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A Proof of the Ryll-Nardzewski Fixed Point Theorem

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If $\mathscr{F}$ is any family of maps of a space X into itself, by a fixed point for $\mathscr{F}$ is meant a point $x_0 \in X$ such that $T(x_0) = x_0$ for each $T \in \mathscr{F}$. Our aim in this note is to present a self-contained proof of the theorem of Ryll-Nardzewski on fixed points of families of affine self-maps of convex sets. The proof, which clarifies and simplifies some of the arguments in [2], is based entirely on two general results in functional analysis: the extended Krein-Milman theorem and the Mazur theorem in locally convex spaces.

We begin with

DEFINITION 1. Let $\mathscr{F}$ be a family of self-maps of a space X. The family $\mathscr{F}$ is called *distal* on X if for each pair $x \neq y$ of distinct points of X, there is an open covering $\{V_\alpha \mid \alpha \in \mathscr{A}\}$ of X such that $T(y) \notin \bigcup \{V_\alpha \mid T(x) \in V_\alpha\}$ for each $T \in \mathscr{F}$.

The requirement of this definition, that for no $T \in \mathscr{F}$ do Tx and Ty belong to a common set V_α, can be stated directly in terms of the cartesian product $X \times X$: for $x \neq y$, the set $\{(Tx, Ty) \mid T \in \mathscr{F}\}$ must be outside some neighbourhood $\bigcup \{V_\alpha \times V_\alpha \mid \alpha \in \mathscr{A}\}$ of the diagonal; the neighbourhood depends on the pair x, y and the notion of distal clearly depends on the topology in X.

If $\mathscr{F}$ is a family of self-maps of X, a subset $A \subset X$ is called $\mathscr{F}$*-invariant* if $T(A) \subset A$ for all $T \in \mathscr{F}$; a closed $A \subset X$ which is $\mathscr{F}$-invariant and has no

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proper closed $\mathscr{F}$-invariant subset is called a *minimal* closed $\mathscr{F}$-invariant subset. The proof of the following result relies on an extended version of the Krein–Milman theorem in locally convex spaces: if the convex closure Conv A of a set A is compact, Conv A has extreme points and if A itself is also compact, those extreme points belong to A.

THEOREM 1. *Let C be a nonempty compact convex set in a locally convex space E, and let $\mathscr{F}$ be a semigroup of continuous affine maps of C into itself. If $\mathscr{F}$ is distal on each minimal closed $\mathscr{F}$-invariant set, then $\mathscr{F}$ has a fixed point.*

Proof. Let $\mathscr{K}$ be the collection of all nonempty compact convex subsets that are $\mathscr{F}$-invariant; since $C \subset K$ this family is not empty. Partially order $\mathscr{K}$ by inclusion; since each descending chain $\{K_\alpha\}$ has the lower bound $\bigcap K_\alpha$, the Kuratowski–Zorn lemma gives a minimal $C_0 \subset C$ in $\mathscr{K}$.

Now let $\mathscr{K}_0$ be the family of all nonempty compact subsets of C_0 that are $\mathscr{F}$-invariant; again by the Kuratowski–Zorn lemma there is a closed minimal $\mathscr{F}$-invariant $X \subset C_0$. We are going to prove that X consists of a single point.

Assume $x \neq y$ were two distinct points in X. Because $(x+y)/2 \in C_0$ and C_0 is $\mathscr{F}$-invariant, we have $A = \{T((x+y)/2) \mid T \in \mathscr{F}\} \subset C_0$, so $\bar{A} \subset C_0$ is compact. Moreover, $\bar{A}$ is $\mathscr{F}$-invariant and, because each T is affine, the convex closure Conv $\bar{A} \subset C_0$ is also $\mathscr{F}$-invariant, so by the minimal property of C_0 we have Conv $\bar{A} = C_0$. Now let z be an extreme point of C_0; because $\bar{A}$ is compact, the extended Krein–Milman theorem shows $z \in \bar{A}$, so there is a net $T_\alpha((x+y)/2) \to z$. The $T_\alpha x$ and $T_\alpha y$ belong to the compact X, so we may assume that $T_\alpha x \to u \in X$ and $T_\alpha y \to v \in X$, so that

$$z = \lim \frac{1}{2} [T_\alpha x + T_\alpha \; y] = \frac{u+v}{2},$$

and, because z is an extreme point, therefore $u = v = z$. This means that for each open covering $\{V_\alpha\}$ of X, any set V_{α_0} that contains u will contain almost all the $T_\alpha x$, $T_\alpha y$, consequently $\mathscr{F}$ is not distal on the closed minimal $\mathscr{F}$-invariant set X. The assumption that X has more than one point has led to a contradiction with our hypothesis. Thus, X must consist of a single point x_0 and, since x_0 is $\mathscr{F}$-invariant, we have $T(x_0) = x_0$ for all $T \in \mathscr{F}$, so it is a fixed point for $\mathscr{F}$. This completes the proof.

As a first consequence, we obtain

THEOREM 2 (F. Hahn [1]). *Let C be a compact convex subset of a locally convex space E, and let $\mathscr{F}$ be a semigroup of continuous affine self-maps of C. If $\mathscr{F}$ is distal on C, then $\mathscr{F}$ has a fixed point.*

Proof. It is clear that a family $\mathscr{F}$ distal on C is distal on any $\mathscr{F}$-invariant subset, so this follows immediately from Theorem 1.

The Ryll-Nardzewski theorem is a generalization and at the same time a consequence of Theorem 1; it involves an interplay between natural topology of a locally convex space (which we will call strong topology for emphasis) and its weak topology. The main general fact needed for the proof is the Mazur theorem in locally convex spaces: a convex set is weakly closed if and only of it is strongly closed; in particular a strongly closed convex subset of a weakly compact set is weakly compact.

THEOREM 3 (C. Ryll-Nardzewski [4]). *Let C be a nonempty weakly compact convex set in a locally convex space E, and let $\mathscr{F}$ be a semigroup of weakly continuous affine self-maps of C. If $\mathscr{F}$ is strongly distal on C, then $\mathscr{F}$ has a fixed point.*

Proof. We must show $\bigcap \{\text{Fix } T \mid T \in \mathscr{F}\} \neq \varnothing$ and we begin by reducing the problem. Noting that each set Fix T is weakly closed, therefore weakly compact, it is enough to show that each finite intersection of the sets Fix T is nonempty. Let then $T_1, \ldots, T_n$ be finitely many members of $\mathscr{F}$, and let $\mathscr{S}$ be the subsemigroup generated by $T_1, \ldots, T_n$; clearly $\mathscr{S}$ is countable, and it suffices to show that $\mathscr{S}$ has a fixed point.

We reduce the problem further. Pick any $c_0 \in C$ and consider the convex closure $Q = \text{Conv}\{T(c_0) \mid T \in \mathscr{S}\}$; because $\mathscr{S}$ is countable, Q is strongly separable. Moreover, because each T is affine, Q is $\mathscr{S}$-invariant and because Q is a closed convex subset of C, it is weakly closed, therefore compact. Thus, replacing C by Q and $\mathscr{F}$ by $\mathscr{S}$, it is enough to prove the theorem with the additional hypothesis that C is strongly separable. We now begin the proof.

Working in the locally convex space E with weak topology, the result will follow from Theorem 1 if we can show that $\mathscr{F}$ is weakly distal on each given weakly closed minimal $\mathscr{F}$-invariant set $X \subset C$.

Let then $x \neq y$ be two elements of X, and let $\{V_\alpha \mid \alpha \in \mathscr{A}\}$ be a strongly open cover satisfying Definition 1. Because E is locally convex, $\{V_\alpha\}$ has an open refinement $\{X \cap B_\beta \mid \beta \in \mathscr{B}\}$, where each B_β is a convex strongly open set in E, and each $X \cap \bar{B}_\beta \subset$ some V_α. By strong separability, this has a countable subcover $\{X \cap B_i \mid i \in \mathbb{Z}\}$. Now, each $\bar{B}_i$ being strongly closed and convex is also weakly closed, therefore $\{X \cap \bar{B}_i \mid i \in \mathbb{Z}\}$ is a countable weakly closed cover of the weakly compact X so, by the Baire theorem, at least one of these sets must contain a weakly open set U, say $U \subset X \cap \bar{B}_i \subset V_\alpha$.

[1] For another proof, which uses, however, some special properties of locally convex spaces, see Namioka–Asplund [3].

We show that the family $\{T^{-1}(U) \mid T \in \mathscr{F}\}$ of weakly open sets satisfies the requirement of Definition 1 for the points $x \neq y$. First, these sets must cover X, otherwise $X - \bigcup \{T^{-1}(U) \mid T \in \mathscr{F}\}$ would be a weakly compact $\mathscr{F}$-invariant proper subset of X, contradicting the minimality of X. Finally, for no $S \in \mathscr{F}$ do Sx and Sy belong to a common set $T^{-1}(U)$, otherwise TSx and TSy would belong to $U \subset X \cap \bar{B}_i \subset V_\alpha$ and, since $TS \in \mathscr{F}$, this contradicts the definition of the strongly open cover $\{V_\alpha\}$.

Thus, the requirements of Definition 1 are satisfied in the weak topology of X, and since x, y are arbitrary, this shows $\mathscr{F}$ is weakly distal on X. Thus, by Theorem 1, the family $\mathscr{F}$ has a fixed point and, as we have observed, this is enough to complete the proof.

As an illustration we deduce the well-known theorem of Kakutani. For this purpose, recall

DEFINITION 2. Let C be a subset of a linear space, and $\mathscr{F}$ a family of self-maps of C. We call $\mathscr{F}$ *equicontinuous* on C if for each given neighbourhood $W(0)$ of the origin, there is a neighbourhood $V(0)$ with the property: whenever $x - y \in V$, then $Tx - Ty \in W$ for all $T \in \mathscr{F}$.

The members of an equicontinuous family are, clearly, necessarily continuous.

THEOREM 4 (S. Kakutani). *Let C be a compact subset of a locally convex space, and let F be a group of affine transformations of C into itself. If $\mathscr{F}$ is equicontinuous on C, then $\mathscr{F}$ has a fixed point.*

Proof. (cf. [1]). This will follow from Theorem 2 (or from Theorem 3, because a strongly compact set is also weakly compact) once we show that $\mathscr{F}$ is distal on C.

Assume $\mathscr{F}$ were not distal on C; then there must be a pair $x \neq y$ of distinct points in C with the property: for each neighbourhood $U(0)$ there is a $T_U \in \mathscr{F}$ such that $T_U(x)$ and $T_U(y)$ belong to a common set of the open cover $\{U + c \mid c \in C\}$ of C. Choose now a $W(0)$ so that $y \notin x + W$; then for each $V(0)$ there is a $U(0)$ with $U - U \subset V$ and we have $T_U x - T_U y \in V$, yet for the function $T_U^{-1} \in \mathscr{F}$ we have $T_U^{-1}(T_U x) - T_U^{-1}(T_U y) = x - y \notin W$. Thus, $\mathscr{F}$ cannot be equicontinuous on C.

Thus, the equicontinuous family $\mathscr{F}$ is distal on C and, by Theorem 2, the proof is complete.

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