Available online at www.sciencedirect.com

SciVerse ScienceDirect

Stochastic Processes and their Applications 122 (2012) 3790–3811

stochastic processes and their applications

www.elsevier.com/locate/spa

Strong mixing properties of max-infinitely divisible random fields

Clément Dombry*, Frédéric Eyi-Minko

Université de Poitiers, Laboratoire de Mathématiques et Applications, UMR CNRS 6086, Téléport 2, BP 30179, F-86962 Futuroscope-Chasseneuil cedex, France

Received 22 January 2012; received in revised form 20 June 2012; accepted 22 June 2012
Available online 30 June 2012

Abstract

Let $\eta = (\eta(t))_{t\in T}$ be a sample continuous max-infinitely random field on a locally compact metric space T. For a closed subset $S \subset T$, we denote by η_S the restriction of η to S. We consider $\beta(S_1, S_2)$, the absolute regularity coefficient between η_{S_1} and η_{S_2}, where S_1, S_2 are two disjoint closed subsets of T. Our main result is a simple upper bound for $\beta(S_1, S_2)$ involving the exponent measure μ of η: we prove that $\beta(S_1, S_2) \leq 2\int \mathbb{P}[\eta \not<_{S_1} f, \eta \not<_{S_2} f]\,\mu(df)$, where $f \not<_S g$ means that there exists $s \in S$ such that $f(s) \geq g(s)$.

If η is a simple max-stable random field, the upper bound is related to the so-called extremal coefficients: for countable disjoint sets S_1 and S_2, we obtain $\beta(S_1, S_2) \leq 4\sum_{(s_1,s_2)\in S_1\times S_2}(2 - \theta(s_1, s_2))$, where $\theta(s_1, s_2)$ is the pair extremal coefficient.

As an application, we show that these new estimates entail a central limit theorem for stationary max-infinitely divisible random fields on $\mathbb{Z}^d$. In the stationary max-stable case, we derive the asymptotic normality of three simple estimators of the pair extremal coefficient.

MSC: primary 60G70; secondary 60G10; 37A25

Keywords: Absolute regularity coefficient; Max-infinitely divisible random field; Max-stable random field; Central limit theorem for weakly dependent random field

* Corresponding author.
E-mail addresses: clement.dombry@math.univ-poitiers.fr (C. Dombry), Frederic.Eyi.minko@math.univ-poitiers.fr (F. Eyi-Minko).

0304-4149/$ - see front matter
doi:10.1016/j.spa.2012.06.013

1. Introduction

Max-stable random fields turn out to be fundamental models for spatial extremes since they arise as the limit of rescaled maxima. More precisely, consider the component-wise maxima

$$\eta_n(t) = \max_{1\le i\le n} \xi_i(t), \quad t \in T,$$

of independent copies $\xi_i, i \ge 1$, of a random field $\xi = (\xi(t))_{t\in T}$. If the random field $\eta_n = (\eta_n(t))_{t\in T}$ converges in distribution, as $n \to \infty$, under suitable affine normalization, then its limit $\eta = \{\eta(t)\}_{t\in T}$ is necessarily max-stable. Therefore, max-stable random fields play a central role in the extreme value theory, just like Gaussian random fields do in the classical statistical theory based on the central limit theorem.

Max-stable processes have been studied extensively in the past decades and many of their properties are well-understood. For example, the structure of their finite dimensional distributions is well known and insightful Poisson point process or spectral representations are available. Also the theory has been extended to max-infinitely divisible (max-i.d.) processes. See for example the seminal works by Resnick [20], de Haan [9,10], de Haan and Pickands [13], Giné et al. [16], Resnick and Roy [21] and many others. More details and further references can be found in the monographs by Resnick [20] or de Haan and Ferreira [11].

The questions of mixing and ergodicity of max-stable random processes indexed by $\mathbb{R}$ or $\mathbb{Z}$ have been addressed recently. First results by Weintraub [29] in the max-stable case have been completed by Stoev [25], providing necessary and sufficient conditions for mixing of max-stable processes based on their spectral representations. More recently, Kabluchko and Schlather [18] have extended these results and obtain necessary and sufficient conditions for both mixing and ergodicity of max-i.d. random processes. They define the dependence function of a stationary max-i.d. random process $\eta = (\eta(t))_{t\in\mathbb{Z}}$ by

$$\tau_a(h) = \log \frac{\mathbb{P}[\eta(0) \le a, \eta(h) \le a]}{\mathbb{P}[\eta(0) \le a]\mathbb{P}[\eta(h) \le a]}, \quad a > \operatorname{essinf} \eta(0), h \in \mathbb{Z}.$$

Then, it holds with $\ell = \operatorname{essinf} \eta(0)$:

- η is mixing if and only if for all $a > \ell$, $\tau_a(n) \to 0$ as $n \to +\infty$;
- η is ergodic if and only if for all $a > \ell$, $n^{-1} \sum_{h=1}^{n} \tau_a(h) \to 0$ as $n \to +\infty$.

Ergodicity is strongly connected to the strong law of large numbers via the ergodic theorem. The above results find natural applications in statistics to obtain strong consistency of several natural estimators based on non-independent but ergodic observations.

Going a step further, we address in this paper the issue of estimating the strong mixing coefficients of max-i.d. random fields. In some sense, ergodicity and mixing state that the restrictions η_{S_1} and η_{S_2} to two subsets S_1, S_2 become almost independent when the distance between S_1 and S_2 goes to infinity. Strong mixing coefficients make this statement quantitative: we introduce two standard mixing coefficients $\alpha(S_1, S_2)$ and $\beta(S_1, S_2)$ that measure how much η_{S_1} and η_{S_2} differ from independence. The rate of decay of those coefficients as the distance between S_1 and S_2 goes to infinity is a crucial point for the central limit theorem (see Appendix A.3). As an application, we consider the asymptotic normality of three simple estimators of the extremal coefficients of a stationary max-stable random field on $\mathbb{Z}^d$ with standard unit Fréchet margins.

Our approach differs from those of Stoev [25] based on spectral representations and of Kabluchko and Schlather [18] based on exponent measures. It relies on the Poisson point process representation of max-i.d. random fields offered by Giné et al. [16] (see also Appendix A.1) and on the notions of extremal and subextremal points recently introduced by the authors [14]. The Palm theory for the Poisson point process and the Campbell–Slyvniak formula are also a key tool (see Appendix A.2).

The structure of the paper is the following: the framework and results are detailed in the next section; Section 3 is devoted to the proofs and an Appendix gathers some more technical details.

2. Framework and results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and T be a locally compact metric space. We denote by $\mathbb{C}(T) = \mathbb{C}(T, [0, +\infty))$ the space of nonnegative continuous functions endowed with the topology of uniform convergence on compact sets and by $\mathcal{C}$ its Borel σ-field. A measure is said to be locally finite if it assigns finite measure to compact sets. Let μ be a locally finite Borel measure on $\mathbb{C}_0(T) = \mathbb{C}(T) \setminus \{0\}$ satisfying

$$\mu\Big[\{f \in \mathbb{C}_0(T);\ \sup_K f > \varepsilon\}\Big] < \infty \quad \text{for all compact } K \subset T \text{ and } \varepsilon > 0, \tag{1}$$

and Φ a Poisson point process on $\mathbb{C}_0(T)$ with intensity μ. More rigorously, we should consider Φ as a random point measure rather than as a random set of points, since there may be points with multiplicities. It is however standard to consider Φ as a random set of points with possible repetitions.

We consider the random process

$$\eta(t) = \max\{\phi(t), \phi \in \Phi\}, \quad t \in T,$$

with the convention that the maximum of the empty set is equal to 0. Condition (1) ensures that the random process η is continuous on T (see [16] and Appendix A.1). Another property is worth noting: η is max-infinitely divisible. This means that for all $n \geq 1$, there exist independent and identically distributed continuous random fields $(\eta_{i,n})_{1\leq i\leq n}$ such that

$$\eta \stackrel{\mathcal{L}}{=} \vee_{i=1}^{n} \eta_{i,n},$$

where $\vee$ stands for pointwise maximum and $\stackrel{\mathcal{L}}{=}$ for equality in distribution. Note that the max-infinite divisibility of η is a simple consequence of the superposition theorem for Poisson point processes. Furthermore, for all $t \in T$, the essential infimum of the random variable $\eta(t)$ is equal to 0. As shown by Giné et al. [16], up to simple transformations, essentially all max-i.d. continuous random processes on T can be obtained in this way (see Appendix A.1). The measure μ is called the exponent measure associated to the max-i.d. process η. It should be stressed that Giné et al. [16] deal with upper semi-continuous functions. For the sake of simplicity, we consider in this paper only continuous processes, even if the main results (Theorems 2.1 and 2.2) can be extended almost directly to cover the case of upper semi-continuous processes.

We now introduce the so-called α- and β-mixing coefficients. For more details on strong mixing conditions, the reader should refer to the recent survey by Bradley [3] or to the monographs [15,22,4–6,8]. For $S \subset T$ a closed subset, we denote by $\mathcal{F}_S$ the σ-field generated by the random variables $\{\eta(s), s \in S\}$ and by $\mathcal{P}_S$ the distribution of the restriction η_S in the set $\mathbb{C}(S)$ of nonnegative continuous functions on S endowed with its Borel σ-field $\mathcal{C}_S$. Let $S_1, S_2 \subset T$ be disjoint closed subsets. The α-mixing coefficient introduced by Rosenblatt [23] between the

σ-fields $\mathcal{F}_{S_1}$ and $\mathcal{F}_{S_2}$ is defined by

$$\alpha(S_1, S_2) = \sup\Big\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|;\ A \in \mathcal{F}_{S_1}, B \in \mathcal{F}_{S_2}\Big\}.$$

The β-mixing coefficient (or absolute regularity coefficient, see [28]) between the σ-fields $\mathcal{F}_{S_1}$ and $\mathcal{F}_{S_2}$ is given by

$$\beta(S_1, S_2) = \sup\Big\{|\mathcal{P}_{S_1\cup S_2}(C) - \mathcal{P}_{S_1} \otimes \mathcal{P}_{S_2}(C)|;\ C \in \mathcal{C}_{S_1\cup S_2}\Big\}. \tag{2}$$

Since S_1 and S_2 are disjoint closed subsets, $\mathbb{C}(S_1 \cup S_2)$ is naturally identified with $\mathbb{C}(S_1) \times \mathbb{C}(S_2)$. We denote by $\|\cdot\|_{\mathrm{var}}$ the total variation of a signed measure. Equivalent definitions of the β-mixing coefficient are

$$\begin{aligned}\beta(S_1, S_2) &= \|\mathcal{P}_{S_1\cup S_2} - \mathcal{P}_{S_1} \otimes \mathcal{P}_{S_2}\|_{\mathrm{var}}\\ &= \frac{1}{2}\sup\left\{\sum_{i=1}^{I}\sum_{j=1}^{J} |\mathbb{P}(A_i \cap B_j) - \mathbb{P}(A_i)\mathbb{P}(B_j)|\right\}\end{aligned}$$

where the supremum is taken over all partitions $\{A_1, \dots, A_I\}$ and $\{B_1, \dots, B_J\}$ of Ω with the A_i's in $\mathcal{F}_{S_1}$ and the B_j's in $\mathcal{F}_{S_2}$. The following inequality is worth noting

$$\alpha(S_1, S_2) \le \frac{1}{2}\beta(S_1, S_2). \tag{3}$$

Our main result is the following.

Theorem 2.1. *Let η be a continuous max-i.d. process on T with exponent measure μ satisfying (1). Then, for all disjoint closed subsets $S_1, S_2 \subset T$,*

$$\beta(S_1, S_2) \le 2\int_{\mathbb{C}_0} \mathbb{P}[f \not<_{S_1} \eta, f \not<_{S_2} \eta]\,\mu(df).$$

In the particular case when S_1 and S_2 are finite or countable (which naturally arise for example if $T = \mathbb{Z}^d$), we can provide an upper bound for the mixing coefficient $\beta(S_1, S_2)$ involving only the 2-dimensional marginal distributions of the process η.

For $(s_1, s_2) \in T^2$, let μ_{s_1,s_2} be the exponent measure of the max-i.d. random vector $(\eta(s_1), \eta(s_2))$ defined on $[0, +\infty)^2$ by

$$\mu_{s_1,s_2}(A) = \mu\big[\{f \in \mathbb{C}_0(T);\ (f(s_1), f(s_2)) \in A\}\big], \quad A \subset [0, +\infty)^2 \text{ Borel set}.$$

Corollary 2.1. *If S_1 and S_2 are finite or countable disjoint closed subsets of T,*

$$\beta(S_1, S_2) \le 2\sum_{s_1\in S_1}\sum_{s_2\in S_2}\int \mathbb{P}[\eta(s_1) \le y_1, \eta(s_2) \le y_2]\,\mu_{s_1,s_2}(dy_1dy_2).$$

Next, we focus on simple max-stable random fields, where the phrase simple means that the marginals are standardized to the standard unit Fréchet distribution,

$$\mathbb{P}[\eta(t) \le y] = \exp[-y^{-1}]1_{\{y>0\}}, \quad y \in \mathbb{R}, t \in T.$$

In this framework, an insight into the dependence structure is given by the extremal coefficients $\theta(S)$, $S \subset T$ compact, defined by the relation

$$\mathbb{P}[\sup_{s\in S} \eta(s) \le y] = \exp[-\theta(S)y^{-1}], \quad y > 0. \tag{4}$$

Theorem 2.2. *Let η be a continuous simple max-stable random field on T.*

For all compact $S \subset T$, the quantity

$$C(S) = \mathbb{E}[\sup\{\eta(s)^{-1};\ s \in S\}] \tag{5}$$

is finite and furthermore:

- *For all disjoint compact subsets $S_1, S_2 \subset T$,*
$$\beta(S_1, S_2) \le 2\Big[C(S_1) + C(S_2)\Big]\Big[\theta(S_1) + \theta(S_2) - \theta(S_1 \cup S_2)\Big].$$
- *For $(S_{1,i})_{i\in I}$ and $(S_{2,j})_{j\in J}$ countable families of compact subsets of T such that $S_1 = \cup_{i\in I} S_{1,i}$ and $S_2 = \cup_{j\in J} S_{2,j}$ are disjoint,*
$$\beta(S_1, S_2) \le 2\sum_{i\in I}\sum_{j\in J}\Big[C(S_{1,i}) + C(S_{2,j})\Big]\Big[\theta(S_{1,i}) + \theta(S_{2,j}) - \theta(S_{1,i} \cup S_{2,j})\Big].$$

In the particular case when S_1 and S_2 are finite or countable, the mixing coefficient $\beta(S_1, S_2)$ can be bounded from above in terms of the extremal coefficient function

$$\theta(s_1, s_2) = \theta(\{s_1, s_2\}), \quad s_1, s_2 \in T.$$

We recall the following basic properties: it always holds $\theta(s_1, s_2) \in [1, 2]$; $\theta(s_1, s_2) = 2$ iff $\eta(s_1)$ and $\eta(s_2)$ are independent; $\theta(s_1, s_2) = 1$ iff $\eta(s_1) = \eta(s_2)$. Thus the extremal coefficient function gives some insight into the 2-dimensional dependence structure of the max-stable field η, although it does not characterize it completely.

Corollary 2.2. *Suppose η is a continuous simple max-stable random field on T. If S_1 and S_2 are finite or countable disjoint closed subsets of T, then*

$$\beta(S_1, S_2) \le 4\sum_{s_1\in S_1}\sum_{s_2\in S_2}[2 - \theta(s_1, s_2)].$$

Remark 2.1. It should be stressed that the proof of Theorem 2.2 relies on the following inequality: if η is a max-stable process with exponent measure μ, then for all disjoint compact subsets $S_1, S_2 \subset T$

$$\int_{\mathbb{C}_0} \mathbb{P}[f \not<_{S_1} \eta, f \not<_{S_2} \eta]\, \mu(df) \le \Big[C(S_1) + C(S_2)\Big]\Big[\theta(S_1) + \theta(S_2) - \theta(S_1 \cup S_2)\Big].$$

In view of Theorem 2.1, this inequality entails the first point of Theorem 2.2. When $S_1 = \{s_1\}$ and $S_2 = \{s_2\}$, we obtain

$$\int \mathbb{P}[\eta(s_1) \le y_1, \eta(s_2) \le y_2]\, \mu_{s_1,s_2}(dy_1 dy_2) \le 2[2 - \theta(s_1, s_2)].$$

This is used in the proof of Corollary 2.2.

As noted in the introduction, our main motivation for considering the strong mixing properties of max-i.d. random fields is to obtain central limit theorems (CLTs) for stationary max-i.d. random fields. In this direction, we focus on stationary random fields on $T = \mathbb{Z}^d$ and our analysis relies on Bolthausen's CLT [2] (see Appendix A.3).

We denote by $|h| = \max_{1\le i\le d} |h_i|$ the norm of $h \in \mathbb{Z}^d$ and by $|S|$ the number of elements of a subset $S \subset \mathbb{Z}^d$. The boundary ∂S of S is the set of elements $h \in S$ such that there is $h' \notin S$ with $d(h, h') = 1$.

A random field $X = (X(t))_{t\in\mathbb{Z}^d}$ is said to be stationary if the law of $(X(t+s))_{t\in\mathbb{Z}^d}$ does not depend on $s \in \mathbb{Z}^d$. We say that a square integrable stationary random field X satisfies the CLT if the following two conditions are satisfied:

(i) the series $\sigma^2 = \sum_{t\in\mathbb{Z}^d} \mathrm{Cov}[X(0), X(t)]$ converges absolutely;
(ii) for all sequences Λ_n of finite subsets of $\mathbb{Z}^d$, which increase to $\mathbb{Z}^d$ and such that $\lim_{n\to\infty} |\partial\Lambda_n|/|\Lambda_n| = 0$, the sequence $|\Lambda_n|^{-1/2}\sum_{t\in\Lambda_n}(X(t) - \mathbb{E}[X(t)])$ converges in law to the normal distribution with mean 0 and variance σ^2 as $n \to \infty$.

Please note that we do not require the limit variance σ^2 to be positive; the case $\sigma^2 = 0$ corresponds to a degenerated CLT where the limit distribution is the Dirac mass at zero. Bolthausen's CLT for stationary mixing random fields together with our estimates of mixing coefficients of max-i.d. random fields yields the following theorem.

Theorem 2.3. *Suppose η is a stationary max-i.d. random field on $\mathbb{Z}^d$ with exponent measure μ and let*

$$\gamma(h) = \int \mathbb{P}[\eta(0) \le y_1, \eta(h) \le y_2]\, \mu_{0,h}(dy_1 dy_2), \quad h \in \mathbb{Z}^d.$$

Let $g : \mathbb{R}^p \to \mathbb{R}$ be a measurable function and $t_1, \ldots, t_p \in \mathbb{Z}^d$ such that

$$\mathbb{E}[g(\eta(t_1), \ldots, \eta(t_p))^{2+\delta}] < \infty \quad \textit{for some } \delta > 0,$$

and assume that

$$\sum_{|h|\ge m} \gamma(h) = o(m^{-d}) \quad \textit{and} \quad \sum_{m=1}^{\infty} m^{d-1} \sup_{|h|\ge m} \gamma(h)^{\delta/(2+\delta)} < \infty. \tag{6}$$

Then the stationary random field X defined by

$$X(t) = g(\eta(t_1 + t), \ldots, \eta(t_p + t)), \quad t \in \mathbb{Z}^d$$

satisfies the CLT.

Condition (6) requires that γ goes fast enough to 0 at infinity. It is met for example if

$$\gamma(h) \le C \cdot |h|^{-b} \quad \text{for some } b > d\max(2, (2+\delta)/\delta) \text{ and } C > 0. \tag{7}$$

If η is simple max-stable, it follows from the proof of Theorem 2.2 and Corollary 2.2 that

$$\gamma(h) \le 2(2 - \theta(0, h)),$$

with θ the extremal coefficient function.

As an application of Theorem 2.3, we consider the estimation of the extremal coefficient for a stationary simple max-stable random field on $\mathbb{Z}^d$. For $h \in \mathbb{Z}^d$, we note $\theta(h) = \theta(0, h)$. Eq. (4)

implies

$$\theta(h) = -y \log p(h, y) \quad \text{with } p(h, y) = \mathbb{P}(\eta(0) \le y,\ \eta(h) \le y),\ y > 0,$$

suggesting the simple estimator

$$\hat{\theta}_n^{(1)}(h) = -y \log \hat{p}_n(h, y) \quad \text{with } \hat{p}_n(h, y) = |\Lambda_n|^{-1} \sum_{t \in \Lambda_n} 1_{\{\eta(t) \le y, \eta(t+h) \le y\}} \tag{8}$$

where Λ_n is a sequence of finite subsets increasing to $\mathbb{Z}^d$ such that $|\partial \Lambda_n|/|\Lambda_n| \to 0$ as $n \to \infty$. The fact that the naive estimator $\hat{\theta}_n^{(1)}(h)$ depends on the threshold level $y > 0$ is not satisfactory. Alternatively, one may consider the following procedures. Smith [24] noticed that $\min(\eta(0)^{-1}, \eta(h)^{-1})$ has an exponential distribution with mean $\theta(h)^{-1}$ and proposed the estimator

$$\hat{\theta}_n^{(2)}(h) = \frac{|\Lambda_n|}{\sum_{t \in \Lambda_n} \min(\eta(t)^{-1}, \eta(t+h)^{-1})}.$$

Cooley et al. [7] introduced the F-madogram defined by

$$\nu_F(h) = \mathbb{E}[|F(\eta(0)) - F(\eta(h))|] \quad \text{with } F(y) = \exp(-1/y) 1_{\{y > 0\}}$$

and showed that it satisfies

$$\nu_F(h) = \frac{1}{2} \frac{\theta(h) - 1}{\theta(h) + 1} \quad \text{or equivalently} \quad \theta(h) = \frac{1 + 2\nu_F(h)}{1 - 2\nu_F(h)}.$$

This suggests the estimator

$$\hat{\theta}_n^{(3)}(h) = \frac{|\Lambda_n| + 2 \sum_{t \in \Lambda_n} |F(\eta(t)) - F(\eta(t+h))|}{|\Lambda_n| - 2 \sum_{t \in \Lambda_n} |F(\eta(t)) - F(\eta(t+h))|}.$$

The following proposition states the asymptotic normality of these estimators.

Proposition 2.1. *Suppose that η is a stationary simple max-stable random field on $\mathbb{Z}^d$ with extremal coefficient function satisfying*

$$2 - \theta(h) \le C \cdot |h|^{-b} \quad \textit{for some } b > 2d \textit{ and } C > 0. \tag{9}$$

Then, the estimators $\hat{\theta}_n^{(i)}(h)$, $i = 1, 2, 3$ are asymptotically normal:

$$|\Lambda_n|^{1/2} \big(\hat{\theta}_n^{(i)}(h) - \theta(h)\big) \Longrightarrow \mathcal{N}(0, \sigma_i^2) \quad \textit{as } n \to \infty$$

with limit variances

$$\sigma_1^2 = y^2 \sum_{t \in \mathbb{Z}^d} \big(\exp[(2\theta(h) - \theta(\{0, h, t, t+h\}))y^{-1}] - 1\big),$$

$$\sigma_2^2 = \theta(h)^4 \sum_{t \in \mathbb{Z}^d} \mathrm{Cov}\big[\min(\eta(0)^{-1}, \eta(h)^{-1}), \min(\eta(t)^{-1}, \eta(t+h)^{-1})\big],$$

$$\sigma_3^2 = (\theta(h) + 1)^4 \sum_{t \in \mathbb{Z}^d} \mathrm{Cov}\big[|F(\eta(0)) - F(\eta(h))|, |F(\eta(t)) - F(\eta(t+h))|\big].$$

Interestingly, the function $y \mapsto \sigma_1^2$ is strictly convex, has limit $+\infty$ as $y \to 0^+$ or $+\infty$ and hence it admits a unique minimizer $y^\star$ corresponding to an asymptotically optimal truncation level for the estimator $\hat{\theta}_n^{(1)}$. Unfortunately, the limit variances σ_2^2 and σ_3^2 are not so explicit so that a comparison between the three is difficult.

We illustrate our results on two classes of stationary max-stable random fields on $\mathbb{R}^d$.

Example 2.1. We consider the Brown–Resnick simple max-stable model (see [19]). Let $(W_i)_{i\geq 1}$ be independent copies of a sample continuous stationary increments Gaussian random field $W = (W(t))_{t\in\mathbb{R}^d}$ with zero mean and variance $\sigma^2(t)$. Independently, let $(Z_i)_{i\geq 1}$ be the nonincreasing enumeration of the points of a Poisson point process $(0, +\infty)$ with intensity $z^{-2}dz$. The associated Brown–Resnick max-stable random field is defined by

$$\eta(t) = \bigvee_{i=1}^{\infty} Z_i \exp[W_i(t) - \sigma^2(t)/2], \quad t \in \mathbb{R}^d.$$

It is known that η is a stationary simple max-stable random field whose law depends only on the negative semi-definite function V, called the variogram of W, and defined by

$$V(h) = \mathbb{E}[(W(t+h) - W(t))^2], \quad h \in \mathbb{R}^d.$$

In this case, the extremal coefficient function is given by

$$\theta(s_1, s_2) = 2\Psi(\sqrt{V(s_2 - s_1)}/2), \quad s_1, s_2 \in \mathbb{R}^d,$$

where Ψ denotes the cdf of the standard normal law. Using the tail equivalent

$$1 - \Psi(x) \sim \frac{e^{-x^2/2}}{x\sqrt{2\pi}} \quad \text{as } x \to +\infty,$$

we see that Eq. (9) holds as soon as

$$\liminf_{h\to\infty} \frac{V(h)}{\sqrt{\log|h|}} > 2\sqrt{d}.$$

This completes the necessary and sufficient conditions for ergodicity or mixing of Brown–Resnick processes given by Kabluchko and Schlather [18].

Example 2.2. Our second class of example is the moving maximum process by de Haan and Pereira [12]. Let $f : \mathbb{R}^d \to [0, +\infty)$ be a continuous density function such that

$$\int_{\mathbb{R}^d} f(x)\,dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^d} \sup_{|h|\leq 1} f(x+h)\,dx < \infty.$$

Let $\sum_{i=1}^{\infty} \delta_{(Z_i, U_i)}$ be a Poisson random measure on $(0, +\infty) \times \mathbb{R}^d$ with intensity $z^{-2}dzdu$. Then the random field

$$\eta(t) = \bigvee_{i=1}^{\infty} Z_i f(t - U_i), \quad t \in \mathbb{R}^d,$$

is a stationary sample continuous simple max-stable random field. The corresponding extremal coefficient function is given by

$$\theta(s_1, s_2) = \int_{\mathbb{R}^d} \max(f(s_1 - x), f(s_2 - x))\,dx, \quad s_1, s_2 \in \mathbb{R}^d.$$

Some computations reveal that Eq. (9) holds true as soon as

$$\limsup_{h\to\infty} \frac{\log f(h)}{\log |h|} < -\kappa_d$$

with $\kappa_1 = 3$ and $\kappa_d = 2(d+1)$ for $d \geq 2$.

3. Proofs

3.1. Strong mixing properties of extremal point processes

In the sequel, we shall write shortly $\mathbb{C}_0 = \mathbb{C}_0(T)$. We denote by $M_p(\mathbb{C}_0)$ the set of locally finite point measures N on $\mathbb{C}_0$ endowed with the σ-algebra generated by the family of mappings $\{N \mapsto N(A),\ A \subset \mathbb{C}_0 \text{ Borel set}\}$. We introduce here the notion of S-extremal points that will play a key role in this work. We use the following notations: if f_1, f_2 are two functions defined (at least) on S, we note

$$\begin{aligned} f_1 =_S f_2 &\quad \text{if and only if } \forall s \in S,\ f_1(s) = f_2(s),\\ f_1 <_S f_2 &\quad \text{if and only if } \forall s \in S,\ f_1(s) < f_2(s),\\ f_1 \not<_S f_2 &\quad \text{if and only if } \exists s \in S,\ f_1(s) \geq f_2(s). \end{aligned}$$

A point $\phi \in \Phi$ is said to be S-subextremal if $\phi <_S \eta$, it is said S-extremal otherwise, i.e. if there exists $s \in S$ such that $\phi(s) = \eta(s)$. In words, a S-subextremal point has no contribution to the maximum η on S.

Definition 3.1. Define the S-extremal random point process Φ_S^+ and the S-subextremal random point process Φ_S^- by

$$\Phi_S^+ = \{\phi \in \Phi;\ \phi \not<_S \eta\} \quad \text{and} \quad \Phi_S^- = \{\phi \in \Phi;\ \phi <_S \eta\}.$$

The fact that Φ_S^+ and Φ_S^- are well defined point processes, i.e. that they satisfy some measurability properties, is proved in [14, Appendix A.3]. Clearly, the restriction η_S depends on Φ_S^+ only:

$$\eta(s) = \max\{\phi(s);\ \phi \in \Phi_S^+\}, \quad s \in S.$$

This implies that the strong mixing coefficient $\beta(S_1, S_2)$ defined by Eq. (2) can be upper bounded by a similar β-mixing coefficient defined on the level of the extremal point process $\Phi_{S_1}^+, \Phi_{S_2}^+$. For $i = 1, 2$, let $P_{\Phi_{S_i}^+}$ be the distribution of $\Phi_{S_i}^+$ on the space of locally finite point measures on $\mathbb{C}_0$ and let $P_{(\Phi_{S_1}^+, \Phi_{S_2}^+)}$ be the joint distribution of $(\Phi_{S_1}^+, \Phi_{S_2}^+)$. We define

$$\beta(\Phi_{S_1}^+, \Phi_{S_2}^+) = \| P_{(\Phi_{S_1}^+, \Phi_{S_2}^+)} - P_{\Phi_{S_1}^+} \otimes P_{\Phi_{S_2}^+} \|_{\mathrm{var}}. \tag{10}$$

It holds

$$\beta(S_1, S_2) \leq \beta(\Phi_{S_1}^+, \Phi_{S_2}^+). \tag{11}$$

The following theorem provides a simple estimate for the β-mixing coefficient on the point process level. It implies Theorem 2.1 straightforwardly and has a clearer interpretation.

Theorem 3.1. • *The following upper bound holds true:*

$$\beta(\Phi^+_{S_1}, \Phi^+_{S_2}) \leq 2\,\mathbb{P}[\Phi^+_{S_1} \cap \Phi^+_{S_2} \neq \emptyset] \tag{12}$$

with

$$\mathbb{P}[\Phi^+_{S_1} \cap \Phi^+_{S_2} \neq \emptyset] \leq \int_{\mathbb{C}_0} \mathbb{P}(f \not<_{S_1} \eta, f \not<_{S_2} \eta)\, \mu(df). \tag{13}$$

• *If the point process Φ is simple (in particular in the max-stable case), the following lower bound holds true:*

$$\beta(\Phi^+_{S_1}, \Phi^+_{S_2}) \geq \mathbb{P}[\Phi^+_{S_1} \cap \Phi^+_{S_2} \neq \emptyset] \tag{14}$$

Clearly, Eqs. (11)–(13) together imply Theorem 2.1.

Remark 3.1. The upper and lower bounds in Theorem 3.1 are of the same order, and hence relatively sharp. It is not clear however how to bound $\beta(S_1, S_2)$ from below and how sharp the upper bound in Theorem 2.1 is.

3.2. Proof of Theorem 3.1

The upper bound for the mixing coefficient $\beta(\Phi^+_{S_1}, \Phi^+_{S_2})$ defined by Eq. (10) relies on a standard coupling argument. There are indeed deep relationships between β-mixing and optimal couplings and we will use the following result (see e.g. [22, Chapter 5]). Note that the lemma holds true for any pairs of random variables, but for the sake of future reference, we state it for extremal point processes.

Lemma 3.1. *On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, suppose that the random variables $(\Phi^{+i}_{S_1}, \Phi^{+i}_{S_2})$, $i = 1, 2$ are such that:*

(i) *the distribution of $(\Phi^{+1}_{S_1}, \Phi^{+1}_{S_2})$ is $P_{(\Phi^+_{S_1}, \Phi^+_{S_2})}$;*

(ii) *the distribution of $(\Phi^{+2}_{S_1}, \Phi^{+2}_{S_2})$ is $P_{\Phi^+_{S_1}} \otimes P_{\Phi^+_{S_2}}$.*

Then,

$$\beta(\Phi^+_{S_1}, \Phi^+_{S_2}) \leq \mathbb{P}\Big[(\Phi^{+1}_{S_1}, \Phi^{+1}_{S_2}) \neq (\Phi^{+2}_{S_1}, \Phi^{+2}_{S_2})\Big].$$

We say that the random variables $(\Phi^{+i}_{S_1}, \Phi^{+i}_{S_2})$, $i = 1, 2$ satisfying (i) and (ii) realize a coupling between the distributions $P_{(\Phi^+_{S_1}, \Phi^+_{S_2})}$ and $P_{\Phi^+_{S_1}} \otimes P_{\Phi^+_{S_2}}$.

In order to construct a suitable coupling, we need the following lemma describing the dependence between Φ^+_S and Φ^-_S.

Lemma 3.2. *Let $S \subset T$ be a closed set. The conditional distribution of Φ^-_S with respect to Φ^+_S is equal to the distribution of a Poisson point process with intensity $1_{\{f <_S \eta\}}\mu(df)$.*

Proof of Lemma 3.2. Note that in the particular case when T is compact and Φ^+_S is finite almost surely, Lemma 3.2 follows from [14, Theorem 2.1 and Corollary 2.1]. For T non compact, the proof needs to be modified in a non straightforward way.

Clearly, the event $\{f <_S \eta\}$ depends only on the restriction η_S and is hence measurable with respect to the σ-field generated by Φ^+_S. In order to prove the statement, let $A_1, \ldots, A_k \subset \mathbb{C}_0$ be disjoint compact sets and $n_1, \dots, n_k \geq 0$. Let $A = \cup_{i=1}^k A_i$ and $n = \sum_{i=1}^k n_i$. We compute the conditional probability with respect to Φ_S^+ of the event

$$\{\Phi_S^-(A_1) = n_1, \dots, \Phi_S^-(A_k) = n_k\}.$$

This event is equal to $\{\Phi_S^- \in B\}$ with $B = \{N \in M_p(\mathbb{C}_0);\ N(A_1) = n_1, \dots, N(A_k) = n_k\}$. We remark that it is realized if and only if there exists a n-tuple $(\phi_1, \dots, \phi_n)$ of atoms of Φ such that:

- the atoms $\phi_1, \dots, \phi_n$ are S-subextremal;
- $\sum_{j=1}^n \delta_{\phi_j} \in B$;
- the point measure $\Phi - \sum_{j=1}^n \delta_{\phi_j}$ has no S-subextremal atom in A, i.e. it belongs to

$$D = \{N \in M_p(\mathbb{C}_0);\ N_S^-(A) = 0\}.$$

Then the n-tuple $(\phi_1, \dots, \phi_n)$ is unique up to permutation of the coordinates. The above observations entail that for all measurable $C \subset M_p(\mathbb{C}_0)$,

$$\begin{aligned}
&\mathbb{P}[\Phi_S^+ \in C, \Phi_S^- \in B] \\
&= \frac{1}{n!}\mathbb{E}\Bigg[\int_{\mathbb{C}_0^n} 1_{\{\Phi_S^+ \in C\}} 1_{\{\forall i \in [\![1,n]\!],\ \phi_i <_S \eta\}} 1_{\left\{\sum_{i=1}^n \delta_{\phi_i} \in B\right\}} 1_{\left\{\Phi - \sum_{i=1}^n \delta_{\phi_i} \in D\right\}} \Phi(d\phi_1) \\
&\quad \times (\Phi - \delta_{\phi_1})(d\phi_2) \cdots \left(\Phi - \sum_{i=1}^{n-1} \delta_{\phi_i}\right)(d\phi_n)\Bigg].
\end{aligned}$$

The Campbell–Slyvniak formula entails

$$\begin{aligned}
&\mathbb{P}[\Phi_S^+ \in C, \Phi_S^- \in B] \\
&= \mathbb{E}\left[1_{\{\Phi_S^+ \in C\}} 1_{\{\Phi_S^-(A) = 0\}} \frac{1}{n!} \int_{\mathbb{C}_0^n} 1_{\{\sum_{i=1}^n \delta_{f_i} \in B\}} \otimes_{i=1}^n \left(1_{\{f_i <_S \eta\}} \mu(df_i)\right)\right].
\end{aligned} \tag{15}$$

Summing this relation over the different values of $n_1, \dots, n_k \in \mathbb{N}$ and the related sets $B = \{N \in M_p(\mathbb{C}_0);\ N(A_1) = n_1, \dots, N(A_k) = n_k\}$, we obtain

$$\mathbb{P}[\Phi_S^+ \in C] = \mathbb{E}\Big[1_{\{\Phi_S^+ \in C\}} 1_{\{\Phi_S^-(A) = 0\}} \exp\big[\mu(\{f \in A;\ f <_S \eta\})\big]\Big].$$

So we can rewrite Eq. (15) as

$$\mathbb{P}[\Phi_S^+ \in C, \Phi_S^- \in B] = \mathbb{E}\Big[1_{\{\Phi_S^+ \in C\}} 1_{\{\Phi_S^-(A) = 0\}} \exp\big[\mu(\{f \in A;\ f <_S \eta\})\big] K(\eta_S, B)\Big],$$

where

$$K(\eta_S, B) = \frac{\exp\big[-\mu(\{f \in A;\ f <_S \eta\})\big]}{n!} \int_{A^n} 1_{\left\{\sum_{i=1}^n \delta_{f_i} \in B\right\}} \otimes_{i=1}^n \left(1_{\{f_i <_S \eta\}} \mu(df_i)\right)$$

is the conditional probability of $\{\Phi_S^- \in B\}$ with respect to Φ_S^+ (note it depends on Φ_S^+ only through the restriction η_S). We recognize the distribution of a Poisson random measure with intensity $1_{\{f <_S \eta\}} \mu(df)$ and this proves Lemma 3.2. □

We now construct the coupling providing the upper bound (12).

Proposition 3.1. *Let* $(\widetilde{\Phi}, \widetilde{\eta})$ *be an independent copy of* (Φ, η) *and define*

$$\widehat{\Phi} = \Phi_{S_1}^+ \cup \{\widetilde{\phi} \in \widetilde{\Phi};\ \widetilde{\phi} <_{S_1} \eta\}. \tag{16}$$

The following properties hold true:

- $\widehat{\Phi}$ *has the same distribution as* Φ *and* $\widetilde{\Phi}$ *and satisfies*

$$\widehat{\Phi}_{S_1}^+ = \Phi_{S_1}^+, \qquad \widehat{\Phi}_{S_1}^- = \{\widetilde{\phi} \in \widetilde{\Phi};\ \widetilde{\phi} <_{S_1} \eta\}; \tag{17}$$

- $(\widehat{\Phi}_{S_1}^+, \widehat{\Phi}_{S_2}^+)$ *and* $(\Phi_{S_1}^+, \widetilde{\Phi}_{S_2}^+)$ *is a coupling between* $P_{(\Phi_{S_1}^+, \Phi_{S_2}^+)}$ *and* $P_{\Phi_{S_1}^+} \otimes P_{\Phi_{S_2}^+}$ *such that*

$$\mathbb{P}\Big[(\widehat{\Phi}_{S_1}^+, \widehat{\Phi}_{S_2}^+) \neq (\Phi_{S_1}^+, \widetilde{\Phi}_{S_2}^+)\Big] \leq 2\,\mathbb{P}[\Phi_{S_1}^+ \cap \Phi_{S_2}^+ \neq \emptyset]. \tag{18}$$

Proof of Proposition 3.1.

- Eq. (17) follows from the construction of $\widehat{\Phi}$: consider

$$\widehat{\eta}(t) = \bigvee_{\phi \in \widehat{\Phi}} \phi(t), \quad t \in T;$$

the maximum η is achieved on S_1 by the S_1-extremal points $\Phi_{S_1}^+$, and the definition (16) ensures that $\widehat{\eta}$ and η are equal on S_1 so that Eq. (17) holds.

Furthermore, conditionally on $\Phi_{S_1}^+$, the distribution of $\{\widetilde{\phi} \in \widetilde{\Phi};\ \widetilde{\phi} <_{S_1} \eta\}$ is equal to the distribution of a Poisson point process with intensity $1_{\{f <_{S_1} \eta\}}\mu(df)$. According to Lemma 3.2, this is the conditional distribution of $\Phi_{S_1}^-$ given $\Phi_{S_1}^+$, whence $(\widehat{\Phi}_{S_1}^+, \widehat{\Phi}_{S_1}^-)$ has the same distribution as $(\Phi_{S_1}^+, \Phi_{S_1}^-)$. We deduce that $\Phi = \Phi_{S_1}^+ \cup \Phi_{S_1}^-$ and $\widehat{\Phi} = \widehat{\Phi}_{S_1}^+ \cup \widehat{\Phi}_{S_1}^-$ have the same distribution.

- The coupling property is easily proved: since Φ and $\widehat{\Phi}$ have the same distribution, the law of $(\widehat{\Phi}_{S_1}^+, \widehat{\Phi}_{S_2}^+)$ is equal to $P_{(\Phi_{S_1}^+, \Phi_{S_2}^+)}$; since Φ and $\widetilde{\Phi}$ are independent, $(\Phi_{S_1}^+, \widetilde{\Phi}_{S_2}^+)$ has law $P_{\Phi_{S_1}^+} \otimes P_{\Phi_{S_2}^+}$.

We are left to prove Eq. (18). Since $\Phi_{S_1}^+ = \widehat{\Phi}_{S_1}^+$, we need to bound the probability $\mathbb{P}\big[\widehat{\Phi}_{S_2}^+ \neq \widetilde{\Phi}_{S_2}^+\big]$ from above. By construction, $\widehat{\Phi}$ is obtained from $\widetilde{\Phi}$ by removing the points $\widetilde{\phi} \in \widetilde{\Phi}$ such that $\widetilde{\phi} \not<_{S_1} \eta$ and adding the points $\phi \in \Phi_{S_1}^+$. Hence, it holds

$$\{\widehat{\Phi}_{S_2}^+ \neq \widetilde{\Phi}_{S_2}^+\} \subset \{\exists \phi \in \Phi_{S_1}^+, \phi \not<_{S_2} \widehat{\eta}\} \cup \{\exists \widetilde{\phi} \in \widetilde{\Phi}_{S_2}^+, \widetilde{\phi} \not<_{S_1} \eta\}.$$

Noting the equality of events

$$\{\exists \phi \in \Phi_{S_1}^+, \phi \not<_{S_2} \widehat{\eta}\} = \{\exists \phi \in \widehat{\Phi}_{S_1}^+ \cap \widehat{\Phi}_{S_2}^+\} = \{\widehat{\Phi}_{S_1}^+ \cap \widehat{\Phi}_{S_2}^+ \neq \emptyset\},$$

we obtain

$$\mathbb{P}[\widehat{\Phi}_{S_2}^+ \neq \widetilde{\Phi}_{S_2}^+] \leq \mathbb{P}[\widehat{\Phi}_{S_1}^+ \cap \widehat{\Phi}_{S_2}^+ \neq \emptyset] + \mathbb{P}[\exists \widetilde{\phi} \in \widetilde{\Phi}_{S_2}^+, \widetilde{\phi} \not<_{S_1} \eta].$$

Since $\widehat{\Phi}$ and Φ have the same law, we have

$$\mathbb{P}[\widehat{\Phi}_{S_1}^+ \cap \widehat{\Phi}_{S_2}^+ \neq \emptyset] = \mathbb{P}[\Phi_{S_1}^+ \cap \Phi_{S_2}^+ \neq \emptyset].$$

Hence, Eq. (18) follows from the upper bound

$$\mathbb{P}[\exists \widetilde{\phi} \in \widetilde{\Phi}_{S_2}^+, \widetilde{\phi} \not<_{S_1} \eta] \leq \mathbb{P}[\Phi_{S_1}^+ \cap \Phi_{S_2}^+ \neq \emptyset]$$

that we prove now. Using symmetry and exchanging the roles of Φ and $\widetilde{\Phi}$ on the one hand and the roles of S_1 and S_2 on the other hand, it is equivalent to prove that

$$\mathbb{P}[\exists\phi \in \Phi^+_{S_1}, \phi \not<_{S_2} \widetilde{\eta}] \leq \mathbb{P}[\Phi^+_{S_1} \cap \Phi^+_{S_2} \neq \emptyset].$$

We conclude the proof by noticing that the inclusion of events

$$\{\exists\phi \in \Phi^+_{S_1}, \phi \not<_{S_2} \widetilde{\eta}\} \subset \{\exists\phi \in \Phi^+_{S_1}, \phi \not<_{S_2} \widehat{\eta}\}$$

entails

$$\mathbb{P}[\exists\phi \in \Phi^+_{S_1}, \phi \not<_{S_2} \widetilde{\eta}] \leq \mathbb{P}[\exists\phi \in \Phi^+_{S_1}, \phi \not<_{S_2} \widehat{\eta}] = \mathbb{P}[\Phi^+_{S_1} \cap \Phi^+_{S_2} \neq \emptyset]. \quad \square$$

We now complete the proof of Theorem 3.1 by proving Eqs. (13) and (14).

Proof of Eq. (13). We observe that

$$\{\Phi^+_{S_1} \cap \Phi^+_{S_2} \neq \emptyset\} = \{\exists\phi \in \Phi, \phi \not<_{S_1} \eta, \phi \not<_{S_2} \eta\}$$

which entails

$$\mathbb{P}[\Phi^+_{S_1} \cap \Phi^+_{S_2} \neq \emptyset] \leq \mathbb{E}\left[\sum_{\phi\in\Phi} 1_{\{\phi\not<_{S_1} \eta, \phi\not<_{S_2} \eta\}}\right].$$

Noting that $\phi \not<_{S_i} \eta$ if and only if $\phi \not<_{S_i} \max(\Phi - \{\phi\})$, we apply the Campbell–Slyvniak formula (see Appendix A.2) and compute

$$\begin{aligned}
\mathbb{E}\left[\sum_{\phi\in\Phi} 1_{\{\phi\not<_{S_1} \eta, \phi\not<_{S_2} \eta\}}\right] &= \mathbb{E}\left[\sum_{\phi\in\Phi} 1_{\{\phi\not<_{S_1} \max(\Phi-\{\phi\}), \phi\not<_{S_2} \max(\Phi-\{\phi\})\}}\right] \\
&= \int_{\mathbb{C}_0} \mathbb{E}[1_{\{f\not<_{S_1} \max(\Phi), f\not<_{S_2} \max(\Phi)\}}]\mu(df) \\
&= \int_{\mathbb{C}_0} \mathbb{P}[f \not<_{S_1} \eta, f \not<_{S_2} \eta]\mu(df). \quad \square
\end{aligned}$$

Proof of Eq. (14). For any measurable subset $C \subset M_p(\mathbb{C}_0) \times M_p(\mathbb{C}_0)$,

$$\beta(\Phi^+_{S_1}, \Phi^+_{S_2}) \geq |\mathbb{P}[(\Phi^+_{S_1}, \Phi^+_{S_2}) \in C] - \mathbb{P}[(\Phi^+_{S_1}, \widetilde{\Phi}^+_{S_2}) \in C]|$$

where $\widetilde{\Phi}$ is an independent copy of Φ. We obtain the lower bound (14) by choosing the subset

$$C = \{(M_1, M_2) \in M_p(\mathbb{C}_0) \times M_p(\mathbb{C}_0);\ M_1 \cap M_2 \neq \emptyset\}.$$

This yields indeed

$$\beta(\Phi^+_{S_1}, \Phi^+_{S_2}) \geq |\mathbb{P}[\Phi^+_{S_1} \cap \Phi^+_{S_2} \neq \emptyset] - \mathbb{P}[\Phi^+_{S_1} \cap \widetilde{\Phi}^+_{S_2} \neq \emptyset]|$$

and in the case when Φ is a simple point process, i.e. when the intensity measure μ has no atom, we have

$$\mathbb{P}[\Phi^+_{S_1} \cap \widetilde{\Phi}^+_{S_2} \neq \emptyset] = \mathbb{P}[\Phi \cap \widetilde{\Phi} \neq \emptyset] = 0. \quad \square$$

3.3. Proof of Corollaries 2.1 and 2.2 and Theorem 2.2

Proof of Corollary 2.1. We have for all $f \in \mathbb{C}_0$,

$$\begin{aligned}\{f \not<_{S_1} \eta, f \not<_{S_2} \eta\} &= \{\exists (s_1, s_2) \in S_1 \times S_2,\, f(s_1) \geq \eta(s_1),\, f(s_2) \geq \eta(s_2)\} \\ &= \cup_{s_1 \in S_1} \cup_{s_2 \in S_2} \{\eta(s_1) \leq f(s_1), \eta(s_2) \leq f(s_2)\}\end{aligned}$$

whence, for S_1 and S_2 finite or countable,

$$\mathbb{P}[f \not<_{S_1} \eta, f \not<_{S_2} \eta] \leq \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \mathbb{P}[\eta(s_1) \leq f(s_1), \eta(s_2) \leq f(s_2)].$$

As a consequence, the integral in Theorem 2.1 satisfies

$$\begin{aligned}&\int_{\mathbb{C}_0} \mathbb{P}[f \not<_{S_1} \eta, f \not<_{S_2} \eta]\, \mu(df) \\ &\leq \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \int_{\mathbb{C}_0} \mathbb{P}[\eta(s_1) \leq f(s_1),\; \eta(s_2) \leq f(s_2)]\, \mu(df) \\ &= \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \int_{[0,+\infty)}^{2} \mathbb{P}[\eta(s_1) \leq y_1, \eta(s_2) \leq y_2]\, \mu_{s_1,s_2}(dy_1 dy_2).\end{aligned}$$

In the last line, we have used the fact that μ_{s_1,s_2} is the image of the measure μ under the mapping $f \mapsto (f(s_1), f(s_2))$. $\square$

Proof of Theorem 2.2. We recall that for a simple max-stable random field, the exponent measure μ is homogeneous of order -1, i.e. $\mu(cA) = c^{-1}\mu(A)$ for all $A \subset \mathbb{C}_0$ Borel set and $c > 0$. Also the assumption of standard unit Fréchet marginals implies

$$\bar{\mu}_t(y) = y^{-1}, \quad t \in T, y > 0.$$

These conditions imply (see [16, Proposition 3.2] or [11, Theorem 9.4.1 and Corollary 9.4.2]) that μ can be written as

$$\mu(A) = \int_0^\infty \int_{\mathbb{C}_0} 1_{\{rf \in A\}} r^{-2} dr \sigma(df)$$

where σ is a probability measure on $\mathbb{C}_0$ such that

$$\int_{\mathbb{C}_0} f(t)\, \sigma(df) = 1 \quad \text{for all } t \in T,$$

and

$$\int_{\mathbb{C}_0} \sup_{s \in S} f(s)\, \sigma(df) < \infty \quad \text{for all compact } S \subset T.$$

Using this, note that for all compact $S \subset T$ and $y > 0$,

$$\begin{aligned}\mathbb{P}[\sup_{s \in S} f(s) \leq y] &= \exp[-\mu(\{f \in \mathbb{C}_0;\; \sup_{s \in S} f(s) > y\})] \\ &= \exp\left[-\int_{\mathbb{C}_0} 1_{\{\sup_{s \in S} rf(s) > y\}} r^{-2} dr \sigma(df)\right] \\ &= \exp\left[-y^{-1} \int_{\mathbb{C}_0} \sup_{s \in S} f(s)\, \sigma(df)\right].\end{aligned}$$

It follows that the extremal coefficient $\theta(S)$ defined by (4) is equal to

$$\theta(S) = \int_{\mathbb{C}_0} \sup_{s\in S} f(s)\,\sigma(df). \tag{19}$$

We now consider $C(S)$ defined by Eq. (5). Since $S \subset T$ compact,

$$C(S) = \mathbb{E}\left[\left(\inf_{s\in S} \eta(s)\right)^{-1}\right],$$

we need to provide a lower bound for $\inf_{s\in S}\eta(s)$. To this aim, we remark that

$$\inf_{s\in S}\eta(s) = \inf_{s\in S}\max_{\phi\in\Phi}\phi(s) \geq \max_{\phi\in\Phi}\inf_{s\in S}\phi(s).$$

The right hand side is a random variable with unit Fréchet distribution since

$$\begin{aligned}\mathbb{P}[\max_{\phi\in\Phi}\inf_{s\in S}\phi(s) \leq y] &= \mathbb{P}[\forall\phi \in \Phi, \inf_{s\in S}\phi(s) \leq y]\\ &= \exp(-\mu(\{f \in \mathbb{C}_0;\ \inf_{s\in S} f(s) > y\}))\end{aligned}$$

and

$$\mu(\{f \in \mathbb{C}_0;\ \inf_{s\in S} f(s) > y\}) = y^{-1}\int_{\mathbb{C}_0}\inf_{s\in S} f(s)\,\sigma(df).$$

Hence, if $\int_{\mathbb{C}_0}\inf_{s\in S} f(s)\,\sigma(df) > 0$, we obtain

$$C(S) = \mathbb{E}\left[\left(\inf_{s\in S}\eta(s)\right)^{-1}\right] \leq \mathbb{E}\Big[(\max_{\phi\in\Phi}\inf_{s\in S}\phi(s))^{-1}\Big] = \left(\int_{\mathbb{C}_0}\inf_{s\in S} f(s)\,\sigma(df)\right)^{-1} < \infty.$$

For arbitrary compact $S \subset T$, we may however have $\int_{\mathbb{C}_0}\inf_{s\in S} f(s)\,\sigma(df) = 0$. But, if $S = B(s_0, \varepsilon)$ is a closed ball with center s_0 and radius ε, the monotone convergence theorem implies

$$\int_{\mathbb{C}_0}\inf_{s\in B(s_0,\varepsilon)} f(s)\,\sigma(df) \to \int_{\mathbb{C}_0} f(s_0)\,\sigma(df) = 1 \quad \text{as } \varepsilon \to 0,$$

so that $\int_{\mathbb{C}_0}\inf_{s\in B(s_0,\varepsilon_0)} f(s)\,\sigma(df) > 0$ and $C(B(s_0,\varepsilon_0)) < \infty$ for ε_0 small enough. The result for general S follows by a compacity argument: there exist $s_1,\ldots,s_k$ and $\varepsilon_1,\ldots,\varepsilon_k$ such that $S \subset \cup_{i=1}^k B(s_i,\varepsilon_i)$. Hence,

$$\sup_{s\in S}\eta(s)^{-1} \leq \max_{1\leq i\leq k}\sup_{s\in B(s_i,\varepsilon_i)}\eta(s)^{-1} \leq \sum_{i=1}^k \sup_{s\in B(s_i,\varepsilon_i)}\eta(s)^{-1}$$

and

$$C(S) = \mathbb{E}\Big[\sup_{s\in S}\eta(s)^{-1}\Big] \leq \sum_{i=1}^k \mathbb{E}\Big[\sup_{s\in B(s_i,\varepsilon_i)}\eta(s)^{-1}\Big] = \sum_{i=1}^k C(B(s_i,\varepsilon_i)) < \infty.$$

This proves the fact that $C(S)$ is finite.

- The upper bound for $\beta(S_1, S_2)$ given by Theorem 2.1 can be expressed as

$$\begin{aligned}\beta(S_1, S_2) &\leq 2\int_{\mathbb{C}_0} \mathbb{P}[f \not<_{S_1} \eta, f \not<_{S_2} \eta]\,\mu(df)\\ &= 2\int_{\mathbb{C}_0}\int_0^\infty \mathbb{P}[r\,f \not<_{S_1} \eta, r\,f \not<_{S_2} \eta]\,r^{-2}dr\sigma(df)\\ &= 2\int_{\mathbb{C}_0}\int_0^\infty \mathbb{E}\Big[1_{\{r\geq \inf_{s_1\in S_1}\eta(s_1)/f(s_1),\, r\geq \inf_{s_2\in S_2}\eta(s_2)/f(s_2)\}}\Big]\,r^{-2}dr\sigma(df)\\ &= 2\int_{\mathbb{C}_0}\mathbb{E}\left[\max\left(\inf_{s_1\in S_1}\frac{\eta(s_1)}{f(s_1)},\ \inf_{s_2\in S_2}\frac{\eta(s_2)}{f(s_2)}\right)^{-1}\right]\sigma(df). \qquad (20)\end{aligned}$$

We then introduce the upper bound

$$\begin{aligned}&\max\left(\inf_{s_1\in S_1}\frac{\eta(s_1)}{f(s_1)}, \inf_{s_2\in S_2}\frac{\eta(s_2)}{f(s_2)}\right)^{-1}\\ &\quad\leq \max(\sup_{s_1\in S_1}\eta(s_1)^{-1}, \sup_{s_2\in S_2}\eta(s_2)^{-1})\ \min(\sup_{s_1\in S_1} f(s_1), \sup_{s_2\in S_2} f(s_2))\end{aligned}$$

whence we deduce

$$\begin{aligned}\beta(S_1, S_2) &= 2\mathbb{E}\Big[\max\big(\sup_{s_1\in S_1}\eta(s_1)^{-1}, \sup_{s_2\in S_2}\eta(s_2)^{-1}\big)\Big]\\ &\qquad\times\int_{\mathbb{C}_0}\min(\sup_{s_1\in S_1} f(s_1), \sup_{s_2\in S_2} f(s_2))\,\sigma(df)\\ &\leq 2\mathbb{E}\Big[\sup_{s_1\in S_1}\eta(s_1)^{-1} + \sup_{s_2\in S_2}\eta(s_2)^{-1}\Big]\\ &\qquad\times\int_{\mathbb{C}_0}\min(\sup_{s_1\in S_1} f(s_1), \sup_{s_2\in S_2} f(s_2))\,\sigma(df)\\ &= 2\big[C(S_1)+C(S_2)\big]\,\big[\theta(S_1)+\theta(S_2)-\theta(S_1\cup S_2)\big].\end{aligned}$$

In the last equality, we use Eq. (5) defining $C(S)$ and Eq. (19) defining $\theta(S)$ together with the following simple equality

$$\begin{aligned}&\min(\sup_{s_1\in S_1} f(s_1), \sup_{s_2\in S_2} f(s_2)) + \max(\sup_{s_1\in S_1} f(s_1), \sup_{s_2\in S_2} f(s_2))\\ &\quad= \sup_{s_1\in S_1} f(s_1) + \sup_{s_2\in S_2} f(s_2).\end{aligned}$$

- The second point is straightforward since

$$\begin{aligned}\mathbb{P}[f \not<_{S_1} \eta, f \not<_{S_2} \eta] &= \mathbb{P}[\exists i\in I,\ f \not<_{S_{1,i}} \eta, \exists j\in J, f \not<_{S_{2,j}} \eta]\\ &\leq \sum_{i\in I}\sum_{j\in J}\mathbb{P}[f \not<_{S_{1,i}} \eta, f \not<_{S_{2,j}} \eta]\end{aligned}$$

so that

$$\begin{aligned}\beta(S_1, S_2) &\leq \sum_{i\in I}\sum_{j\in J}\int_{\mathbb{C}_0}\mathbb{P}[f \not<_{S_{1,i}} \eta, f \not<_{S_{2,j}} \eta]\,\mu(df)\\ &\leq 2\sum_{i\in I}\sum_{j\in J}\big[C(S_{1,i})+C(S_{2,j})\big]\,\big[\theta(S_{1,i})+\theta(S_{2,j})-\theta(S_{1,i}\cup S_{2,j})\big]. \quad \square\end{aligned}$$

Proof of Corollary 2.2. This is a straightforward consequence of the second point of Theorem 2.2 with $S_1 = \cup_{s_1 \in S_1}\{s_1\}$ and $S_2 = \cup_{s_2 \in S_2}\{s_2\}$. It holds indeed

$$\beta(S_1, S_2) \leq 2 \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \left[C(\{s_1\}) + C(\{s_2\})\right] \left[\theta(\{s_1\}) + \theta(\{s_2\}) - \theta(\{s_1\} \cup \{s_2\})\right]$$

with

$$\theta(\{s_1\}) = \theta(\{s_2\}) = 1, \qquad \theta(\{s_1\} \cup \{s_2\}) = \theta(s_1, s_2)$$

and

$$C(\{s_1\}) = C(\{s_2\}) = 1.$$

The last equality follows from the fact that, for all $s \in S$, $\eta(s)$ has a standard unit Fréchet distribution and hence $\eta(s)^{-1}$ has an exponential distribution with mean 1. Hence we obtain

$$\beta(S_1, S_2) \leq 4 \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} (2 - \theta(s_1, s_2)). \quad \square$$

3.4. *Proof of Theorem 2.3 and Proposition 2.1*

Proof of Theorem 2.3. According to Bolthausen's CLT for stationary mixing random fields (see Appendix A.3), it is enough to prove that the mixing coefficients $\alpha_{k,l}(m)$ defined by Eq. (21) with $X(t) = g(\eta(t_1 + t), \ldots, \eta(t_p + t))$ satisfy Eqs. (22)–(24).

For $S \subset \mathbb{Z}^d$, we define $\widetilde{S} = \cup_{i=1}^{p}\{s + t_i,\ s \in S\}$. The inclusion of σ-fields

$$\sigma(\{X(s), s \in S\}) \subset \sigma(\{\eta(s), s \in \widetilde{S}\})$$

entails a comparison of the related α-mixing coefficients: for disjoint $S_1, S_2 \subset \mathbb{Z}$,

$$\alpha^X(S_1, S_2) \leq \alpha^\eta(\widetilde{S}_1, \widetilde{S}_2),$$

where the superscript X or η denotes that we are computing the α-mixing coefficient of the random field X or η respectively. Furthermore,

$$|\widetilde{S}_i| \leq p|S_i|, \quad i = 1, 2 \quad \text{and} \quad d(\widetilde{S}_1, \widetilde{S}_2) \geq d(S_1, S_2) - \Delta,$$

with $\Delta = \max_{1 \leq i < j \leq p} d(t_i, t_j)$ the diameter of $\{t_1, \ldots, t_p\}$. Hence, with obvious notations,

$$\alpha^X_{k,l}(m) \leq \alpha^\eta_{pk,pl}(m - \Delta), \quad k, l \in \mathbb{N} \cup \{\infty\}, m \geq \Delta + 1.$$

Similar to the proof of Corollary 2.2, we get

$$\alpha^X_{k,l}(m) \leq \alpha^\eta_{pk,pl}(m - \Delta) \leq p^2 kl \sup_{|t| \geq m - \Delta} \gamma(t), \quad k, l \in \mathbb{N}, m \geq \Delta + 1,$$

and

$$\alpha^X_{1,\infty}(m) \leq \alpha^\eta_{p,\infty}(m - \Delta) \leq p \sum_{|t| \geq m - \Delta} \gamma(t), \quad m \geq \Delta + 1.$$

In view of this, Assumption (6) entails Eqs. (22)–(24), so that the random field X satisfies Bolthausen's CLT. $\square$

Proof of Proposition 2.1. Let $h \in \mathbb{Z}^d$. We apply Theorem 2.3 to the stationary random field

$$X(t) = 1_{\{\eta(t)\le y, \eta(t+h)\le y\}}, \quad t \in \mathbb{Z}^d.$$

Clearly $\mathbb{E}[X(t)] = \exp[-\theta(h)/y]$ and $\mathbb{E}[|X|^{2+\delta}] < \infty$ for all $\delta > 0$. Assumption (9) together with $\gamma(t) \le 4(2-\theta(t))$ ensures that Eq. (6) is satisfied for δ large enough. Hence the estimator $\hat{p}_n(h,y)$ is asymptotically normal:

$$|\Lambda_n|^{1/2}\Big(\hat{p}_n(h,y) - p(h,y)\Big) \Longrightarrow \mathcal{N}(0,\beta_1^2)$$

with limit variance

$$\beta_1^2(y) = \sum_{t\in\mathbb{Z}^d} \mathrm{Cov}[X(0), X(t)]$$
$$= \sum_{t\in\mathbb{Z}^d} \Big(\exp(\theta(\{0,h,t,t+h\})/y) - \exp(2\theta(h))/y\Big) > 0.$$

The δ-method entails the asymptotic normality of the estimator $\hat{\theta}_n^y(h) = -y\log \hat{p}_n(h,y)$:

$$|\Lambda_n|^{1/2}\Big(\hat{\theta}_n^y(h) - \theta(h)\Big) \Longrightarrow \mathcal{N}(0,\sigma_1^2)$$

with limit variance

$$\sigma_1^2 = y^2\exp(2\theta(h)/y)\beta_1^2 = y^2\sum_{t\in\mathbb{Z}^d}\Big(\exp[(2\theta(h) - \theta(\{0,h,t,t+h\}))/y] - 1\Big).$$

The proof of the asymptotic normality of $\hat{\theta}_n^{(2)}$ and $\hat{\theta}_n^{(3)}$ is very similar and we give only the main lines. Using Theorem 2.3, we prove that

$$\hat{q}_n = |\Lambda_n|^{-1}\sum_{t\in|\Lambda_n|}\min(\eta(t)^{-1}, \eta(t+h)^{-1})$$

is an asymptotic normal estimator of $\theta(h)^{-1}$:

$$|\Lambda_n|^{1/2}(\hat{q}_n - \hat{\theta}(h)^{-1}) \Longrightarrow \mathcal{N}(0,\beta_2^2)$$

with limit variance

$$\beta_2^2 = \sum_{t\in\mathbb{Z}^d}\mathrm{Cov}\big[\min(\eta(0)^{-1},\eta(h)^{-1}), \min(\eta(t)^{-1},\eta(t+h)^{-1})\big].$$

The δ-method entails the asymptotic normality of $\theta_n^{(2)}(h) = 1/\hat{q}_n(h)$ with limit variance

$$\sigma_2^2 = \theta(h)^4\beta_1^2.$$

Similarly,

$$\hat{\nu}_{F,n}(h) = |\Lambda_n|^{-1}\sum_{t\in|\Lambda_n|}|F(\eta(t)) - F(\eta(t+h))|$$

is an asymptotic normal estimator of $\nu_F(h) = \mathbb{E}[|F(\eta(0)) - F(\eta(h))|]$:

$$|\Lambda_n|^{1/2}(\hat{\nu}_{F,n}(h) - \nu_F(h)) \Longrightarrow \mathcal{N}(0,\beta_3^2)$$

with limit variance

$$\beta_3^2 = \sum_{t\in\mathbb{Z}^d}\mathrm{Cov}\big[|F(\eta(0)) - F(\eta(h))|, |F(\eta(t)) - F(\eta(t+h))|\big].$$

The δ-method entails the asymptotic normality of

$$\theta_n^{(3)}(h) = \frac{1 + 2\hat{\nu}_{F,n}(h)}{1 - 2\hat{\nu}_{F,n}(h)}$$

with limit variance

$$\sigma_3^2 = (\theta(h) + 1)^4 \beta_3^2. \quad \square$$

Appendix. Auxiliary results

A.1. Structure of max-i.d. random processes

The structure of sample continuous random processes on a compact metric space was elucidated by Giné et al. [16]. Further results by Vatan [27] cover the discrete case $T = \mathbb{Z}^d$. We give here similar results when T is a locally compact metric space, typically $T = \mathbb{R}^d$. Such extensions have been considered for max-stable models on $\mathbb{R}$ (see [11, Chapter 9.6]) but we have found no reference in the max-i.d. case.

Let η be a continuous max-i.d. random process on $\mathbb{C}(T, \mathbb{R})$. Define its vertex function $h : T \to [-\infty, +\infty)$ by

$$h(t) = \operatorname{essinf} \eta(t) = \sup\{x \in \mathbb{R};\ \mathbb{P}(\eta(t) \geq x) = 1\}.$$

We will always assume that h is continuous. We can then suppose without loss of generality that $h \equiv 0$. Indeed, if h is continuous and finite, we may consider $\eta - h$ which is a continuous max-i.d. random field with zero vertex function; and if h is not finite everywhere, we may consider $\exp(\eta) - \exp(h)$ which is max-i.d. with zero vertex function.

We denote by $\mathbb{C}(T) = \mathbb{C}(T, [0, +\infty))$ the space of nonnegative continuous functions on T endowed with the topology of uniform convergence on compact sets and set $\mathbb{C}_0(T) = \mathbb{C}(T) \setminus \{0\}$.

Theorem A.1. • *Let $\eta = (\eta(t))_{t \in T}$ be a continuous max-i.d. process on T with vertex function $h \equiv 0$. There exists a unique locally finite Borel measure on $\mathbb{C}_0$ satisfying condition (1), called the exponent measure of η, such that*

$$\log \mathbb{P}\Big[\cap_{i=1}^k \{\eta(t_i) \leq y_i\}\Big] = -\mu\Big[\cup_{i=1}^k \{f \in \mathbb{C}_0;\ f(t_i) > y_i\}\Big]$$

for all $k \geq 1, t_1, \ldots, t_k \in T$ and $y_1, \ldots, y_k > 0$.

• *Conversely, for any locally finite Borel measure on $\mathbb{C}_0$ satisfying condition (1), there exists a continuous max-i.d. process η on T with vertex function $h \equiv 0$ and exponent measure μ. It can be constructed as follows: let Φ be a Poisson point process on $\mathbb{C}_0$ with intensity μ and define*

$$\eta(t) = \max\{\phi(t), \phi \in \Phi\}, \quad t \in T.$$

Proof of Theorem A.1. Let $(T_n)_{n \geq 1}$ be an increasing sequence of compact sets such that $T = \cup_{n \geq 1} T_n$. We suppose furthermore that T_n is included in the interior set of T_{n+1}. The space $\mathbb{C}(T) = \mathbb{C}(T, \mathbb{R}^+)$ of nonnegative continuous functions on T endowed with the topology of uniform convergence on compact sets can be seen as the projective limit of the sequence of spaces $\mathbb{C}(T_n, \mathbb{R}^+)$ endowed with the topology of uniform convergence. For $m \geq n \geq 1$, we define the natural projections

$$\pi_n : \mathbb{C}(T) \to \mathbb{C}(T_n) \quad \text{and} \quad \pi_{n,m} : \mathbb{C}(T_m) \to \mathbb{C}(T_n).$$

For each $n \geq 1$, the restriction $\pi_n(\eta) = \eta_{T_n}$ is a continuous max-i.d. process on the compact space T_n and according to [16], there exists a locally finite exponent measure μ_n on $\mathbb{C}_0(T_n) = \mathbb{C}(T_n) \setminus \{0\}$ satisfying equation

$$\log \mathbb{P}\Big[\cap_{i=1}^k \{\eta(t_i) \leq y_i\}\Big] = -\mu_n\Big[\cup_{i=1}^k \{f \in \mathbb{C}_0;\ f(t_i) > y_i\}\Big]$$

for all $k \geq 1, t_1, \ldots, t_k \in T_n$ and $y_1, \ldots, y_k > 0$. Furthermore, for all $\varepsilon > 0$

$$\mu_n[\mathcal{S}_{n,\varepsilon}] < \infty \quad \text{where } \mathcal{S}_{n,\varepsilon} = \Big\{f \in \mathbb{C}(T_n);\ \sup_{T_n} f > \varepsilon\Big\}.$$

Let $n_0 \geq 1$ and $\varepsilon > 0$ be fixed. For $n \geq n_0$, define the finite Borel measure by

$$\tilde{\mu}_n^{n_0,\varepsilon}[A] = \mu_n[A \cap \pi_{n_0,n}^{-1}\mathcal{S}_{n_0,\varepsilon}], \quad A \subset \mathbb{C}(T_n) \text{ Borel set.}$$

Clearly, the following compatibility conditions hold true: for $m \geq n \geq n_0$,

$$\tilde{\mu}_n^{n_0,\varepsilon} = \tilde{\mu}_m^{n_0,\varepsilon}\pi_{n,m}^{-1}.$$

Note that since $\mathbb{C}_0(T)$ is a Polish space, every locally finite measure is inner regular and hence a Radon measure. Theorem 5.1.1 in [1] states the existence of projective limits of Radon measures, and it implies the existence of a finite Radon measure $\tilde{\mu}^{n_0,\varepsilon}$ on $\mathbb{C}(T)$ such that

$$\tilde{\mu}_n^{n_0,\varepsilon} = \tilde{\mu}^{n_0,\varepsilon}\pi_n^{-1}, \quad n \geq n_0.$$

It is then easily checked that the measure μ on $\mathbb{C}_0(T)$ defined by

$$\mu[A] = \sup\{\tilde{\mu}^{n_0,\varepsilon}[A];\ n_0 \geq 1, \varepsilon > 0\}, \quad A \subset \mathbb{C}_0(T) \text{ Borel set}$$

is locally finite and enjoys the required properties. $\square$

A.2. *Slyvniak's formula*

The Palm theory deals with conditional distribution for point processes. We recall here one of the most famous formula of the Palm theory, known as Slyvniak's theorem. This will be the main tool in our computations. For a general reference on Poisson point processes, the Palm theory and their applications, the reader is invited to refer to the monograph [26] by Stoyan, Kendall and Mecke.

The following formula is obtained thanks to Campbell's theorem and Slyvniak's theorem together, and is sometimes referred to as the Campbell–Slyvniak formula. For our purpose, we state it for $\mathbb{C}_0$ valued point processes. Let $M_p(\mathbb{C}_0)$ be the set of locally finite point measures N on $\mathbb{C}_0$ endowed with the σ-algebra generated by the family of mappings $\{N \mapsto N(A), A \subset \mathbb{C}_0 \text{ Borelset}\}$.

Theorem A.2 (*Campbell–Slyvniak Formula*)**.** *Let Φ be a Poisson point process on $\mathbb{C}_0$ with intensity measure μ. For all measurable functions $F : \mathbb{C}_0^k \times M_p(\mathbb{C}_0) \to [0, +\infty)$,*

$$\mathbb{E}\left[\int_{\mathbb{C}_0^k} F\left(\phi_1, \ldots, \phi_k, \Phi - \sum_{i=1}^k \delta_{\phi_i}\right) \Phi(d\phi_1)\,(\Phi - \delta_{\phi_1})(d\phi_2) \cdots \left(\Phi - \sum_{j=1}^{k-1} \delta_{\phi_j}\right)(d\phi_k)\right]$$
$$= \int_{\mathbb{C}_0^k} \mathbb{E}[F(f_1, \ldots, f_k, \Phi)]\,\mu^{\otimes k}(df_1, \ldots, df_k).$$

A.3. A central limit theorem for weakly dependent processes

Since the pioneer work of Ibragimov [17], many versions of the central limit theorem for weakly dependent processes have been developed under various strong mixing conditions. We present here a central limit theorem for stationary mixing random fields due to Bolthausen [2]. Let $(X_k)_{k\in\mathbb{Z}^d}$ be a real valued stationary random field and recall the definition of the α-mixing coefficient (21). If $\Lambda \subset \mathbb{Z}^d$, we denote by $|\Lambda|$ the number of elements in Λ and by $\partial\Lambda$ the set of elements $k \in \Lambda$ such that there is $l \notin \Lambda$ with $d(k,l) = 1$. Let Λ_n be a fixed increasing sequence of finite subsets of $\mathbb{Z}^d$, which increases to $\mathbb{Z}^d$ and such that $\lim_{n\to\infty} |\partial\Lambda_n|/|\Lambda_n| = 0$. Let $\Sigma_n = \sum_{h\in\Lambda_n}(X_h - \mathbb{E}[X_h])$.

For subsets $S_1, S_2 \subset \mathbb{Z}^d$, we define

$$d(S_1,S_2) = \min\{|s_2 - s_1|;\ s_1 \in S_1, s_2 \in S_2\}.$$

Bolthausen's central limit theorem is based on the mixing coefficients

$$\alpha_{kl}(m) = \sup\Big\{\alpha(S_1,S_2);\ |S_1| = k, |S_2| = l, d(S_1,S_2) \geq m\Big\} \tag{21}$$

defined for $m \geq 1$ and $k,l \in \mathbb{N}\cup\{\infty\}$.

Theorem A.3. *Suppose that the following three conditions are satisfied:*

$$\alpha_{1\infty}(m) = o(m^{-d}); \tag{22}$$

$$\sum_{m=1}^{\infty} m^{d-1}\alpha_{kl}(m) < \infty \quad \textit{for all } k \geq 1, l \geq 1 \textit{ such that } k+l \leq 4; \tag{23}$$

$$\mathbb{E}\big[|X_h|^{2+\delta}\big] < \infty \quad \textit{and} \quad \sum_{m=1}^{\infty} m^{d-1}|\alpha_{11}(m)|^{\delta/(2+\delta)} < \infty \quad \textit{for some } \delta > 0. \tag{24}$$

Then the series $\sigma^2 = \sum_{h\in\mathbb{Z}^d}\mathrm{Cov}[X_0,X_h]$ converges absolutely and if furthermore $\sigma^2 > 0$,

$$\frac{\Sigma_n}{\sigma|\Lambda_n|^{1/2}} \Longrightarrow \mathcal{N}(0,1), \quad \textit{as } n \to \infty.$$

References

[1] S. Bochner, Harmonic Analysis and the Theory of Probability, University of California Press, Berkeley, Los Angeles, 1955.

[2] E. Bolthausen, On the central limit theorem for stationary mixing random fields, Ann. Probab. 10 (4) (1982) 1047–1050.

[3] R.C. Bradley, Basic properties of strong mixing conditions. A survey and some open questions, Probab. Surv. 2 (2005) 107–144 (electronic), Update of, and a supplement to, the 1986 original.

[4] R.C. Bradley, Introduction to Strong Mixing Conditions Vol. 1, Kendrick Press, Heber City, UT, 2007.

[5] R.C. Bradley, Introduction to Strong Mixing Conditions Vol. 2, Kendrick Press, Heber City, UT, 2007.

[6] R.C. Bradley, Introduction to Strong Mixing Conditions Vol. 3, Kendrick Press, Heber City, UT, 2007.

[7] D. Cooley, P. Naveau, P. Poncet, Variograms for spatial max-stable random fields, in: Dependence in Probability and Statistics, in: Lecture Notes in Statist., vol. 187, Springer, New York, 2006, pp. 373–390.

[8] J. Dedecker, P. Doukhan, G. Lang, R. José Rafael León, S. Louhichi, C. Prieur, Weak Dependence: With Examples and Applications, in: Lecture Notes in Statistics, vol. 190, Springer, New York, 2007.

[9] L. de Haan, A characterization of multidimensional extreme-value distributions, Sankhyā Ser. A 40 (1) (1978) 85–88.

[10] L. de Haan, A spectral representation for max-stable processes, Ann. Probab. 12 (4) (1984) 1194–1204.
[11] L. de Haan, A. Ferreira, Extreme value theory, in: Springer Series in Operations Research and Financial Engineering, Springer, New York, 2006, An Introduction.
[12] L. de Haan, T.T. Pereira, Spatial extremes: models for the stationary case, Ann. Statist. 34 (1) (2006) 146–168.
[13] L. de Haan, J. Pickands III, Stationary min-stable stochastic processes, Probab. Theory Related Fields 72 (4) (1986) 477–492.
[14] C. Dombry, F. Eyi-Minko, Regular conditional distributions of max-infinitely divisible processes, 2011, Preprint Hal-00627375. Available at: http://hal.archives-ouvertes.fr/hal-00627375/fr/.
[15] P. Doukhan, Mixing: Properties and Examples, in: Lecture Notes in Statistics, vol. 85, Springer-Verlag, New York, 1994.
[16] E. Giné, M.G. Hahn, P. Vatan, Max-infinitely divisible and max-stable sample continuous processes, Probab. Theory Related Fields 87 (2) (1990) 139–165.
[17] I.A. Ibragimov, Some limit theorems for stationary processes, Teor. Veroyatn. Primen. 7 (1962) 361–392.
[18] Z. Kabluchko, M. Schlather, Ergodic properties of max-infinitely divisible processes, Stochastic Process. Appl. 120 (3) (2010) 281–295.
[19] Z. Kabluchko, M. Schlather, L. de Haan, Stationary max-stable fields associated to negative definite functions, Ann. Probab. 37 (5) (2009) 2042–2065.
[20] S.I. Resnick, Extreme Values, Regular Variation and Point Processes, in: Springer Series in Operations Research and Financial Engineering, Springer, New York, 2008, Reprint of the 1987 original.
[21] S.I. Resnick, R. Roy, Random USC functions, max-stable processes and continuous choice, Ann. Appl. Probab. 1 (2) (1991) 267–292.
[22] E. Rio, Théorie asymptotique des processus aléatoires faiblement dépendants, in: Mathématiques & Applications (Berlin), vol. 31, Springer-Verlag, Berlin, 2000.
[23] M. Rosenblatt, A central limit theorem and a strong mixing condition, Proc. Natl. Acad. Sci. USA 42 (1956) 43–47.
[24] R. Smith, Max-stable processes and spatial extremes, Unpublished Manuscript, 1990.
[25] S.A. Stoev, Max-stable processes: representations, ergodic properties and statistical applications, in: Dependence in Probability and Statistics, in: Lecture Notes in Statist., vol. 200, Springer, Berlin, 2010, pp. 21–42.
[26] D. Stoyan, W.S. Kendall, J. Mecke, Stochastic geometry and its applications, in: Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics, John Wiley & Sons Ltd., Chichester, 1987, With a foreword by D.G. Kendall.
[27] P. Vatan, Max-infinite divisibility and max-stability in infinite dimensions, in: Probability in Banach Spaces, V, Medford, Mass., 1984, in: Lecture Notes in Math., vol. 1153, Springer, Berlin, 1985, pp. 400–425.
[28] V.A. Volkonskiĭ, Y.A. Rozanov, Some limit theorems for random functions. I, Theory Probab. Appl. 4 (1959) 178–197.
[29] K.S. Weintraub, Sample and ergodic properties of some min-stable processes, Ann. Probab. 19 (2) (1991) 706–723.