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On the nontrivial projection problem

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Abstract

The *nontrivial projection problem* asks whether every finite-dimensional normed space admits a well-bounded projection of nontrivial rank and corank or, equivalently, whether every centrally symmetric convex body (of arbitrary dimension) is approximately affinely equivalent to a direct product of two bodies of nontrivial dimensions. We show that this is true "up to a logarithmic factor."

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1. Introduction and the main results

A series of well-known open problems in the asymptotic theory of normed spaces is concerned with the existence, in any finite-dimensional normed space (of dimension greater than one), of well-bounded projections of non-trivial rank and corank. One possible formulation is as follows.

The nontrivial projection problem. Do there exist $C \geqslant 1$ and a sequence $k_n \to \infty$ such that for every n-dimensional normed space X there is a projection P on X with

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(i) $\|P\| \leqslant C$,
(ii) $\min\{\operatorname{rank} P, \operatorname{rank}(I-P)\} \geqslant k_n$?

Versions of this question were explicitly posed by Pisier ([24], 1983) and Milman ([17], 1986) in their ICM talks; see also [30]. In geometric terms, the problem asks whether an arbitrary n-dimensional, centrally symmetric convex body is approximately ("up to a universal constant C") affinely equivalent to a direct product of two bodies whose dimensions are at least k_n.

To put the problem in perspective, for a subspace E of a Banach space X denote

$$\lambda(E, X) := \inf\bigl\{\|P\|\colon\ P \text{ is a projection from } X \text{ onto } E\bigr\}.$$

We then have (Kadets and Snobar [10], 1971)

$$\dim E = k \quad \Rightarrow \quad \lambda(E, X) \leqslant \sqrt{k}$$

or, more precisely (König and Tomczak-Jaegermann [11], 1990),

$$\lambda(E, X) \leqslant \sqrt{k} - c/\sqrt{k}$$

for all $k > 1$ and some universal (and explicit) numerical constant $c > 0$.

The estimates above hold for all subspaces, and sometimes cannot be substantially improved. First, in the *infinite-dimensional* context, there is the remarkable example of Pisier ([23], 1983).

Pisier's space. *There exists a Banach space X and $c > 0$ such that for any finite rank projection P on X one has $\|P\| \geqslant c\sqrt{\operatorname{rank} P}$.*

Next, it follows from the work on the finite-dimensional basis problem (Gluskin [6], 1981; Szarek [29], 1983) that, in general, we may not be able to find any projection on X whose rank and corank are of the same order as $\dim X$ and whose norm is $o(\sqrt{\dim X})$. Further, the statement from the problem cannot hold with k_n substantially larger than $\sqrt{n}$ (more precisely, with $k_n \gg \sqrt{n \log n}$).

However, all these results do not exclude a positive answer to the following (sample) question.

Generalized Auerbach system. Does there exist $C \geqslant 1$ such that for any $n \in \mathbb{N}$, for any n-dimensional normed space X, and for any integer m with $\sqrt{n} < m \leqslant n$, the space X can be split into a direct sum of m subspaces $E_1, \ldots, E_m$ of approximately equal dimensions, and such that if P_j is the projection onto E_j that annihilates all E_i's with $i \neq j$, then $\max_{1 \leqslant j \leqslant m} \|P_j\| \leqslant C$?

A positive answer would of course imply a positive solution to the nontrivial projection problem. The classical Auerbach lemma asserts that if $m = n$, then the answer is "yes, with $C = 1$."

In the positive direction, it has been known for quite a while that in some cases bounds on the norm sharper than $\sqrt{\min\{\operatorname{rank} P, \operatorname{rank}(I-P)\}}$ can be obtained, primarily via arguments based on K-convexity (Figiel and Tomczak-Jaegermann [4], 1979; Pisier [20], 1980). Based on that point of view and on the arguments and results from [23], Pisier posed [22,23] modified variants of the nontrivial projection problem. One possible formulation is the following version of the *uniformly complemented ℓ_p^n conjecture* of Lindenstrauss [14].

The modified problem. Given a sequence (X_n) of finite-dimensional normed spaces with $\dim X_n \to \infty$, does there exist $p \in \{1, 2, \infty\}$, a constant $C \geqslant 1$ and sequences $m_k \to \infty$ and $n_k \to \infty$ such that X_{n_k} contains a subspace which is C-complemented and C-isomorphic to $\ell_p^{m_k}$?

It is worthwhile to note that, up to the precise value of the constant, the conditions on the subspace can be conveniently rephrased as "*the identity on ℓ_p^k C-factors through X_{n_k}*." (We refer to the next section and to Section 5 for definitions of concepts that may be unfamiliar to a non-specialist reader.)

An affirmative answer to the modified problem would follow from an affirmative answer to the following (see, e.g., [21,22]).

The cotype-cotype conjecture. *If a Banach space X has (an appropriate) approximation property and if both X and its dual X^* have nontrivial cotype, then X is K-convex.*

Pisier's example mentioned earlier shows that *some* approximation hypothesis is necessary. The setting that is of interest to us is finite-dimensional, with dimension-free estimates on the parameters involved, and so the issues related to approximation properties will not enter the discussion.

In the present paper we shall prove the following result in the direction of the nontrivial projection problem.

Theorem 1. *There exist $C > 0$ and a sequence $k_n \to \infty$ such that, for every $n \geqslant 2$ and for every n-dimensional normed space X, there is a projection P on X with*

(i) $\|P\| \leqslant C(1 + \log k_n)^2$,
(ii) $\min\{\operatorname{rank} P, \operatorname{rank}(I - P)\} \geqslant k_n$.

Moreover, the range of the projection P is C-isomorphic to an ℓ_p-space for some $p \in \{1, 2, \infty\}$.

Remarks. (a) The argument shows that one can choose (k_n) to grow as (roughly) $\exp(\sqrt{\log n}\,)$.

(b) A slightly weaker but more compact statement than the assertion of the Theorem is "*the identity on $\ell_p^{k_n}$ can be $C(1 + \log k_n)^2$-factored through X*."

(c) Already in this last form, the assertion is nearly optimal (even for the class of ℓ_q^n spaces), except for the exact values of the powers of $\log k_n$ in (i). Similarly, k_n cannot be substantially larger than the quantity given in Remark (a) above. This is explained in Section 5; see also the remark following the proof of the theorem.

The proof of the theorem is based on a dichotomy which yields either

(1) a reasonably complemented copy of $\ell_2^{k_n}$ via an argument based on K-convexity and the ℓ-ellipsoid (essentially as in [4]) *or*
(2) a good copy of $\ell_\infty^{k_n}$ in X or in X^*, necessarily well-complemented (the latter implies existence of a well-complemented copy of $\ell_1^{k_n}$ in X). This part is based on a result of Alon–Milman with a refinement due to Talagrand, on restricted invertibility results in the spirit of Bourgain–Tzafriri, on a blocking argument due to James, and on various tricks of the trade developed over the last 25 years.

After this paper was submitted, it has been brought to our attention that results that partly parallel our technical statements, Propositions 2 and 3, can be found in an earlier article [27] which, however, had a different focus.

2. Notation and preliminaries

We use the standard notation from the Banach space theory. In particular, we denote Banach (or normed) spaces by X, Y, etc., and by $B_X, B_Y, \dots$ their (closed) unit balls. An operator means a bounded linear operator. For an operator $T : X \to Y$, its operator norm is denoted by $\|T : X \to Y\|$ or just by $\|T\|$. For isomorphic Banach spaces X and Y, their Banach–Mazur distance is defined by $d(X, Y) = \inf \|T\| \|T^{-1}\|$, where the infimum is taken over all isomorphisms T from X onto Y; we say that X is λ-isomorphic to Y if $d(X, Y) \leqslant \lambda$. A subspace F of X is λ-complemented if there exists a projection from X onto F of norm less than or equal to λ.

For finite-dimensional normed spaces, the essentially equivalent language of symmetric convex bodies is natural and often very useful. (By a symmetric convex body $K \subset \mathbb{R}^n$ we will mean a convex compact set with non-empty interior which is centrally symmetric with respect to the origin.) By $\|\cdot\|_K$ we denote the gauge of K; then $X = (\mathbb{R}^n, \|\cdot\|_K)$ is an n-dimensional normed space such that $K = B_X$. Any n-dimensional normed space can be represented in such a form in many different (although isometric) ways. If $K_1 \subset \mathbb{R}^{n_1}$, $K_2 \subset \mathbb{R}^{n_2}$ are symmetric convex bodies and X_1, X_2 are the corresponding normed spaces, for an operator $T : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ the operator norm $\|T : X_1 \to X_2\|$ will be also denoted by $\|T : K_1 \to K_2\|$ or (for example) by $\|T : K_1 \to X_2\|$.

By $|\cdot|$ we denote the Euclidean norm on $\mathbb{R}^n$ and we use the representation $\ell_2^n = (\mathbb{R}^n, |\cdot|)$. The Euclidean ball in $\mathbb{R}^n$ and the inner product are denoted by B_2^n and $\langle \cdot, \cdot \rangle$. For a subspace $E \subset \mathbb{R}^n$, we denote by P_E the orthogonal projection on E. The polar body K° is defined by $K^\circ := \{x \in \mathbb{R}^n \mid |\langle x, y \rangle| \leqslant 1 \text{ for all } y \in K\}$. As is well known, the normed space $(\mathbb{R}^n, \|\cdot\|_{K^\circ})$ can be canonically identified with the dual space $(\mathbb{R}^n, \|\cdot\|_K)^*$.

We now recall the following less standard concept which will be useful further on. Given a normed space Y and a linear operator $S : \ell_2^n \to Y$, the ℓ-norm of S is defined via $\ell(S) := (\int_{\mathbb{R}^n} \|Sx\|^2 \, d\mu_n)^{1/2}$, where μ_n is the standard Gaussian measure on $\mathbb{R}^n$. In other words,

$$\ell(S) = \left(\mathbb{E} \left\| \sum_{i=1}^n g_i S v_i \right\|^2 \right)^{1/2},$$

where (v_i) is an arbitrary orthonormal basis of ℓ_2^n and (g_i)—an i.i.d. sequence of $N(0, 1)$ Gaussian random variables ($\mathbb{E}$ stands for the expected value). It is well known and easy to verify that the ℓ-norm satisfies $\|S\| \leqslant \ell(S)$ and it has the ideal property $\ell(SA) \leqslant \ell(S) \|A : \ell_2^n \to \ell_2^n\|$. We refer the reader to [32] or [25] for more details.

For a symmetric convex body $K \subset \mathbb{R}^n$ we set

$$\ell(K) = \left(\mathbb{E}\|g\|_K^2\right)^{1/2},$$

where $g = (g_1, \dots, g_n) \in \mathbb{R}^n$ is the standard Gaussian vector. (In other words, $\ell(K) = \ell(J)$ where $J : \ell_2^n \to K$ is the formal identity operator.) It is known that one may find a linear image

$\tilde{K} = uK$ (with $u : \mathbb{R}^n \to \mathbb{R}^n$ one-to-one and onto), called by some authors the ℓ-position of K, which in particular satisfies

$$\ell(\tilde{K}) = \ell\big((\tilde{K})^\circ\big) \leqslant C\sqrt{n(1+\log n)}, \tag{1}$$

where C is a universal constant. Clearly, the normed space induced by $\tilde{K}$ is isometric to the one associated with K.

The inequality (1) lies at the core of our arguments. It is obtained by combining results of [21] and [4], in turn based on [13], and exploits deep connections to K-convexity; this is where the ℓ-position/ℓ-ellipsoid come in.

As in inequality (1) and earlier in the introduction, the symbols c, C, c', C_1, etc. will stand in what follows for *universal* positive constants, independent of the particular instance of the problem that is being considered (most notably independent of the dimension). However, the same symbol may represent different numerical values in different parts of the paper.

3. Proof of Theorem 1

The argument will be based on two propositions corresponding to the two alternatives of the dichotomy mentioned in Section 1.

Proposition 2. *Let $K_1, K_2 \subset \mathbb{R}^n$ be symmetric convex bodies such that $K_1 \subset \alpha B_2^n$ and $K_2 \supset \beta^{-1} B_2^n$ and let $\sqrt{m} \leqslant c \min\{\ell(K_1^\circ)/\alpha, \ell(K_2)/\beta\}$. Then, for most of subspaces F of $\mathbb{R}^n$ of dimension m (in the sense of the Haar measure on the corresponding Grassmannian),*

(i) $\|P_F : K_1 \to K_2\| \leqslant C\ell(K_1^\circ)\ell(K_2)/n$,
(ii) $\exists r > 0$ *such that* $r(B_2^n \cap F) \subset K_2 \cap F \subset Cr(B_2^n \cap F)$.

Here is a sketch of the proof based on Milman's version of the Dvoretzky theorem ([16] or [18, Chapter 4]). First, if $m \leqslant (c\ell(K_2)/\beta)^2$, then, for most of subspaces F of dimension m, the section $K_2 \cap F$ is approximately a Euclidean ball of radius $r = \sqrt{n}/\ell(K_2)$, which yields (ii). Dually, if $m \leqslant (c\,\ell(K_1^\circ)/\alpha)^2$, then $P_F K_1$ is—again, for most F's—approximately a Euclidean ball of radius $R = \ell(K_1^\circ)/\sqrt{n}$. If F is such that both of the above hold, then $\|P_F : K_1 \to K_2\|$ is approximately R/r, whence (i) follows.

We point out that most authors use in similar arguments *spherical* rather that *Gaussian* averages; this is why our formulae involve $\sqrt{n}$ factors that are absent, e.g., in [18].

The second technical result that we need is the following.

Proposition 3. *Let $K_0 = B_Y \subset \mathbb{R}^n$ be such that for any subspace $E \subset \mathbb{R}^n$ with* $\operatorname{codim} E < k$ *we have $\|P_E : Y \to \ell_2^n\| \geqslant a$. Then there exists a subspace Z of Y such that $d(Z, \ell_1^m) \leqslant C$ with $m := \dim Z \geqslant c\,k^{1/\gamma}$, where $\gamma = 2\log_2(16\sqrt{2\pi}\,\ell(K_0^\circ)/a)$ and a projection $Q : Y \to Z$ with $\|Q\| \leqslant C$.*

We postpone the proof of Proposition 3 until the next section and direct our attention to Theorem 1. The argument will split naturally into three parts corresponding to different choices of $p \in \{1, 2, \infty\}$, which will in turn depend on the values of certain parameters related to the geometry of X.

Set $K = B_X$ and $k = \lceil n/4 \rceil$. Let $a \in [1, \sqrt{n}]$ (a will be later specified to be roughly $\sqrt{n}/\exp\sqrt{\log n}$).

Assume now that for every subspace $E \subset \mathbb{R}^n$ with $\operatorname{codim} E < k$ we have $\|P_E : X \to \ell_2^n\| \geqslant a$. Accordingly, Proposition 3 applies for $Y = X$, yielding a well-complemented m-dimensional subspace of X, well-isomorphic to ℓ_1^m, with $m \geqslant ck^{1/\gamma} \geqslant c'n^{1/\gamma}$, where $\gamma = 2\log_2(16\sqrt{2\pi}\,\ell(K_0^\circ)/a)$.

Similarly, if, for every subspace $E \subset \mathbb{R}^n$ with $\operatorname{codim} E < k$ the estimate $\|P_E : X^* \to \ell_2^n\| \geqslant a$ holds, then the same argument produces a well-complemented subspace of X^* well-isomorphic to ℓ_1^m. By duality, this yields a well-complemented subspace of X well-isomorphic to ℓ_∞^m, with the bound for m involving now $\gamma = 2\log_2(16\sqrt{2\pi}\,\ell(K)/a)$.

If neither of these conditions is satisfied, then there exist subspaces $E_1, E_2 \subset \mathbb{R}^n$ of codimension $< n/4$ such that the appropriate norms of projections P_{E_1}, P_{E_2} do not exceed a, and so also $\|P_H : X \to \ell_2^n\| \leqslant a$ and $\|P_H : X^* \to \ell_2^n\| \leqslant a$, where $H = E_1 \cap E_2$. In geometric terms, this is equivalent to the inclusions

$$P_H K \subset aB_2^n, \qquad K \cap H \supset a^{-1}\big(B_2^n \cap H\big),$$

the latter of which is the dual reformulation of $P_H K^\circ \subset aB_2^n$. We are thus in a position to apply Proposition 2 with $K_1 = P_H K$, $K_2 = K \cap H$, $\alpha = \beta = a$ and H playing the role of $\mathbb{R}^n$. (Note that $\dim H > n/2$.) This yields existence of a C-Euclidean section $K_2 \cap F = K \cap F$, whose dimension m is of order $(\min\{\ell((P_H K)^\circ), \ell(K \cap H)\})^2/a^2$. Moreover, $P_F K = P_F(P_H K) \subset \lambda(K \cap H) \subset \lambda K$, where $\lambda \leqslant C\ell((P_H K)^\circ)\ell(K \cap H)/n$. In other words, F is a λ-complemented C-Euclidean subspace of X.

It remains to collect estimates on ranks and norms of the projections and choose an optimal value for a. This will also require choosing an appropriate representation of X on $\mathbb{R}^n$, namely the ℓ-position, so that condition (1) of Section 2 is satisfied. In particular, we will have

$$\ell(K \cap H) \leqslant \ell(K) = \sqrt{n\kappa},$$
$$\ell\big((P_H K)^\circ\big) = \ell\big(K^\circ \cap H\big) \leqslant \ell\big(K^\circ\big) = \sqrt{n\kappa},$$

where $\kappa \leqslant C(1 + \log n)$. On the other hand,

$$\frac{n}{2} < \dim H \leqslant \ell\big((P_H K)^\circ\big)\ell(P_H K) \leqslant \ell\big(K^\circ \cap H\big)\ell(K \cap H),$$

and so we also have lower estimates

$$\ell(K \cap H) \geqslant \frac{1}{2}\sqrt{\frac{n}{\kappa}}, \qquad \ell\big((P_H K)^\circ\big) = \ell\big(K^\circ \cap H\big) \geqslant \frac{1}{2}\sqrt{\frac{n}{\kappa}}.$$

The lower bounds for dimensions of subspaces become

$$\frac{c'n}{\kappa a^2}, \qquad c'n^{1/2\log_2(32\sqrt{n\kappa}/a)},$$

for $p = 2$ and $p = 1$ or ∞, respectively. Choosing $a = \sqrt{n}/\exp\sqrt{\log n}$ and remembering the upper bound on κ we easily check that both of these quantities are $\geqslant k_n := c\exp(\frac{1}{2}\sqrt{\log n}\,)$. On the other hand, the upper bound on the norm of projection in the case $p = 2$ is clearly $C_1\kappa \leqslant C_2(1 + \log n) \leqslant C_3(1 + \log k_n)^2$, which concludes the proof of Theorem 1 together with the bound on k_n given in Remark (a).

Our final comment concerns optimality of the estimate for the norms of the projections in terms of their rank. By choosing differently the threshold value a, we can increase the dimension of the C-Euclidean subspace F, while keeping the norm of P_F bounded by $C\log n$. This way we can assure that, in all cases, the norm of the projection P is $\leqslant C'\log(\operatorname{rank} P)\log\log(\operatorname{rank} P)$. The price we pay is a decrease in the dimensions of the ℓ_1^m or ℓ_∞^m subspaces, and the common lower bound for ranks of projections is only a power of $\log n$ instead of $\exp(c\sqrt{\log n})$. (The power of $\log n$ can be chosen arbitrarily, at the cost of increasing the constant C'.)

4. Proof of Proposition 3

We start by defining (by induction) two sequences $x_1, x_2, \dots, x_k$ and $y_1, y_2, \dots, y_k$ with certain extremal properties. (The argument is similar to that in [15].) First, let $x_1 = y_1 \in K_0$ be such that $|x_1| = a_1 := \max_{x\in K_0}|x|$. For consistence with future notation set $F_1 = \mathbb{R}^n$. Next, suppose that $1 < j \leqslant k$ and that x_i, y_i for $i < j$ have already been defined. Set $F_j := [x_1, x_2, \dots, x_{j-1}]^\perp$ and choose $y_j \in K_0$ so that $|P_{F_j} y_j| = \|P_{F_j} : K_0 \to \ell_2^n\| =: a_j$. Set $x_j = P_{F_j} y_j$; then the sequence (x_j) is orthogonal with $|x_j| = a_j$. Finally, define an orthonormal sequence (u_j) by $u_j := x_j/a_j$, $j = 1, 2, \dots, k$. Note that, by hypothesis and construction, $a_1 \geqslant a_2 \geqslant \dots \geqslant a_k \geqslant a$.

Pick an interval $I \subset \{1, \dots, k\}$ with $|I| \geqslant k/(1+\log_2\frac{a_1}{a_k})$ such that $a_i \leqslant 2a_{i'}$ for all $i, i' \in I$. Set $F := [x_i]_{i\in I}$, and let $a' = a_{\min I}$. In the sequel we will analyze the convex set $\tilde{K}_0 := P_F K_0$—viewed as a convex body in F—and sequences $\tilde{y}_j := P_F y_j$. By construction, all $\tilde{y}_j$'s are elements of $\tilde{K}_0$. Moreover, since $\|P_F : K_0 \to B_2^n\| \leqslant \|P_{F_{\min I}} : K_0 \to B_2^n\| = a'$, it follows that for all $w \in F$,

$$|w| \leqslant a'\|w\|_{\tilde{K}_0} \tag{2}$$

and, in particular, $|\tilde{y}_j| \leqslant a'$ for $j \in I$. On the other hand, since $P_{F_j} u_j = P_F u_j = u_j$ for $j \in I$, it follows that for such j

$$\langle \tilde{y}_j, u_j\rangle = \langle P_F y_j, u_j\rangle = \langle y_j, u_j\rangle = \langle P_{F_j} y_j, u_j\rangle = \langle x_j, u_j\rangle = a_j \geqslant a'/2.$$

Accordingly, we are in a position to apply Bourgain–Tzafriri restricted invertibility principle [3] in the form presented in [2, Lemma B] to conclude that there exists a set $\sigma \subset I$ such that $s := |\sigma| \geqslant ck/(1+\log_2\frac{a_1}{a_k})$ and verifying, for any sequence of scalars $(t_j)_{j\in\sigma}$,

$$\left|\sum_{j\in\sigma} t_j \tilde{y}_j\right| \geqslant \frac{a'}{8}\left(\sum_{j\in\sigma}|t_j|^2\right)^{1/2}.$$

To reduce the clutter of subscripts, we will assume that $\sigma = \{1, 2, \dots, s\}$. Let $(z_j)_{j=1}^s$ be the sequence in $[\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_s]$ that is biorthogonal to $(\tilde{y}_j)_{j=1}^s$, then

$$\left|\sum_{j=1}^{s} t_j z_j\right| \leqslant \frac{8}{a'}\left(\sum_{j=1}^{s}|t_j|^2\right)^{1/2} \tag{3}$$

for any sequence of scalars (t_j).

Next, consider two polar bodies: $\tilde{K}_0^\circ$, the polar of $\tilde{K}_0$ *inside* F, and K_0°, the polar of K_0 (in $\mathbb{R}^n$). Since $\tilde{K}_0$ is an orthogonal projection of K_0, i.e., $\tilde{K}_0 = P_F K_0$, it follows that $\tilde{K}_0^\circ$ is a section of K_0°, namely $\tilde{K}_0^\circ = K_0^\circ \cap F$. Thus, given that $\|\tilde{y}_j\|_{\tilde{K}_0} \leqslant 1$ and $z_j \in F$, it follows that $\|z_j\|_{K_0^\circ} = \|z_j\|_{\tilde{K}_0^\circ} \geqslant 1$ for $1 \leqslant j \leqslant s$.

Consider now the quantity $M := (\mathbb{E}\|\sum_{j=1}^s g_i z_i\|_{K_0^\circ}^2)^{1/2}$ and define a linear map $T : \ell_2^n \to \ell_2^n$ by $Te_j = z_j$ for $j = 1, 2, \ldots, s$ and $Te_j = 0$ for $j > s$; it then follows from (3) that $\|T\| \leqslant 8/a'$. Accordingly, denoting by J the identity map considered as an operator from ℓ_2^n to $Y^* = (\mathbb{R}^n, \|\cdot\|_{K_0^\circ})$, and using the definition and properties of the ℓ-norm discussed in Section 2, we conclude that

$$M = \left(\mathbb{E}\left\|\sum_{j=1}^n g_i T e_i\right\|_{K_0^\circ}^2\right)^{1/2} = \ell(JT)$$

$$\leqslant \ell(J)\left\|T : \ell_2^n \to \ell_2^n\right\| \leqslant \ell(J)\frac{8}{a'} = \frac{8}{a'}\ell\big(K_0^\circ\big). \tag{4}$$

We now want to appeal to [31] to extract from (z_i) a subsequence resembling an ℓ_∞ basis. To this end, we need to consider the modified average $M_1 := \mathbb{E}\|\sum_{j=1}^s \varepsilon_i z_i\|_{K_0^\circ}$, where (ε_i) is an i.i.d. sequence of Bernoulli random variables. As is well known, $M_1 \leqslant (\mathbb{E}\|\sum_{j=1}^s \varepsilon_i z_i\|_{K_0^\circ}^2)^{1/2} \leqslant \sqrt{\frac{\pi}{2}} M$ (see, e.g., [32, Proposition 25.2]), which combined with (4) yields

$$M_1 = \mathbb{E}\left\|\sum_{j=1}^s \varepsilon_i z_i\right\|_{K_0^\circ} \leqslant \sqrt{\frac{\pi}{2}}\frac{8}{a'}\ell\big(K_0^\circ\big) \leqslant \frac{4\sqrt{2\pi}}{a}\ell\big(K_0^\circ\big). \tag{5}$$

Another quantity that is needed to appeal to [31] is $w := \max\|\sum_{j=1}^s \varepsilon_i z_i\|_{K_0^\circ}$ (i.e., the maximum over all choices of $\varepsilon_i = \pm 1$). Dualizing estimate (2) and using (3), we obtain, for all such (ε_i),

$$\left\|\sum_{j=1}^s \varepsilon_i z_i\right\|_{K_0^\circ} = \left\|\sum_{j=1}^s \varepsilon_i z_i\right\|_{\tilde{K}_0^\circ} \leqslant a'\left|\sum_{j=1}^s \varepsilon_i z_i\right| \leqslant 8s^{1/2}.$$

We are now ready to use the following result from [31].

Fact 4. *Let $(z_i)_{i=1}^s$ be a sequence in a normed space. Set $M_1 = \mathbb{E}\|\sum_{j=1}^s \varepsilon_i z_i\|$ and $w = \max\|\sum_{j=1}^s \varepsilon_i z_i\|$. Then there exists a subset $\tau \subset \{1, 2, \ldots, s\}$ with $|\tau| \geqslant sM_1/2w$ such that, for any scalars (t_i),*

$$\left\|\sum_{i\in\tau} t_i z_i\right\| \leqslant 4M_1 \max_{i\in\tau}|t_i|.$$

Specified to our context, Fact 4 yields τ with $|\tau| \geqslant c'(\frac{k}{1+\log_2\frac{a_1}{a_k}})^{1/2} M_1$.

The next step is a well-known blocking argument due to R.C. James.

Fact 5. *Let $v_1, v_2, \dots, v_{m^2}$ be elements of a normed space V with $\|v_j\| \geqslant 1$ for all j verifying, for some $\beta \geqslant 1$, $\|\sum_{j=1}^{m^2} t_j v_j\| \leqslant \beta \max_j |t_j|$ for all sequences of scalars (t_j). Then there exist $v_1', v_2', \dots, v_m' \in V$ with $\|v_i'\| \geqslant 1$ for all i such that $\|\sum_{i=1}^{m} t_i v_i'\| \leqslant \beta^{1/2} \max_i |t_i|$ for all sequences of scalars (t_i).*

The proof of Fact 5 is based on the following dichotomy. If there is a subset $\sigma \subset \{1, 2, \dots, m^2\}$ with $|\sigma| = m$, for which $\|\sum_{i\in\sigma} \pm v_i\| \leqslant \beta^{1/2}$ for all choices of signs, then the collection $\{v_i\colon i \in \sigma\}$ works. If not, then for each such σ there is $v_\sigma = \beta^{-1/2}\sum_{i\in\sigma} \pm v_i$ with $\|v_\sigma\| > 1$; partitioning the set $\{1, 2, \dots, m^2\}$ into subsets $\sigma_1, \sigma_2, \dots, \sigma_m$ with $|\sigma_j| = m$ for all j we are led to a collection $\{v_{\sigma_1}, v_{\sigma_2}, \dots, v_{\sigma_m}\}$ which has the required property.

The procedure implicit in Fact 5 can clearly be iterated. Applying it $d = \lfloor \log_2 \log_2(4M_1)\rfloor$ times to our sequence $(z_i)_{i\in\tau}$ we are led to $z_1', z_2', \dots, z_l'$ such that $\|z_i'\|_{K_0^\circ} \geqslant 1$ for $i = 1, 2, \dots, l$ and that, for all sequences (t_i),

$$\left\|\sum_{i=1}^{l} t_i z_i'\right\|_{K_0^\circ} \leqslant \omega \max_i |t_i|, \tag{6}$$

where $\omega \leqslant (4M_1)^{1/2^d} < 4$. Moreover, the length l of the sequence satisfies

$$l \geqslant \left\lfloor |\tau|^{1/2^d} \right\rfloor \geqslant \left\lfloor |\tau|^{1/\log_2 M_1} \right\rfloor \geqslant c'' \left(\frac{k}{1 + \log_2 \frac{a_1}{a_k}} \right)^{1/2 \log_2(4M_1)} \tag{7}$$

(note that in our setting we clearly have $M_1 \geqslant 1$ and so $d \geqslant 1$).

The last step is based on another result from [31].

Fact 6. *Let $(z_i')_{i=1}^{l}$ be a sequence in a normed space such that $\|z_i'\| \geqslant 1$ for $i = 1, 2, \dots, l$. Set $w' = \max \|\sum_{i=1}^{l} \varepsilon_i z_i'\|$. Then there exists a subset $\tau' \subset \{1, 2, \dots, l\}$ with $|\tau'| \geqslant l/8w'$ such that, for any scalars (t_i),*

$$\left\|\sum_{i\in\tau'} t_i z_i\right\| \geqslant \frac{1}{2} \max_{i\in\tau'} |t_i|.$$

In our setting, by (6), $m := |\tau'| > l/32$. On the other hand, also by (6), the subspace of Y^* spanned by $z_i', i \in \tau'$, is 8-isomorphic to ℓ_∞^m and hence automatically 8-complemented in Y^*. The conclusion of Proposition 3 follows then by duality, the only point needing clarification being the lower bound on m. To elucidate this last issue, we note that the exponent $1/\gamma = 1/2 \log_2(16\sqrt{2\pi}\,\ell(K_0^\circ)/a)$ from the proposition coincides with the lower bound on the exponent $1/2\log_2(4M_1)$ in (7) given by (5). Furthermore, $a_k \geqslant a$ and

$$a_1 = \left\|Id : K_0 \to \ell_2^n\right\| = \left\|Id : \ell_2^n \to K_0^\circ\right\| \leqslant \ell\bigl(K_0^\circ\bigr),$$

hence $a_1/a_k \leqslant \ell(K_0^\circ)/a$ and so, taking again into account the form of the lower bound on $1/2\log_2(4M_1)$ that we are using, we conclude that the effect of the quantity $1 + \log_2(a_1/a_k)$ in (7) reduces to a multiplicative numerical constant (about 0.91 under the worst case scenario).

5. Near optimality, and finite-dimensional subspaces of L_q

The purpose of this section is to substantiate Remark (c), which followed Theorem 1 and which asserted that the theorem as stated cannot be essentially improved, even if X varies only over the class of ℓ_q^n spaces. To see this, denote by $\gamma_q(Y)$ the factorization constant of Id_Y, the identity on Y, through an L_q-space (i.e., $\gamma_q(Y) := \inf\{\|u\|\|v\|: u : Y \to L_q,\ v : L_q \to Y,\ v \circ u = Id_Y\}$), and similarly $\gamma_q^{(n)}(Y)$—the factorization constant of Id_Y through ℓ_q^n. We then have

Fact 7. *If $q \geqslant 2$, then*

1. $\gamma_q(\ell_\infty^k) \geqslant k^{1/q}$,
2. $\gamma_q(\ell_1^k) \geqslant c\sqrt{k}$,
3. $\gamma_q(\ell_2^k) \geqslant c \min\{\sqrt{k}, \sqrt{q}\}$,
4. $\gamma_q^{(n)}(\ell_2^k) \geqslant c\sqrt{k}/n^{1/q}$,

where $c > 0$ is a universal constant.

The "near optimality" of the statement in Theorem 1 follows now from the fact that if k and (large, but not too large) $q > 2$ are appropriately related, then all of the quantities in the statements 1–3 must be at least $(\log k)^{1/2}$ (modulo lower order factors; note that Theorem 1 gives an upper estimate with exponent 2 in place of 1/2). Specifically, if k is sufficiently large and if $q = \log k / \log\log k$, then $k^{1/q} = \log k$ and so the smallest of the lower bounds is the second expression from 3, i.e., $c\sqrt{q} = c\sqrt{\log k / \log\log k}$.

The second part of Remark (c) addressed the "near optimality" of our estimate on the growth of $k_n = c \exp(\frac{1}{2}\sqrt{\log n})$. One possible way of stating this assertion more precisely is: *if for every n-dimensional space X the factorization constant of $Id_{\ell_p^k}$ through X is, for some $p \in \{1, 2, \infty\}$, smaller than* $\exp(\sqrt{\log k})$, *then* $k < \exp(C(\log n)^{2/3})$. The argument involves balancing the bounds from statements 1 and 4 and goes roughly as follows. Consider $X = \ell_q^n$, where $q = \sqrt{\log k}$. Then $k^{1/q} = \exp(\sqrt{\log k})$, which excludes $p = \infty$ (and $p = 1$ if k is sufficiently large, which we may assume). Next, given that $q = \sqrt{\log k}$, a straightforward calculation shows that $k \geqslant \exp((4\log n)^{2/3})$ implies (in fact is equivalent to) $n^{1/q} \leqslant k^{1/4}$ and subsequently implies $c\sqrt{k}/n^{1/q} \geqslant ck^{1/4} \gg \exp(\sqrt{\log k})$. This excludes $p = 2$. If we want to exclude factorization constants smaller than a power of $\log k$ (say, $(1 + \log k)^A$, as opposed to $\exp(\sqrt{\log k})$), the calculation will be slightly more involved and the resulting restriction on the growth of (k_n) will be (up to constants depending on A appearing in several places in the exponent) of the form $\exp(\sqrt{\log n \log\log n})$.

The argument above may exist in the literature or is a folklore; it certainly follows from well-known results and methods. (Indeed, similar considerations might have motivated various versions of *the modified problem*; note that it is easy to see that the answer to that problem, as stated in the introduction, is affirmative if we restrict our attention to spaces $X_n = \ell_{q_n}^{m_n}$.) Similarly, the estimates from Fact 7 are well known to specialists. In fact, the exact values of most (or perhaps even all) quantities involved there have been computed. However, the results are spread over the literature and often are not explicitly stated. For completeness, we will sketch derivations of Fact 7 from better known results. (For definitions of unexplained concepts and for cited facts we refer the reader to [25] or [32].)

1. It is an elementary fact that $d(\ell_\infty^k, \ell_2^k)$, the Banach–Mazur distance between ℓ_∞^k and ℓ_2^k, equals $k^{1/2}$. A less elementary, but classical estimate (see [12]) is that for any k-dimensional subspace $F \subset L_q$ we have $d(\ell_2^k, F) \leqslant k^{|1/2-1/q|}$. Combining these two results we infer that, for any such F, $d(\ell_\infty^k, F) \geqslant k^{1/q}$. *A fortiori*, $\gamma_q(\ell_\infty^k) \geqslant k^{1/q}$.

2. By duality, $\gamma_q(\ell_1^k) = \gamma_{q^*}(\ell_\infty^k)$, where $q^* = q/(q-1) \in [1,2]$ is the dual exponent. Now, the cotype 2 constant of ℓ_∞^k is $\sqrt{k}$, while the cotype 2 constants of spaces L_r, $1 \leqslant r \leqslant 2$, are bounded by a universal constant, say C. By the ideal property of the cotype 2 constant it follows that, for such r, $\gamma_r(\ell_\infty^k) \geqslant C^{-1}\sqrt{k}$, and the asserted estimate follows.

4. The exact value of $\gamma_\infty(\ell_2^k)$, the projective constant of ℓ_2^k, is well known [9,28], in particular we have $\gamma_\infty(\ell_2^k)/\sqrt{k} \in (\sqrt{2/\pi}, 1]$ for all $k \in \mathbb{N}$ ([32, Theorem 32.9(ii)]; in modern parlance, this is a consequence of the "little" Grothendieck theorem). Consequently, for any $n \in \mathbb{N}$, $\gamma_\infty^{(n)}(\ell_2^k) \geqslant \sqrt{2/\pi}\,\sqrt{k}$. This settles the case $q = \infty$, and the general case follows since $d(\ell_\infty^n, \ell_q^n) = n^{1/q}$.

3. Again, the estimates for (and even the exact values of) $\gamma_q(\ell_2^k)$ are known to specialists, but finding them in the literature seems to require combining formulae from several sources. First, $\gamma_q(\ell_2^k) = n/\pi_q(Id_{\ell_2^n})\pi_{q^*}(Id_{\ell_2^n})$ [8,26]; this follows from the duality theory for the γ_q ideal norm (see, e.g., [32, Theorem 13.4]) and from symmetries of the Hilbert space (cf. [32, §16]). (In fact we need here only the *lower* bound on $\gamma_q(\ell_2^k)$, which follows just from the duality theory.) Next, the exact values of, and/or the estimates for $\pi_r(Id_{\ell_2^n})$ can be found in [5,7] or in [32, Theorem 10.3]. And here is a more transparent argument which gives just a sightly weaker estimate with $\sqrt{q}$ replaced by $\sqrt{q/\log q}$. (This has only minor effect on our applications of Fact 7: the lower bound $c\sqrt{\log k/\log\log k}$ becomes $c\sqrt{\log k}/\log\log k$.) If $\dim Y = k$, then $\gamma_q^{(n)}(Y) \leqslant 4\gamma_q(Y)$ for some $n \leqslant (Ck)^k$. This is because every k-dimensional subspace of L_q is contained in a larger subspace of dimension $n \leqslant (Ck)^k$, whose Banach–Mazur distance to ℓ_q^n is less than (say) 2, and which is 2-complemented in L_p [19]. Now, if $k \leqslant q/\log q$, then $k\log(Ck) \leqslant q$ (at least for sufficiently large q) and so, for n as above, $n^{1/q} \leqslant ((Ck)^k)^{1/q} = \exp(k\log(Ck)/q) \leqslant e$. We now appeal to statement 4 to deduce that, for all such k and n,

$$\gamma_q(\ell_2^k) \geqslant \frac{1}{4}\gamma_q^{(n)}(\ell_2^k) \geqslant \frac{1}{4}\sqrt{\frac{2}{\pi}}\frac{\sqrt{k}}{n^{1/q}} \geqslant \frac{1}{4e}\sqrt{\frac{2}{\pi}}\sqrt{k},$$

as claimed. The remaining case $k > q/\log q$ follows then from the fact that, for fixed q, the sequence $\gamma_q(\ell_2^k)$, $k = 1, 2, \ldots$, is (clearly) nondecreasing.

Note. The second argument above would yield the precise version of statement 3 if we knew that every k-dimensional subspace of L_q is contained in a larger subspace whose dimension is (at most) exponential in k and which is, say, 2-isomorphic to ℓ_q^N and 2-complemented. It would be of (independent) interest to clarify this issue, which is relevant to well studied "uniform approximation function" of L_p-spaces (see [1] and its references for the background and related results).

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