phi_2$.

LEMMA 6.1. $H_1 = H_2{}^*$, *and*

(1) $\dim(\chi_1 |_C)_{b_0} = 2e + 1 + \dim H_1 + \dim H_2 = 2^n + 1$,

(2) $\dim)\chi_i |_C)_{b_0} = e + \dim H_i = 2^{n-1} - 1$, $i = 2, 3$.

Furthermore,

$$D_0{}^{\mathrm{I}} = \begin{array}{c} \begin{array}{ccc} \phi_0 & \phi_1 & \phi_2 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 2e+1 & 1 & 1 \\ e & 1 & 0 \\ e & 0 & 1 \\ 2e & 1 & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi^{(k)}. \end{array}$$

Proof. By Theorem 1.4, $1 + \chi_2 + \chi_3 = \chi_1$ on the set O, while $1 + \chi^{(k)} = \chi_1$. Also, $\chi_2 = \bar{\chi}_3$. This gives us the following decomposition matrix:

$$D_0^{\mathrm{I}} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} \phi_0 & \phi_1 & \phi_2 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 2e+1 & f+g & f+g \\ e & f & g \\ e & g & f \\ 2e & f+g & f+g \end{pmatrix} \end{array} \begin{array}{l} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi^{(k)}. \end{array}$$

Using the above lemma, we get

$$\det C^{\mathrm{I}} = 2^n(f^2 - g^2)^2. \tag{62}$$

Thus, $(f^2 - g^2)^2$ is a 2-power. On the other hand, since $D_0{}^{\mathrm{I}}$ has rank 3, $f^2 - g^2 \not\equiv 0 \pmod{2}$. Thus, $f^2 - g^2 = \pm 1$, and we may as well assume that $f = 1$, $g = 0$. By the corollary of Theorem 3.4, we get (1) and (2) directly.

Next, we consider

Case II. Let $\chi_2 = e_1\phi_0 + f\phi_1 + g\phi_2$, $\chi_3 = e_2\phi_0 + h\phi_1 + i\phi_2$.

LEMMA 6.2.

(1) $\dim(\chi_2 |_C)_{b_0} = e_1 + f \dim H_1 + g \dim H_2 = 2^{n-1} + 1$,

(2) $\dim(\chi_3 |_C)_{b_0} = e_2 + h \dim H_1 + i \dim H_2 = 2^{n-1} + 1$.

Furthermore,

$$D_0^{\mathrm{II}} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} \phi_0 & \phi_1 & \phi_2 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ e_1+e_2-1 & f+h & g+i \\ e_1 & f & g \\ e_2 & h & i \\ e_1+e_2 & f+h & g+i \end{pmatrix} \end{array} \begin{array}{l} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi^{(k)}, \end{array}$$

where

$$f_i - gh = 1, \quad f+g \leqslant 2, \quad h, i \leqslant 2, \quad h+i \leqslant 3. \tag{63}$$

Proof. In this case, $1 + \chi_1 = \chi_2 + \chi_3 = \chi^{(k)}$ on the set O. Here, the decomposition matrix is

$$D_0^{\mathrm{II}} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} \phi_0 & \phi_1 & \phi_2 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ e_1+e_2-1 & f+h & g+i \\ e_1 & f & g \\ e_2 & h & i \\ e_1+e_2 & f+h & g+i \end{pmatrix} \end{array} \begin{array}{l} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi^{(k)}. \end{array}$$

Again, by applying the above lemma,

$$\det C_0^{\mathrm{II}} = 2^n(fi - gh)^2, \tag{64}$$

from which we deduce by the same argument that $fi - gh = \pm 1$, say $fi - gh = 1$. Now, suppose that F_1 and F_2 are algebraic conjugate. Then, as in Case I, $f = i$ and $g = h$, and again, $f = i = 1$ and $g = h = 0$, so under this assumption, the first part of the lemma certainly holds. If they are not algebraic conjugate, we may apply the corollary of Lemma 5.2 and get immediately (63) from the above equations.

Finally, we describe

Case III. Let $\chi_1 = e_1\phi_0 + f\phi_1 + g\phi_2$, $\chi_3 = e_2\phi_0 + h\phi_1 + i\phi_2$

LEMMA 6.3.

(1) $\dim(\chi_1 |_C)_{b_0} = e_1 + f \dim H_1 + g \dim H_2 = 2^n - 1$,

(2) $\dim(\chi_3 |_C)_{b_0} = e_2 + h \dim H_1 + i \dim H_2 = 2^{n-1} + 1$.

Furthermore,

$$D_0^{\mathrm{III}} = \begin{array}{c} \begin{array}{ccc} \phi_0 & \phi_1 & \phi_2 \end{array} \\ \left\{\begin{array}{ccc} 1 & 0 & 0 \\ e_1 & f & g \\ e_1 + e_2 - 1 & f + h & g + i \\ e_2 & h & i \\ e_1 - 1 & f & g \end{array}\right\} \end{array} \begin{array}{l} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi^{(k)}, \end{array}$$

where

$$fi - gh = 1, \quad h + i \leqslant 3, \quad f + g \leqslant 7. \tag{65}$$

Proof. Same as the proof of Lemma 6.2. The upper bounds may be strengthened for $n = 3$.

Remark. Case I corresponds to $PSL(2, q)$, $q \equiv 3$, (mod(4)) if G is simple (although of course we do not depend on any classification theorem). By Alperin, the decomposition matrix for the principal block of these groups is

$$D_0^{\mathrm{I}} = \begin{array}{c} \begin{array}{ccc} \phi_0 & \phi_1 & \phi_2 \end{array} \\ \left\{\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right\} \end{array} \begin{array}{l} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi^{(k)}, \end{array}$$

so by Lemma 6.1, $\dim H_i = 2^{n-1} - 1$, $i = 1, 2$. We note that this implies that $V_i = P$.

Case II corresponds to $PSL(2, q)$, $q \equiv 1$, (mod(4)). Again by Alperin,

$$D_0^{\mathrm{II}} = \begin{array}{c c} \begin{array}{ccc} \phi_0 & \phi_1 & \phi_2 \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} & \begin{array}{l} \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi^{(k)}. \end{array} \end{array}$$

By Lemma 6.2, $\dim H_i = 2^{n-1}$ for these groups. Unfortunately, this does not give any information about the vertices.

Case III corresponds to A_7 and $n = 3$. A_7 has two blocks, the principal and one of defect 2 with a copy of $Z_2 \times Z_2$ as defect group, and the decomposition matrix has the following form

$$\begin{array}{c c} & \begin{array}{cccccc} 1 & 14 & 20 & 6 & 4_1 & 4_2 \end{array} \\ \begin{array}{r} 1 \\ 15 \\ 35 \\ 21 \\ 14_a \\ 6 \\ 14_b \\ 10_1 \\ 10_2 \end{array} & \begin{pmatrix} 1 & 0 & 0 & & & \\ 1 & 1 & 0 & & & \\ 1 & 1 & 1 & & & \\ 1 & 0 & 1 & & & \\ 0 & 1 & 0 & & & \\ & & & 1 & 0 & 0 \\ & & & 1 & 1 & 1 \\ & & & 1 & 1 & 0 \\ & & & 1 & 0 & 1 \end{pmatrix}. \end{array}$$

We remark that the nonprincipal block has the same form as the principal block of any group with an elementary abelian Sylow 2-subgroup of order 4 that is of type II, as we shall see in the following section.

For A_7 we find that $\dim H_1 = 6$, and $\dim H_2 = 4$.

Remark. We shall not reconsider the upper bounds for the decomposition numbers for small n, although they may be considerably strengthened, only the special case $n = 2$, where we will give an interesting complete solution to the problem, essentially due to Glauberman, as mentioned in the Introduction. It should be pointed out though, that as far as the determination of the decomposition numbers is concerned, this may be completely omitted by a general result of Fong.

7. The Case $n = 2$

If P is elementary abelian of order 4, only Case I and II may occur as mentioned in Section 1.

Case I (is trivial).

THEOREM 7.1.

$$D_0^{\mathrm{I}} = \begin{array}{cc} \begin{array}{ccc} \phi_0 & \phi_1 & \phi_2 \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{array}{l} \chi_0 \\ \chi_1 \\ 23 \\ \chi_3 . \end{array} \end{array}$$

Moreover, F_i *has vertex* P *and* H_i *is of dimension* 1.

Proof. By Lemma 6.1, (2), $e + \dim H_i = 1$, $i = 1, 2$.

Case II.

THEOREM 7.2.

$$D_0^{\mathrm{II}} = \begin{array}{cc} \begin{array}{ccc} \phi_0 & \phi_1 & \phi_2 \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} & \begin{array}{l} \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 . \end{array} \end{array}$$

Moreover, H_i *is indecomposable of dimension* 2, *and* F_1 *and* F_2 *are algebraic conjugate. Finally*, F_i *has vertex* P.

Proof. By Lemma 6.2, (1) and (2),

$$e_1 + f \dim H_1 + g \dim H_2 = e_2 + h \dim H_1 + i \dim H_2 = 3, \qquad (66)$$

where $fi - gh = 1$. As $\chi_1 = \chi_2 + \chi_3 - \chi_0$, $e_1 = e_2 = 0$ is impossible. Thus, we may assume that $e_1 \geqslant 1$, say. Since $f \neq 0$, $\dim H \leqslant 2$.

A. H_1 is indecomposable of dimension 2.

Proof. Assume not. Then $H_1 = I_c$, or $H = I_c \oplus I_c$, I_c the trivial C-module, since K lies in the kernel of any indecomposable representation in b_0. In any case, $F_1 \mid I_C{}^G$ by a theorem of Green (see [8]), since by Theorem 5.1, F_1 has vertex P. Therefore,

$$F_0 \oplus F_1 \mid I_C{}^G \qquad (67)$$

By the Mackey decomposition,

$$I_C{}^G|_{N(P)} = (I_C|_{C\cap N(P)})^{N(P)} \oplus \sum_{\gamma} W \ , \tag{68}$$

where W_γ has vertex of order 1 or 2, and therefore, has even dimension, for all γ. But $C \cap N(P) = P \times R$, where R is the 2-complement of $C(P)$. Furthermore, $|N(P) : C(P)| = 3$. Thus

$$(I_C|_{C\cap N(P)})^{N(P)} = I_{N(P)} \oplus A \oplus B, \tag{69}$$

is of dimension 3, and $I_{N(P)}$, A and B are the modules of the irreducible representations of $N(P)$ with $C(P)$ as kernel. It now follows that

$$I_C{}^G|_{N(P)} = I_{N(P)} \oplus A \oplus B \oplus \sum_{\gamma} W_\gamma, \tag{70}$$

and therefore,

$$F_1|_{N(P)} \oplus F_2|_{N(P)} \mid I_{N(P)} \oplus A \oplus B \oplus \sum_{\gamma} W_\gamma . \tag{71}$$

Since F_1 has vertex P, $F_1|_{N(P)}$ must contain A or B, but not both, by the Green correspondence. However, as $A = B^*$ and $A = f^G_{N(P)}(F_1)$, $F_1 = F_2{}^*$ then, a contradiction since all characters in b_0 are real.

This statement may be omitted by referring to [5, Lemma 1], which states, in general, that if $F = F^*$ is an irreducible representation over a field of characteristic 2, then the dimension of F is even. Therefore, it follows immediately from this lemma that dim $H_1 = 2$, $i = 1, 2$.

B. F_1 and F_2 are algebraically conjugate.

Proof. H_1 is a trivial K-module, so we may as well consider $H_1|_P$, which can be described by

$$\begin{Bmatrix} 1 & 0 \\ \Theta & 1 \end{Bmatrix}. \tag{72}$$

We want to prove that Θ does not take its values in $GF(2)$. By A, Θ is not trivial. Let $j = j_1$, j_2 and j_3 be the involutions of P. Then

$$\Theta(j_1) + \Theta(j_2) + \Theta(j_3) = 0. \tag{73}$$

If Θ takes its values in $GF(2)$, $\Theta(j_0) = 0$ for some j_0, while Θ takes the value 1 on the other involutions. Now, define a $\langle j_0, K\rangle$-module by the corresponding representation M

$$M(j_0) = I, \qquad M(K) = I. \tag{74}$$

Then, M^C is given by $M^C(j_0) = I$, $M^C(K) = I$, and

$$M^C(j_0) = \begin{pmatrix} 0 & M(1) \\ M(1) & 0 \end{pmatrix}. \tag{75}$$

It is now obvious that Θ determines H_1 uniquely, i.e., $H_1 = M^C$. Hence, if Θ takes its values in $GF(2)$, H_1, and therefore, also, F_1 has $\langle j_0 \rangle$ as vertex, a contradiction by Theorem 5.1.

Theorem 7.2 follows immediately.

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