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Theory and Applications

The shortest distance among points in general position

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Abstract

We prove that among n points in the plane in general position, the shortest distance can occur at most $(2+3/7)n$ times. We also give a construction where the shortest distance occurs more than $(2+5/16)n - 10\lfloor\sqrt{n}\rfloor$ times.

1. Introduction

The following well known result of Harborth [5] settles a conjecture of Reutter [7] (see also [6]).

Theorem 1 (Harborth). *Let $f(n)$ denote the maximum number of times the minimum distance can occur among n points in the plane. Then*

$$f(n) = \lfloor 3n - \sqrt{12n-3} \rfloor.$$

The extremal configuration for Theorem 1 is a hexagonal piece of a regular triangular lattice.

Definition. A set of n points is said to be *in general position* if no three of them are on the same line.

Peter Brass [3] raised the following question: at most how many times can the minimum distance occur among n points in general position.

Theorem 2. *Let $g(n)$ denote the maximum number of times the minimum distance can occur among n points in general position in the plane. Then*

$$\left(2+\frac{5}{16}\right)n - 10\lfloor\sqrt{n}\rfloor < g(n) \leqslant \left(2+\frac{3}{7}\right)n.$$

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2. Proof of Theorem 2

1. First we prove that $g(n) \leqslant (2 + 3/7)n$.

Let P be a set of n points in the plane in general position. Suppose without loss of generality that the smallest distance occurring among these points is 1. Consider the graph $G(P)$ whose vertices are the points of P and two vertices are connected iff their distance is 1.

For any $O \in P$ let $d(O)$ denote the degree of O in $G(P)$. Clearly, $d(O) \leqslant 5$. We prove that any point of degree 5 has a neighbor of degree at most 4, except of one special case.

In the sequel, let O be fixed, $d(O) = 5$, let $A_1, A_2, \ldots, A_5$ denote its neighbors listed in clockwise order.

We will have two cases according to the subgraph G_O of $G(P)$ induced by O and its five neighbors.

Case 1. At least one of the neighbors of O, say, A_5, has degree 1 in G_O.

Case 2. All five neighbors of O have degree at least two in G_O.

Since the points are in general position, $G(P)$ cannot have the graph shown in Fig. 1(0) as a subgraph. Thus, in Case 2, G_O is isomorphic to the graph shown in Fig. 1(2).

Case 1. We prove by contradiction that A_5 has degree at most four.

Suppose $d(A_5) = 5$. The neighbors of A_5 are O, B_1, B_2, B_3 and B_4. Since 1 is the smallest distance, $\angle B_i A_5 B_{i+1} \geqslant \pi/3$, $\angle A_i A_5 A_{i+1} \geqslant \pi/3$ $(i = 1,2,3)$, so $\angle A_1 O A_4 \geqslant \pi$. But $|B_1 A_4| \geqslant 1$, $|B_4 A_1| \geqslant 1$, so $\angle A_4 O A_1 \geqslant \angle B_1 A_5 B_4 \geqslant \pi$.

Hence, $\angle A_1 O A_4 = \pi$, a contradiction, because then A_1, O and A_4 are on the same line (Fig. 2).

Case 2 has two subcases.

Case 2.a. G_O is the graph shown in Fig. 1(2), A_4 and A_5 have only one common neighbor, O. See Fig. 3(a).

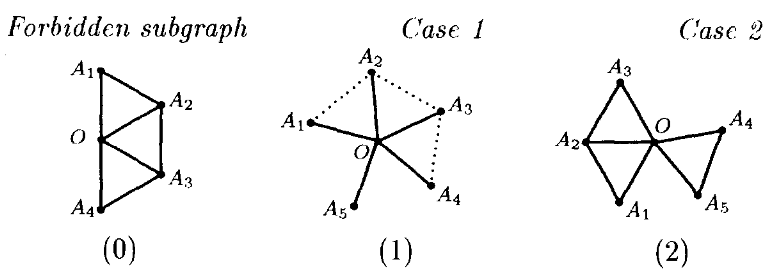

Fig. 1.

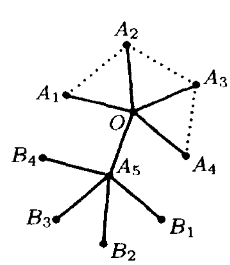

Fig. 2.

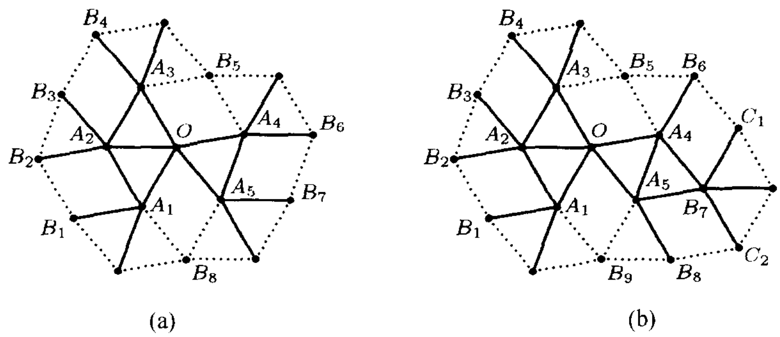

Fig. 3.

Suppose that all neighbors of O have degree five.

Let B_1 be the neighbor of A_1 preceding A_2 in the clockwise order; let B_2 and B_3 be neighbors of A_2; let B_4 be the neighbor of A_3 following A_2 in the clockwise order. Extend the triangle A_3A_4O to a rhombus by adding the point B_5. Let B_6 be the neighbor of A_4 preceding A_5 in clockwise order, B_7 be the neighbor of A_5 following A_4 in the clockwise order. Finally, extend the triangle A_5A_1O to a rhombus by adding the point B_8.

$|B_1B_2|$, $|B_3B_4|$ and $|B_6B_7| \geqslant 1$, so $\angle B_1A_1A_2 + \angle A_1A_2B_2$, $\angle B_3A_2A_3 + \angle A_2A_3B_4$ and $\angle B_6A_4A_5 + \angle A_4A_5B_7 \geqslant \pi$. By easy calculations we get

$$\angle B_8A_1B_1 + \angle B_2A_2B_3 + \angle B_4A_3B_5 + \angle B_5A_4B_6 + \angle B_7A_5B_8 \leqslant 3\pi.$$

But if all A_i had degree 5, then $\angle B_8A_1B_1$, $\angle B_4A_3B_5$, $\angle B_5A_4B_6$ and $\angle B_7A_5B_8 \geqslant 2\pi/3$, $\angle B_2A_2B_3 \geqslant \pi/3$, thus,

$$\angle B_8A_1B_1 + \angle B_2A_2B_3 + \angle B_4A_3B_5 + \angle B_5A_4B_6 + \angle B_7A_5B_8 \geqslant 3\pi.$$

Consequently,

$$\angle B_8A_1B_1 + \angle B_2A_2B_3 + \angle B_4A_3B_5 + \angle B_5A_4B_6 + \angle B_7A_5B_8 = 3\pi,$$

$\angle B_8A_1B_1$, $\angle B_4A_3B_5$, $\angle B_5A_4B_6$ and $\angle B_7A_5B_8 = 2\pi/3$, $\angle B_2A_2B_3 = \pi/3$, $\angle B_1A_1A_2 + \angle A_1A_2B_2$, $\angle B_3A_2A_3 + \angle A_2A_3B_4$ and $\angle B_6A_4A_5 + \angle A_4A_5B_7 = \pi$, but then $|B_1B_2|$, $|B_3B_4|$ and $|B_6B_7| = 1$.

It follows, that $B_2A_2 \| OA_4$, $A_2O \| A_4B_6$ and since all these four segments are of length 1, B_2, O and B_6 are on the same line, a contradiction.

Up to this point we know that if O is of types 1 or 2.a, then it has a neighbor of degree at most 4.

Definition. Two vertices of $G(P)$, A and B are called *special second neighbors* if there are two other vertices, C and D, such that AC, AD, CD, CB and DB are all edges of $G(P)$, i.e., $|AC|$, $|AD|$, $|CD|$, $|CB|$ and $|DB| = 1$.

Case 2.b. G_O is the graph shown in Fig. 1(2), A_4 and A_5 have two common neighbors, O and B_7 (so O and B_7 are special second neighbors). See Fig. 3(b).

Suppose that all neighbors of O have degree five, and the degree of B_7, the special second neighbor of O, is also five.

The points B_1, B_2, B_3, B_4 and B_5 are defined as in Case 2.a. Let B_6 be the neighbor of A_4 preceding B_7 in the clockwise order; let B_8 be the neighbor of A_5 following B_7 in the clockwise order. Extend the triangle A_5OA_1 to a rhombus by adding the point B_9. Finally, let C_1 be the neighbor of B_7 following A_4 in the clockwise order, and let C_2 be the neighbor of B_7 preceding A_5 in the clockwise order. Observe that $B_6 \neq C_1$ and $B_8 \neq C_2$.

Since $|B_6C_1|$, $|C_2B_8|$, $|B_1B_2|$ and $|B_3B_4| \geqslant 1$, we have that $\angle A_4B_7C_1 + \angle B_6A_4B_7$, $\angle C_2B_7A_5 + \angle B_7A_5B_8$, $\angle B_1A_1A_2 + \angle A_1A_2B_2$ and $\angle B_3A_2A_3 + \angle A_2A_3B_4 \geqslant \pi$.

So, by simple calculations,

$$\angle C_1B_7C_2 + \angle B_8A_5B_9 + \angle B_9A_1B_1 + \angle B_2A_2B_3 + \angle B_4A_3B_5 + \angle B_5A_4B_6 \leqslant 3\pi.$$

By the assumption that all A_i and B_7 has degree five, $\angle B_8A_5B_9$, $\angle B_2A_2B_3$ and $\angle B_5A_4B_6 \geqslant \pi/3$, $\angle C_1B_7C_2$, $\angle B_9A_1B_1$ and $\angle B_4A_3B_5 \geqslant 2\pi/3$, therefore,

$$\angle C_1B_7C_2 + \angle B_8A_5B_9 + \angle B_9A_1B_1 + \angle B_2A_2B_3 + \angle B_4A_3B_5 + \angle B_5A_4B_6 \geqslant 3\pi.$$

Again we obtain a regular configuration: $\angle B_8A_5B_9$, $\angle B_2A_2B_3$, $\angle B_5A_4B_6 = \pi/3$, $\angle C_1B_7C_2$, $\angle B_9A_1B_1$, $\angle B_4A_3B_5 = 2\pi/3$, $|B_6C_1|$, $|C_2B_8|$, $|B_1B_2|$ and $|B_3B_4| = 1$.

It follows, that $B_1A_1 \| OA_4$ and $A_1O \| A_4B_6$, and that all of them are unit segments. Therefore, B_1, O and B_6 are on the same line, contradiction.

To sum up, if a vertex of $G(P)$ had degree 5, one of its neighbors or special second neighbors has degree at most 4.

Assign each 5-degree vertex of $G(P)$ to one of its neighbors or special second neighbors of degree at most 4. Since the points are in general position, any vertex has at most two special second neighbors. So to each vertex of degree at most 4, there will be assigned at most six vertices—four neighbors and two special second neighbors. Therefore, the average degree of the vertices of $G(P)$ is at most $(6 \cdot 5 + 4)/7 = 34/7$, i.e., $G(P)$ has at most $17n/7 = n(2 + 3/7)$ edges.

2. Next we prove by a construction that $g(n) \geqslant (2 + 5/16)n - 10\lfloor\sqrt{n}\rfloor$.

Let a be a unit vector.

Definition. For any vector x, let $\arg(x)$ denote the counterclockwise angle from a to x.

Let $\varepsilon > 0$ small; b, c and d be unit vectors, $\arg(b) = -\pi/6 + \varepsilon$, $\arg(c) = \pi/3$, $\arg(d) = \pi/2 + \varepsilon$. Let p and q be positive integers. Finally, let $u_1, u_2, \ldots, u_{p-1}$ and $v_1, v_2, \ldots, v_{q-1}$ be unit vectors, $u_{2i+1} = a$, $u_{4i+2} = b$, $v_{2i+1} = c$, $v_{4i+2} = d$, $0 > \arg(u_{4i}) > -\pi/6 + \varepsilon$, $\pi/3 < \arg(v_{4i}) < \pi/2 + \varepsilon$.

We choose the exact values of $\arg(u_{4i})$ and $\arg(v_{4i})$ later. Define a configuration P_{pq} of pq points, as follows:

$$P_{pq} = \{p_{ij} \mid 0 < i \leqslant p,\ 0 < j \leqslant q\},$$

where

$$p_{ij} = v_1 + \cdots + v_{i-1} + u_1 + \cdots + u_{j-1}.$$

This configuration is similar to a deformed square lattice, where the "horizontal" edges are the vectors a, b or u_{4i}, and the "vertical" edges are c, d or v_{4i}. So the squares are deformed into rhombuses whose angles are between $\pi/3$ and $2\pi/3$. Therefore the shortest distance between the points is the edge length of the rhombuses, which is 1. Define the graph $G(P_{pq})$ as before.

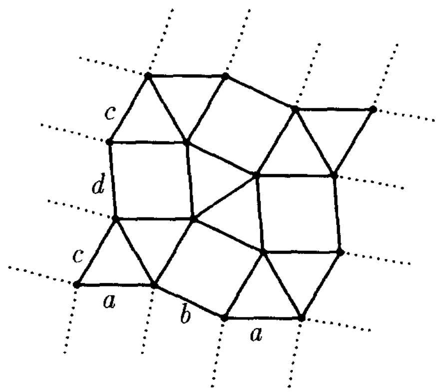

Fig. 4. Configuration $P_{4,4}$.

In this "lattice" of pq points there are $p(q-1)$ "vertical" and $(p-1)q$ "horizontal" edges. Notice that $\angle ac = \pi/3$, $\angle bd = 2\pi/3$, so in the rhombuses where one of the sides is a the other one is c or one of the sides is b the other one is d, one of the diagonals is also of distance 1. Since $u_{2i+1} = a$, $v_{2i+1} = c$, there are $\lfloor p/2 \rfloor \cdot \lfloor q/2 \rfloor$ rhombuses of sides a and c, and $\lfloor p/4 \rfloor \cdot \lfloor q/4 \rfloor$ rhombuses of sides b and d. Each of these rhombuses mean one additional edge in $G(P_{pq})$, so for the edges of $G(P_{pq})$ we have

$$\begin{aligned} \left|E\left(G(P_{pq})\right)\right| &\geqslant p(q-1) + (p-1)q + \left\lfloor \frac{p}{2} \right\rfloor \cdot \left\lfloor \frac{q}{2} \right\rfloor + \left\lfloor \frac{p}{4} \right\rfloor \cdot \left\lfloor \frac{q}{4} \right\rfloor \\ &\geqslant 2pq - p - q + \frac{pq}{4} - \frac{p}{2} - \frac{q}{2} + \frac{pq}{16} - \frac{p}{4} - \frac{q}{4} = \left(2 + \frac{5}{16}\right)pq - \frac{7}{4}(p+q). \end{aligned}$$

Claim. *For any positive integers p and q, we can choose the values of* $\arg(u_{4i})$ *and* $\arg(v_{4j})$ *such that P_{pq} is in general position.*

Proof. It is enough to prove the Claim in the case when both p and q are divisible by 4, i.e., when $p = 4r$ and $q = 4s$.

We prove the Claim by induction on r and s. In the repeated pattern in Fig. 4, which is actually the configuration P_{44}, easy to see, that there are no three points on a line.

Suppose, we could choose the values $\arg(u_{4i})$ and $\arg(v_{4j})$ such that in the configuration $P_{4(r-1),4s}$ the points are in general position.

Construct $P_{4r,4s}$ by choosing the value of $\arg(u_{4(r-1)})$.

For any value of $\arg(u_{4(r-1)})$, the point set we try to add to $P_{4(r-1),4s}$ is

$$R = \{p_{ij} \mid 4(r-1) < i \leqslant 4r,\ 0 < j \leqslant 4s\},$$

a translate of the configuration

$$\{p_{ij} \mid 4(r-2) < i \leqslant 4(r-1),\ 0 < j \leqslant 4s\},$$

which is in general position by assumption. The change of $\arg(u_{4(r-1)})$ results a translation of R. If for a certain value of $\arg(u_{4(r-1)})$ the configuration $P_{4r,4s}$ happens to be not in general position, then either a line determined by the points $P_{4(r-1),4s}$ contains a point of R or, conversely, a line determined by the points of R contains a point of $P_{4(r-1),4s}$. But both events can occur for only finite many values of $\arg(u_{4(r-1)})$, therefore there is a value $0 > \arg(u_{4(r-1)}) > -\pi/6 + \varepsilon$, such that the configuration $P_{4r,4s}$ is in general position. By the same argument, we can step from $s-1$ to s. So for any pair r, s, there exists a configuration of points in general position, $P_{4r,4s}$. $\square$

Finally, for any $n > 3$, construct the configuration C_n.

Let $p = q = \lfloor\sqrt{n}\rfloor$. Take the configuration P_{pq} of $\lfloor\sqrt{n}\rfloor^2$ points, put the remaining points far from this configuration so that the points are still in general position. The smallest distance among the points is 1, and

$$\begin{aligned}\left|E(G(C_n))\right| &\geqslant \left(2+\frac{5}{16}\right)\lfloor\sqrt{n}\rfloor^2 - \frac{7}{2}\lfloor\sqrt{n}\rfloor \\ &\geqslant \left(2+\frac{5}{16}\right)n - \left(4+\frac{5}{8}\right)\lfloor\sqrt{n}\rfloor - 2 - \frac{5}{16} - \frac{7}{2}\lfloor\sqrt{n}\rfloor \\ &\geqslant \left(2+\frac{5}{16}\right)n - \left(8+\frac{1}{8}\right)\lfloor\sqrt{n}\rfloor - 2 - \frac{5}{16} \geqslant \left(2+\frac{5}{16}\right)n - 10\lfloor\sqrt{n}\rfloor. \qquad \square\end{aligned}$$

Remarks. (1) It is not hard to see that for points in general position the upper bound cannot be achieved. That is, it is impossible, that there are six points of degree five assigned to each point of degree four.

(2) In the construction of C_n we put the remaining points far from the configuration P_{pq} ($p, q = \lfloor\sqrt{n}\rfloor$). Placing them more carefully, we could create a few more unit distances, but this would improve the lower bound by $O(\sqrt{n})$ only.

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References

[1] P. Brass, The maximum number of second smallest distances in finite planar graphs, Discrete Comput. Geom. 7 (1992) 371–379.
[2] P. Brass, Beweis einer Vermutung von Erdős und Pach aus der kombinatorischen Geometrie, Ph.D. Thesis, Department of Discrete Mathematics, Technical University Braunschweig, Braunschweig, 1992.
[3] P. Brass, Personal communication.
[4] P. Erdős, Some combinatorial and metric problems in geometry, in: K. Böröczky and G. Fejes Tóth, eds., Intuitive Geometry, Colloq. Math. Soc. János Bolyai 48 (North-Holland, Amsterdam, 1987) 167–177.
[5] H. Harborth, Solution to problem 664A, Elem. Math. 29 (1974) 14–15.
[6] J. Pach and P. Agarwal, Combinatorial Geometry (Wiley, New York, 1995).
[7] D. Reutter, Problem 664A, Elem. Math. 27 (1972) 19.
[8] K. Vesztergombi, On the distribution of distances in finite sets in the plane, Discrete Math. 57 (1985) 129–146.