Subfunctions and Third Order Differential Inequalities

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1. INTRODUCTION

In what follows x, y, z, and w are variables in R, the real numbers, and f is a real valued function. We consider the equation

\[ y'' = f(x, y, y', y'') \]  \hspace{1cm} (1.1)

concerning which subsets of the following assumptions are made:

(A) \( f \) is a continuous function of \((x, y, z, w)\) for \((x, y, z, w) \in (a, b) \times R^3\).

(B) If \( y_1 \) and \( y_2 \) are solutions of (1.1) such that \( y_1(x_i) = y_2(x_i) \) for \( i = 0, 1 \) and 2 where \( x_0, x_1 \) and \( x_2 \) are any three points satisfying \( a < x_0 < x_1 < x_2 < b \), then \( y_1(x) \leq y_2(x) \) on \( x_0 \leq x \leq x_2 \).

(C) All solutions of all initial value problems for (1.1) extend throughout \((a, b)\).

DEFINITION 1.1. A real valued function \( \phi \) defined on an interval \( I \subseteq (a, b) \) is said to be a subfunction with respect to solutions of (1.1) in \( \phi(x) < \phi(x) \) on \([x_0, x_1]\) and \( \phi(x) \geq \phi(x) \) on \([x_1, x_2]\) for any \( x_0 < x_1 < x_2 \), \([x_0, x_2] \subseteq I\) and solution \( y \) of (1.1) with \( y(x_i) = \phi(x_i) \) for \( i = 0, 1, 2 \).

DEFINITION 1.2. A real valued function \( \phi \) defined on an interval \( I \subseteq (a, b) \) is said to be a subfunction in the small with respect to solutions of (1.1) in

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case for any compact interval \([c, d] \subset I\) there is a \(\delta > 0\) such that \(\phi\) is a subfunction with respect to solutions of (1.1) on every subinterval of \([c, d]\) of length less than or equal to \(\delta\).

Our primary concern is with the following question: If \(f\) satisfies conditions (A), (B), and (C) and if \(\phi \in C^0(I)\) and satisfies \(\phi''' \geq f(x, \phi, \phi', \phi'')\) on an interval \(I \subset (a, b)\), is it true that \(\phi\) is a subfunction on \(I\) with respect to solutions of (1.1)? In Section 3 of this paper we prove that, if \(\phi \in C^0(I)\) is a subfunction in the small on an interval \(I \subset (a, b)\), then \(\phi\) is a subfunction on \(I\), that is, \(\phi\) is a subfunction in the large on \(I\). Thus to obtain an affirmative answer to the above question it suffices to show that, if \(f\) satisfies (A), (B), and (C) and if \(\phi \in C^0(I)\) satisfies \(\phi''' \geq f(x, \phi, \phi', \phi'')\) on an interval \(I \subset (a, b)\), then \(\phi\) is a subfunction in the small on \(I\) with respect to solutions of (1.1). Whether or not this is the case remains an open question.

In Section 4 we prove that with the above assumptions \(\phi\) is a subfunction in the small on \(I \subset (a, b)\) if \(\phi''' \geq f(x, \phi, \phi', \phi'')\) on \(I\). This result is then used to obtain an affirmative answer to the above question in the case in which \(f(x, y, z, w)\) satisfies a Lipschitz condition on compact subsets with respect to the variables \(y, z, w\).

For a comprehensive survey of subfunctions and second order differential inequalities the reader should see [1]. Results similar to those in this paper but for second order equations are contained in [2], [3] and [4]. The first such treatment of third order equations seems to be in [5] where the hypotheses included uniqueness of solutions of initial value problems and existence of solutions of all three point boundary value problems as well as the conditions (A), (B) and (C). In addition, there would seem to be an error on page 213 of [5] which makes the proof invalid unless, in terms of our notation, \(\phi''' \geq f(x, \phi, \phi', \phi'')\).

2. Preliminary Results

**Lemma 2.1.** Assume \(f\) satisfies condition (A). Let \(\phi, \psi\) be of class \(C^2\) on \([x_0 - \eta, x_0 + \eta] \subset (a, b)\) with \(\phi(x_0) = \psi(x_0), \phi'(x_0) = \psi'(x_0)\) and \(\phi''(x_0) \leq \psi''(x_0)\). Then there is a \(\delta, 0 < \delta \leq \eta\), such that all solutions \(y\) of (1.1) with initial conditions \(y(x_0) = y_0 = \phi(x_0), y'(x_0) = y_1 = \phi'(x_0), \) and \(y''(x_0) = y_2 = \frac{1}{\delta}[\phi''(x_0) + \psi''(x_0)]\) exist on \([x_0 - \delta, x_0 + \delta]\) and satisfy \(\phi(x) < y(x) < \psi(x)\) for \(0 < |x - x_0| \leq \delta\).

**Proof.** Let \(8\rho = \phi''(x_0) - \psi''(x_0)\) and choose \(\delta_0, 0 < \delta_0 \leq \eta\) such that \(|\phi''(x) - \phi''(x_0)| \leq \rho\) and \(|\psi''(x) - \psi''(x_0)| \leq \rho\) for \(|x - x_0| \leq \delta_0\). Let \(M\) be a bound for \(f(x, y, y', y'')\) on the compact set
\[
K = \{(x, y, y', y''): |x - x_0| \leq \delta_0, |y - y_0| \leq 1, |y' - y_1| \leq 1, |y'' - y_2| \leq 1\}.
\]
It follows from the relations
\[ y(x) = y_0 + \int_{x_0}^{x} y'(t) \, dt \]
\[ y'(x) = y_1 + \int_{x_0}^{x} y''(t) \, dt \]
\[ y''(x) = y_2 + \int_{x_0}^{x} f(t, y(t), y'(t), y''(t)) \, dt \]

that all solutions of the stated initial value problem exist on the closed interval 
\([x_0 - \delta_1, x_0 + \delta_1]\), where
\[ \delta_1 = \min \left\{ \delta_0, \frac{1}{M_1}, \frac{1}{1 + |y_2|}, \frac{1}{1 + |y_1|} \right\}. \]

Then, if
\[ \delta = \min(\delta_1, \rho/M_1), \]

it follows that for all solutions \(y\) of the initial value problem
\[ |y''(x) - y_2| \leq \rho \]
for \(|x - x_0| \leq \delta\). Hence, for all solutions \(y\) of the initial value problem
\[ y''(x) - \phi''(x) \geq 2\rho \]
and
\[ \psi''(x) - y''(x) \geq 2\rho \]
on \([x_0 - \delta, x_0 + \delta]\). The conclusion of the lemma follows.

**Lemma 2.2.** Assume \(f\) satisfies condition (A) and let \([c, d] \subset (a, b)\) be a fixed subinterval. Let constants \(M_0 > 0, M_1 > 0, M_2 > 0\) be given. Then there is a \(\delta = \delta(M_0, M_1, M_2) > 0\) such that the boundary value problem
\[ y''' = f(x, y, y', y'') \]
\[ y(x_0) = y_0, \quad y(x_1) = y_1, \quad y(x_2) = y_2 \]
\[ \text{for } c \leq x_0 < x_1 < x_2 \leq d \]
has a solution of class \(C^3\) on \([x_0, x_2]\) provided that
\[ |y_i| \leq M_0 \text{ for } i = 0, 1, 2, \quad |(y_0 - y_1)/(x_0 - x_1)| \leq M_1, \quad |(y_2 - y_1)/(x_2 - x_1)| \leq M_1, \]
\[ \frac{|y_2 - y_1|}{x_2 - x_1} \leq M_2, \quad \frac{|y_1 - y_0|}{x_1 - x_0} \leq M_2 \]
and \(x_2 - x_0 \leq \delta\).
**Proof.** The proof consists of a standard application of the Schauder-Tychonoff fixed point theorem similar to that for a two point boundary value problem for a second order equation given in [1] p. 309.

**Corollary 2.1.** Assume \( f \) satisfies condition (A) and let \( \phi \) be of class \( C^2 \) on \([c, d] \subset (a, b)\). Then there is a \( \delta > 0 \) such that the boundary value problem (2.1) where \( c \leq x_0 < x_1 < x_2 \leq d \) and \( y_i = \phi(x_i) \) for \( i = 0, 1, 2 \) has a solution of class \( C^3 \) on \([x_0, x_2]\) provided \( x_2 - x_0 \leq \delta \).

**Lemma 2.3.** Assume \( f \) satisfies condition (A) and let \([c, d] \subset (a, b)\) be a fixed subinterval. Let constants \( M_0 > 0, M_1 > 0, M_2 > 0 \) be given. Then there exists a \( \delta = \delta(M_0, M_1, M_2) > 0 \) such that the boundary value problem

\[
y'' = f(x, y, y', y'')
y(x_0) = y_0, \quad y'(x_0) = y_1, \quad y(x_1) = y_2,
\]

where \( c \leq x_0 < x_1 \leq d \) has a solution of class \( C^3 \) on \([x_0, x_1]\) provided that

\[
|y_0| \leq M_0, \quad |y_2| \leq M_0, \quad |y_0 - y_2|_{x_0 - x_1} \leq M_1, \quad |y_1| \leq M_1,
\]

\[
|y_1 - y_0 - y_2|_{x_0 - x_1} / |x_0 - x_1| \leq M_2,
\]

and \( x_1 - x_0 \leq \delta \).

**Proof.** Again, a standard application of the Schauder-Tychonoff fixed point theorem gives the result.

**Corollary 2.2.** Assume \( f \) satisfies condition (A) and let \( \phi \) be of class \( C^2 \) on \([c, d] \subset (a, b)\). Then there is a \( \delta > 0 \) such that the boundary value problem (2.2) where \( c \leq x_0 < x_1 \leq d \), \( y_0 = \phi(x_0), y_2 = \phi(x_1) \) and \( y_1 = \phi'(x_0) \) has a solution of class \( C^3 \) on \([x_0, x_1]\) provided \( x_1 - x_0 \leq \delta \).

There are results analogous to Lemma 2.3 and Corollary 2.2 for the boundary value problem

\[
y'' = f(x, y, y', y'')
y(x_0) = y_0, \quad y(x_1) = y_1, \quad y'(x_1) = y_2.
\]

These results will not be stated here and when we wish to refer to them we will refer instead to Lemma 2.3 or Corollary 2.2.

**Definition 2.1.** A real valued function \( \phi \) of class \( C^1 \) on an interval \( I \subset (a, b) \) is said to be a two point subfunction with respect to solutions of (1.1) in case \( y(x) \geq \phi(x) \) on \([x_0, x_1]\) for any \([x_0, x_1] \subset I \) and solution \( y \) of (1.1)
with \( y(x_0) = \phi(x_0), \ \phi'(x_0) = y'(x_0), \) and \( y(x_1) = \phi(x_1) \) and if, in addition, \( y(x) \leq \phi(x) \) on \([x_0, x_1]\) for any \([x_0, x_1] \subseteq I\) and solution \( y \) of (1.1) with \( y(x_0) = \phi(x_0), \ y(x_1) = \phi(x_1), \) and \( y'(x_1) = \phi'(x_1) \).

**Definition 2.2.** A real valued function \( \phi \) of class \( C^1 \) on an interval \( I \subseteq (a, b) \) is said to be a two point subfunction in the small with respect to solutions of (1.1) in case for any compact interval \([c, d] \subseteq I\) there is a \( \delta > 0 \) such that \( \phi \) is a two point subfunction with respect to solutions of (1.1) on every subinterval of \([c, d]\) of length less than or equal to \( \delta \).

**Definition 2.3.** A real valued function \( \phi \) of class \( C^3 \) on an interval \( I \subseteq (a, b) \) is said to be a lower solution of (1.1) in case \( \phi''(x) > f(x, \phi(x), \phi'(x)) \) for \( x \in I \). Upper solutions are similarly defined but with the inequality reversed. If the strict inequality holds at each point \( x \in I \) we say \( \phi \) is a strict lower or upper solution.

We will at times speak of a function \( \phi \) being a subfunction with respect to upper solutions of (1.1) or being a subfunction in the small with respect to upper solutions of (1.1) or being a two point subfunction with respect to upper solutions of (1.1) or being a two point subfunction in the small with respect to upper solutions of (1.1). By these statements we mean \( \phi \) satisfies Definition 1.1, Definition 1.2, Definition 2.1, or Definition 2.2 respectively where in each case the solutions \( y \) are replaced by upper solutions.

We will state our definitions and results in terms of subfunctions and solutions or in terms of subfunctions and upper solutions. Corresponding results for superfunctions and solutions or lower solutions can easily be formulated.

### 3. Subfunctions

**Theorem 3.1.** Assume that \( f \) satisfies conditions (A), (B), and (C). Then solutions of two point boundary value problems for (1.1) are unique; that is, if \( a < x_0 < x_1 < b \) and \( y_1(x) \) and \( y_2(x) \) are both solutions of (2.2) or of (2.3), then it follows that \( y_1(x) = y_2(x) \) on \([x_0, x_1]\).

**Proof.** We consider the case where \( y_1(x_0) = y_2(x_0), \ y_1'(x_0) = y_2'(x_0), \) and \( y_1(x_1) = y_2(x_1) \). First assume \( y_1''(x_0) \neq y_2''(x_0) \), and to be specific assume \( y_1''(x_0) > y_2''(x_0) \). Then by Lemma 2.1 there is a \( \delta > 0 \) with \( a < x_0 - \delta < x_0 + \delta < x_1 \) such that all solutions \( y(x) \) of the initial value problem for (1.1) with initial conditions

\[
\begin{align*}
y(x_0) &= y_0 = y_1(x_0), & y'(x_0) &= y_1'(x_0), \\
y''(x_0) &= y_2 = \frac{1}{2}[y_1''(x_0) + y_2''(x_0)]
\end{align*}
\]  

(3.1)
satisfy
\[ y'_2(x) < y(x) < y'_1(x) \]
for \(0 < |x - x_0| \leq \delta\). Let \(\{\epsilon_n\}\) be a monotone decreasing sequence of positive numbers converging to zero and let \(z_n\) be a solution of (1.1) with initial values
\[ z_n(x_0) = y_0, \quad z'_n(x_0) = y_1 + \epsilon_n, \quad z''_n(x_0) = y_2 . \tag{3.2} \]
Then \(\{z_n\}\) contains a subsequence converging uniformly on \([x_0 - \delta, x_0 + \delta]\) to a solution of (1.1), (3.1). Hence, for sufficiently large \(n\), there is a solution \(z_n\) of (1.1), (3.2) such that
\[ y_2(x_0 - \delta) < z_n(x_0 - \delta) < y_1(x_0 - \delta) \]
and
\[ y_2(x_0 + \delta) < z_n(x_0 + \delta) < y_1(x_0 + \delta) . \]
Since \(z_n(x_0) = y_2(x_0) = y'_2(x_0)\) and \(z'_n(x_0) = y'_1(x_0) > y'_2(x_0)\), it follows that there are \(t_1, t_2\) with \(x_0 - \delta < t_1 < x_0 < t_2 < x_0 + \delta\) such that \(z_n(t_1) = y_2(t_1)\), and \(z_n(t_2) = y_1(t_2)\). Since \(y_1(x_1) = y_2(x_2)\) at \(x_1 > x_0 + \delta\), it follows that any extension of \(z_n(x)\) intersects either \(y_1(x)\) or \(y_2(x)\) again on \([x_0 + \delta, b]\). Since \(z_n(x) \neq y_2(x)\) on \([t_1, x_0]\) and \(z_n(x) \neq y_1(x)\) on \([x_0, t_2]\), this contradicts condition (B).

We conclude that, if \(y_1(x) \neq y_2(x)\) on \([x_0, x_1]\), then \(y^{(i)}_1(x_0) = y^{(i)}_2(x_0)\) for \(i = 0, 1, 2\). However, if \(y^{(i)}_1(x_0) = y^{(i)}_2(x_0)\) for \(i = 0, 1, 2\) and \(y_1(x) \neq y_2(x)\) on \([x_0, x_1]\), then for \(a < x_2 < x_0\), \(u(x)\) and \(v(x)\) defined by
\[ u(x) = v(x) = y_1(x) \quad \text{on} \quad [x_2, x_0], \]
\[ u(x) = y_1(x) \quad \text{on} \quad (x_0, x_1], \]
and
\[ v(x) = y_2(x) \quad \text{on} \quad (x_0, x_1] \]
would be solutions on \([x_2, x_1]\) which would again contradict condition (B). We conclude that \(y_1(x) \equiv y_2(x)\) on \([x_0, x_1]\).

**Theorem 3.2.** Assume that \(f\) satisfies conditions (A), (B), and (C). Let \(\phi \in C^2(I)\) be a subfunction in the small with respect to solutions of (1.1) on \(I \subset (a, b)\). Then \(\phi\) is a two point subfunction in the small on \(I\) with respect to solutions of (1.1).

**Proof.** Because of similarity in the arguments, we prove only that \(\phi\) is a subfunction in the small with respect to solutions of (1.1) satisfying \(y(x_0) = \phi(x_0), y'(x_0) = \phi'(x_0), y(x_1) = \phi(x_1)\) with \(x_0 < x_1\).

Let \([c, d] \subset I\) be given and let \(\delta = \min\{\delta_1, \delta_2, \delta_3\}\) where \(\delta_1, \delta_2,\) and \(\delta_3\) are as in Definition 1.2, Corollary 2.1, and Corollary 2.2, respectively. Then
we claim that \( \phi \) is a two point subfunction on any subinterval of \([c, d]\) of length not exceeding \( \delta \). Suppose this is not the case, then there exists an interval \([x_0, x_1]\) \( \subseteq [c, d] \) with \( x_1 - x_0 \leq \delta \) and a solution \( y_0 \) of (1.1) with

\[
y_0(x_0) = \phi(x_0), \quad y_0'(x_0) = \phi'(x_0), \quad y_0(x_1) = \phi(x_1)
\]

which is such that \( y_0(x) < \phi(x) \) for some points in \((x_0, x_1)\). If \( t \) is such a point, \( y_0(x) < \phi(x) \) on \((x_0, t)\) for, if \( y_0(t) = \phi(t) \) for some \( x_0 < t < t_1 \) then from the definition of subfunction in the small it would follow that \( y_0(x) \leq \phi(x) \) on \([x_0, t]\) and \( y_0(x) \geq \phi(x) \) on \([t, x_1]\) which contradicts \( y_0(t) < \phi(t) \).

Consequently, we can assume \( y_0(x) < \phi(x) \) on \((x_0, x_1)\).

Let \( y_1 \) be the solution of (1.1) with \( y_1(x_0) = \phi(x_0), \quad y_1(x_1) = \phi(x_1) \), and \( y_1(x_2) = \phi(x_2) \) where \( x_2 = \frac{1}{2}[x_0 + x_1] \). Then, by the choice of \( \delta \), \( y_1(x) \leq \phi(x) \) on \([x_0, x_2]\) and \( y_1(x) \geq \phi(x) \) on \([x_2, x_1]\). If \( y_1(x) < y_0(x) \) for some \( x_0 < x < x_2 \), then there is an \( x_3 \), \( x_0 < x_3 < x_2 \), at which \( y_1(x_3) = y_0(x_0) \). This contradicts condition (B), consequently,

\[
y_0(x) \leq y_1(x) \leq \phi(x)
\]

on \([x_0, x_2]\) which implies

\[
y_0'(x_0) = y_1'(x_0).
\]

It then follows from Theorem 3.1 that \( y_0(x) = y_1(x) \) on \([x_0, x_1]\) which contradicts \( y_0(x_0) < \phi(x_0) = y_1(x_0) \). We conclude that \( y_0(x) \geq \phi(x) \) on \([x_0, x_1]\) and that \( \phi \) is a two point subfunction in the small on \( I \).

**Theorem 3.3.** Assume that \( f \) satisfies conditions (A), (B), and (C). Let \( \phi \in C^0(I) \) be a subfunction in the small with respect to solutions of (1.1) on \( I \subseteq (a, b) \). Then \( \phi \) is a subfunction with respect to solutions of (1.1) on \( I \).

**Proof.** If the conclusion does not hold then there is an interval \([c, d]\) \( \subseteq I \) and a solution \( y \) of (1.1) such that \( y(c) = \phi(c), y(d) = \phi(d) \), and \( y(t_0) = \phi(t_0) \) for some \( c < t_0 < d \) but either there is a point \( t_1 \) with \( c < t_1 < t_0 \) and \( y(t_1) > \phi(t_1) \) or else there is a point \( t_2 \) with \( t_0 < t_2 < d \) and \( y(t_2) < \phi(t_2) \). Because of similarity, we consider only the first case. For this fixed interval \([c, d]\) we now choose \( \delta = \min\{\delta_1, \delta_2, \delta_3\} \) where \( \delta_1, \delta_2 \) and \( \delta_3 \) come from Definition 1.2, Corollary 2.1, and Corollary 2.2, respectively.

The basic idea of the proof is analogous to the induction argument used to prove Theorem 1 in [4]. That is, we show that we can find a subinterval of \([c, d]\) of length less than or equal to \( d - c - \delta/2 > 0 \) on which \( \phi \) is not a subfunction with respect to solutions of (1.1). Repeated applications then show that for each positive integer \( n \), \( d - c - n\delta/2 > 0 \) which is an obvious contradiction.

Assume that this is not the case, that is, assume that \( \phi \) is a subfunction with
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respect to solutions of (1.1) on any subinterval of \([c, d]\) of length not exceeding \(d - c - \delta/2\).

At this point the reader should observe that, since \(\phi\) is a subfunction in the small for solutions of (1.1), there is no loss of generality in assuming that \(y(x) > \phi(x)\) for \(c < x < t_0\) and that either \(y(x) > \phi(x)\) for \(t_0 < x < d\), \(y(x) < \phi(x)\) for \(t_0 < x < d\), or \(y(x) = \phi(x)\) for \(t_0 < x < d\).

Since \(\delta \leq \delta_1\), we see from Definition 1.2 that \(d - c > \delta\) or else the behavior of \(y\) on \([c, d]\) contradicts the assumption that \(\phi\) is a subfunction in the small with respect to solutions of (1.1). Thus either \(d - t_0 > \delta/2\) or \(t_0 - c > \delta/2\).

We consider first the case where \(d - t_0 > \delta/2\) and treat the case where both \(d - t_0 > \delta/2\) and \(t_0 - c > \delta/2\) later. Because of the similarity in the arguments we consider only the case where \(d - t_0 \leq \delta/2\).

By Corollary 1.2 there exists a solution \(u\) of (1.1) on \([d - \delta, d]\) which satisfies

\[
u(d) = \phi(d), \quad u(d - \delta/2) = \phi(d - \delta/2), \quad u(d - \delta) = \phi(d - \delta).
\]

Let \(u\) also designate an extension of this solution to \((a, b)\). By Definition 1.2 we must have \(u(x) \leq \phi(x)\) on \(d - \delta \leq x \leq d - \delta/2\) and \(u(x) \geq \phi(x)\) on \(d - \delta/2 < x < d\), hence, \(u(d - \delta) \leq \phi'(d - \delta)\). Since \(u(d - \delta) < y(d - \delta)\) while \(u\) and \(y\) are equal at two distinct points in \([d - \delta/2, d]\), it follows from (B) that \(u(x) < y(x)\) on \((a, d - \delta)\). In particular \(u(c) < y(c) = \phi(c)\). From this we conclude that \(u(x) \leq \phi(x)\) on \([c, d - \delta]\). For, if \(u(x) > \phi(x)\) at some points of \((c, d - \delta)\), there would be an \(x_0 \in (c, d - \delta)\) such that \(u(x_0) = \phi(x_0)\) and \(u(x) > \phi(x)\) at some points of \((c, d - \delta)\). This would contradict our assumption that \(\phi(x)\) is a subfunction on any subinterval of \([c, d]\) of length not exceeding \(d - c - \delta/2\). Thus \(u(x) \leq \phi(x)\) on \([c, d - \delta]\) from which it follows that \(u'(d - \delta) = \phi'(d - \delta)\). It then follows from Theorem 3.2 that \(u(x) = \phi(x)\) on \([d - \delta, d - \delta/2]\) and \(u''(d - \delta) = \phi''(d - \delta)\). If \(d - \delta/2 \leq c\), let \(v\) be the solution of (1.1) with

\[
v(c) = \phi(c), \quad v(d - \delta) = \phi(d - \delta), \quad v(d - \delta/2) = \phi(d - \delta/2).
\]

It follows from Definition 1.2 that \(v(x) \leq \phi(x)\) on \([c, d - \delta]\) and \(v(x) \geq \phi(x)\) on \([d - \delta, d - \delta/2]\). Also as a consequence of condition (B) we conclude that \(v(x) \geq u(x)\) on \([c, d - \delta]\). It follows that

\[
v'(d - \delta) = u'(d - \delta) = \phi'(d - \delta)
\]

and

\[
v''(d - \delta) = u''(d - \delta) = \phi''(d - \delta).
\]
Then \( w(x) \) defined by

\[
\begin{align*}
    w(x) &= v(x) \quad \text{on } [c, d - \delta] \\
    w(x) &= u(x) \quad \text{on } [d - \delta, d]
\end{align*}
\]

is a solution of (1.1) on \([c, d]\) and \( w(x) \nRightarrow y(x) \) on \([c, d]\) which contradicts condition (B). Thus \( d - 3\delta/2 > c \).

Now let \( v(x) \) be the solution of (1.1) with

\[
\begin{align*}
    v(d - 3\delta/2) &= \phi(d - 3\delta/2), \\
    v(d - \delta) &= \phi(d - \delta), \\
    v(d - \delta/2) &= \phi(d - \delta/2).
\end{align*}
\]

Then again by Definition 1.2 \( v(x) \leq \phi(x) \) on \([d - 3\delta/2, d - \delta]\) and \( v(x) \nRightarrow \phi(x) \) on \([d - \delta, d - \delta/2]\). By condition (B) we conclude again that \( v(x) \nRightarrow u(x) \) on \([d - 3\delta/2, d - \delta]\) from which it follows that

\[
\begin{align*}
    v'(d - \delta) &= u'(d - \delta) = \phi'(d - \delta) \\
    v''(d - \delta) &= u''(d - \delta) = \phi''(d - \delta).
\end{align*}
\]

Then the function \( w(x) \) defined by

\[
\begin{align*}
    w(x) &= v(x) \quad \text{on } [d - 3\delta/2, d - \delta] \\
    w(x) &= u(x) \quad \text{on } [d - \delta, d]
\end{align*}
\]

is a solution of (1.1) on \([d - 3\delta/2, d]\) and has an extension to \((a, b)\). From the same arguments as used above we conclude that \( w(x) \leq \phi(x) \) on \([c, d - 3\delta/2]\). In a finite number of steps we arrive at a solution \( z(x) \) of (1.1) such that \( z(c) = \phi(c), z(x) \equiv u(x) \) on \([d - \delta, d]\), and \( z(x) \leq \phi(x) \) on \([c, d - \delta/2]\). Thus \( z(x) \nRightarrow y(x) \) which contradicts (B) and in the case \( d - t_0 \leq \delta/2 \) we conclude that there is a subinterval on \([c, d]\) of length not exceeding \( d - c - \delta/2 \) on which \( \phi \) is not a subfunction.

Now assume that \( d - t_0 > \delta/2 \) and \( t_0 - c > \delta/2 \) and assume again that \( \phi \) is a subfunction on any subinterval of \([c, d]\) of length not exceeding \( d - c - \delta/2 \). First, if \( y(x) \equiv \phi(x) \) on \([t_0, d]\), then this case can be reduced to the above case with \( d - t_0 \leq \delta/2 \) by selecting a new \( d \). Next let \( u(x) \) be the solution of (1.1) with

\[
\begin{align*}
    u(t_0 - \delta/2) &= \phi(t_0 - \delta/2), \\
    u(t_0) &= \phi(t_0), \\
    u(t_0 + \delta/2) &= \phi(t_0 + \delta/2).
\end{align*}
\]

Then \( u(x) \leq \phi(x) \) on \([t_0 - \delta/2, t_0]\) and \( u(x) \nRightarrow \phi(x) \) on \([t_0, t_0 + \delta/2]\). Assume that \( y(x) < \phi(x) \) on \((t_0, d)\). Then, making use of the process employed above and the assumption that \( \phi \) is a subfunction on subintervals of \([c, d]\) of length not exceeding \( d - c - \delta/2 \), we obtain an extension of \( u(x) \) to \((a, t_0 + \delta/2]\) such that \( u(x_0) < y(x_0) \) for some \( x_0 \) in \([c, t_0 - \delta/2]\). Employing
the same extension process to the right, we then obtain an extension \( u(x) \) to \((a, b)\) such that also \( u(x_i) = y(x_i) \) for some \( x_i \in (t_0 + \delta/2, d] \). Since \( u(t_0) = y(t_0) \) and \( u(x) \neq y(x) \) on \( [x_0, x_1] \) this contradicts (B) and we again conclude that \( \phi(x) \) is not a subfunction on some subinterval of \([c, d]\) of length not exceeding \( d - c - \delta/2 \).

Finally, if \( y(x) > \phi(x) \) on \((t_0, d)\), then \( y'(t_0) = \phi'(t_0) \). Let \( u(x) \) be the solution as described above. Then either \( u(x) = y(x) \) at two distinct points in \([t_0, t_0 + \delta/2]\) or \( u'(t_0) = y'(t_0) \). Again with the assumption that \( \phi \) is a subfunction on subintervals of \([c, d]\) of length not exceeding \( d - c - \delta/2 \) we can construct an extension of \( u(x) \) to \((a, t_0 + \delta/2]\) such that \( u(x_0) = y(x_0) \) at some \( x_0 \in [c, t_0 - \delta/2) \). This contradicts (B) in one case and Theorem 3.1 in the other case.

Thus in all cases \([c, d]\) must contain an interval of length not exceeding \( d - c - \delta/2 \) on which \( \phi \) is not a subfunction. As stated at the beginning of the proof this leads to the ultimate contradiction that \( d - c - n\delta > 0 \) for all positive integers \( n \) and we conclude that \( \phi \) is a subfunction on \( I \).

**Theorem 3.4.** Assume that \( f \) satisfies conditions (A), (B), and (C). Let \( \phi \in C^2(I) \) be a subfunction in the small with respect to solutions of (1.1) on \( I \subset (a, b) \). Then \( \phi \) is a two point subfunction with respect to solutions of (1.1) on \( I \).

**Proof.** If the conclusion does not hold then there is an interval \([c, d]\) for which \( \phi \) is not a two point subfunction on \([c, d]\). Since this argument can then be repeated, we are led to the contradiction \( d - c - n\delta > 0 \) for every positive integer \( n \).

By Corollary 2.2 there exists a solution \( u \) of (1.1) on \([c, c + \delta]\) which satisfies \( u(c) = \phi(c) \), \( u(c + \delta) = \phi(c + \delta) \) and \( u'(c + \delta) = \phi'(c + \delta) \). By Theorem 3.2 we see that \( u(x) \leq \phi(x) \) for \( c \leq x \leq c + \delta \). Also, either \( u'(c) < \phi'(c) \) and \( u(x) = y(x) \) at two distinct points \( x \) in \([c, c + \delta]\) or else \( u'(c) = \phi'(c) \) and \( u(x) = \phi(x) \) on \([c, c + \delta]\) by Theorem 3.2. By (C), \( u \) may be extended to \((a, b)\). If there is any \( x \in (c + \delta, d) \) for which \( u(x) < \phi(x) \) then either \( u(x) = \phi(x) \) for some \( x \in (c + \delta, d) \) or else \( u(x) \) equals \( y(x) \) for some \( x \in (c + \delta, d) \). In the first instance the problem reduces to an interval
of length less than or equal to $d - c - \delta$ as desired while the second case either contradicts the conclusion of Theorem 3.1 or else it contradicts condition (B).

Thus we may assume that $u(x) \geq \phi(x)$ for $c + \delta \leq x \leq d$. If $d - c - \delta \leq \delta$, then we let $v$ be the solution of (1.1) satisfying

$$v(c + \delta) = \phi(c + \delta), \quad v'(c + \delta) = \phi'(c + \delta), \quad v(d) = \phi(d).$$

From Theorem 3.1 it follows that $v(x) \leq u(x)$ for $c + \delta \leq x \leq d$ and since it can be shown that $u''(c + \delta) = \phi''(c + \delta)$, it follows necessarily that $v''(c + \delta) = \phi''(c + \delta)$ also. The function $w$ defined by

$$w(x) = u(x) \quad c \leq x \leq c + \delta$$

$$w(x) = v(x) \quad c + \delta < x \leq d$$

is now a solution of (1.1) on $[c, d]$ and either Theorem 3.1 or condition (B) is violated. If $d - c - \delta > \delta$, then let $v$ be the solution of (1.1) which satisfies

$$v(c + \delta) = \phi(c + \delta), \quad v'(c + \delta) = \phi'(c + \delta), \quad v(c + 2\delta) = \phi(c + 2\delta).$$

From Theorem 3.1 it follows that $v(x) \leq u(x)$ for $c + \delta \leq x \leq c + 2\delta$ and since $u''(c + \delta) = \phi''(c + \delta)$ we have $v''(c + \delta) = \phi''(c + \delta)$, also. Now $w$ defined by

$$w(x) = u(x) \quad c \leq x \leq c + \delta$$

$$w(x) = v(x) \quad c + \delta < x \leq c + 2\delta$$

is a solution of (1.1) on $[c, c + 2\delta]$. By (C), $w$ can now be extended to $(a, b)$. If $w(x) < \phi(x)$ for some $x \in (c + \delta, d]$ then the proof may be finished as before. Otherwise, $w(x) \geq \phi(x)$ on $[c + \delta, d]$. The above process may now be applied again to the interval $[c + 2\delta, d]$ or $[c + 2\delta, c + 3\delta]$ depending on whether $d - c - 2\delta \leq \delta$ or $d - c - 2\delta > \delta$. A finite number of such steps finishes the proof.

**Theorem 3.5.** Assume that $f$ satisfies conditions (A), (B), and (C). Let $\phi \in C^2(I)$ be a two point subfunction in the small with respect to solutions of (1.1) on $I \subset (a, b)$. Then $\phi$ is a subfunction with respect to solutions of (1.1) on $I$.

**Proof.** By Theorem 3.3 it suffices to show that $\phi$ is a subfunction in the small for solutions of (1.1) on $I$. Let $[c, d] \subset I$ and choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ where $\delta_1, \delta_2$ and $\delta_3$ come from Definition 2.2, Corollary 2.1, and Corollary 2.2, respectively. We will show that this $\delta$ works for Definition 1.2. If not, there is an interval $[x, \beta] \subset [c, d]$ with $\beta - x \leq \delta$, a point $t_0$ with $x < t_0 < \beta$ and a solution $y$ of (1.1) on $[x, \beta]$ such that

$$y(x) = \phi(x), \quad y(t_0) = \phi(t_0), \quad y(\beta) = \phi(\beta).$$
and either there is a point $x$, $\alpha < x < t_0$ with $y(x) > \phi(x)$ or else there is a point $x$, $t_0 < x < \beta$ and $y(x) < \phi(x)$. We treat only the first case, the other being similar.

Note that we may assume without loss of generality that $y(x) > \phi(x)$ for $\alpha < x < t_0$. Also, we must have $y'(t_0) < \phi'(t_0)$ since $\phi$ is a two point subfunction in the small for solutions of (1.1). Thus we may assume $y(x) < \phi(x)$ for $t_0 < x < \beta$. Let $u$ be the solution of (1.1) satisfying

$$u(x) = \phi(x), \quad u(t_0) = \phi(t_0), \quad u'(t_0) = \phi'(t_0)$$

and let $v$ be the solution of (1.1) satisfying

$$v(t_0) = \phi(t_0), \quad v'(t_0) = \phi'(t_0), \quad v(\beta) = \phi(\beta),$$

both of which exist by Corollary 2.1. We note that $u(x) \leq \phi(x)$ for $\alpha < x < t_0$ and that $v(x) \geq \phi(x)$ for $t_0 < x < \beta$ since $\phi$ is a two point subfunction in the small for solutions of (1.1). From this we see that $u''(t_0) \leq \phi''(t_0)$ and $v''(t_0) \geq \phi''(t_0)$. Now equality cannot hold in both these inequalities or else $v$ defined by

$$w(x) = u(x) \quad \alpha \leq x \leq t_0,$$

$$w(x) = v(x) \quad t_0 < x \leq \beta$$

is a solution of (1.1) on $[\alpha, \beta]$ and condition (B) is violated. Thus, for example, suppose $u''(t_0) < \phi''(t_0)$. Then either there is an $x$ such that $t_0 < x \leq \beta$ and $u(x) = y(x)$ or else there is an $x$ such that $t_0 < x \leq \beta$ and $u(x) = \phi(x)$. In the first instance condition (B) is violated and in the second $\phi$ would not be a two point subfunction in the small for solutions of (1.1). This completes the proof.

4. Differential Inequalities

**Theorem 4.1.** Assume that $f$ satisfies conditions (A), (B), and (C). Let $\phi \in C^3(I)$ be a strict lower solution on $I$. Then $\phi$ is a subfunction with respect to solutions of (1.1) on $I$.

**Proof.** By Theorem 3.3 it suffices to show that $\phi$ is a subfunction in the small for solutions of (1.1) on $I$.

Let $[c, d]$ be a compact subinterval of $I$. Then, since $\phi \in C^3(I)$ and $\phi'' > f(x, \phi, \phi', \phi'')$ on $[c, d]$, it follows by standard continuity and compactness arguments that there is a $\delta > 0$ such that for any $x_0 \in [c, d]$ and any solution $y(x)$ of (1.1) with $y^{(i)}(x_0) = \phi^{(i)}(x_0)$ for $i = 0, 1, 2$ it follows that $\phi''(x) > y''(x)$ on $[x_0 - \delta, x_0 + \delta] \cap I$. Hence, for any such solution $y(x) > \phi(x)$ on $[x_0 - \delta, x_0 + \delta] \cap I$ and $y(x) < \phi(x)$ on $(x_0, x_0 + \delta] \cap I$. We claim that this $\delta$ will serve in Definition 1.2 for the interval $[c, d]$. Suppose
that this is not the case. Then there is an interval $[\alpha, \beta] \subset [c, d]$ and a solution $y(x)$ such that $\beta - \alpha \leq \delta$, $y(\alpha) = \phi(\alpha)$, $y(\beta) = \phi(\beta)$, $y(x_0) = \phi(x_0)$ for some $\alpha < x_0 < \beta$, and either $y(x) > \phi(x)$ at some points of $(\alpha, x_0)$ or $y(x) < \phi(x)$ at some points of $(x_0, \beta)$. To be specific we consider the first case, and, without loss of generality we can assume that $y(x) > \phi(x)$ on $(\alpha, x_0)$. Let $u(x)$ be a solution of (1.1) with $u^{(i)}(x_0) = \phi^{(i)}(x_0)$ for $i = 0, 1, 2$. Then by the choice of $\delta$, $u(x) > \phi(x)$ on $[\alpha, x_0)$ and $u(x) < \phi(x)$ on $(x_0, \beta)$. There are two cases to consider: $y'(x_0) < \phi'(x_0)$ and $y'(x_0) = \phi'(x_0)$. It is clear that, if $y'(x_0) < \phi'(x_0)$, then $y(x)$ and $u(x)$ violate condition (B). If $y'(x_0) = \phi'(x_0)$, then $y''(x_0) \geq \phi''(x_0)$ because of the choice of $\delta$ and the fact that $y(\alpha) = \phi(\alpha)$. Hence, it follows that $y''(x_0) > \phi''(x_0) = u''(x_0)$ which means that $y(x)$ and $u(x)$ contradict the conclusion of Theorem 3.1. Thus we conclude that Definition 1.2 is satisfied and $\phi$ is a subfunction in the small on $I$.

**Theorem 4.2.** Assume that $f$ satisfies conditions (A), (B), and (C). Let $\phi \in C^2(I)$ be a strict lower solution on $I$. Then $\phi$ is a two point subfunction with respect to solutions of (1.1) on $I$.

**Proof.** This follows directly from Theorem 3.4 and Theorem 4.1.

**Theorem 4.3.** Assume that $f$ satisfies conditions (A), (B), and (C). Assume also that $f$ satisfies a Lipschitz condition with respect to $y, z_1, \text{ and } w$ on compact subsets of $(a, b) \times \mathbb{R}^3$. Let $\phi \in C^2(I)$ be a lower solution on $I$. Then $\phi$ is a subfunction with respect to solutions of (1.1) on $I$.

**Proof.** Assume that $\phi$ is not a subfunction on $I$. Then there is an interval $[c, d] \subset I$ and a solution $y(x)$ of (1.1) such that $y(c) = \phi(c)$, $y(d) = \phi(d)$, $y(x_0) = \phi(x_0)$ for some $c < x_0 < d$, and either $y(x) > \phi(x)$ at some points of $(c, x_0)$ or $y(x) < \phi(x)$ at some points of $(x_0, d)$. We consider only the case in which $y(x) > \phi(x)$ at some points in $(c, x_0)$ and in this case we can assume that $y(x) > \phi(x)$ on $(c, x_0)$.

It follows from Theorem 3.1 that for $\epsilon > 0$ the solution $u(x)$ of (1.1) with $u(d) = y(d)$, $u'(d) = y'(d)$, and $u''(d) = y''(d) - \epsilon$ satisfies $u(x) < y(x)$ on $(a, d)$. Furthermore, for $\epsilon$ sufficiently small $u(x) > \phi(x)$ at some points of $(c, x_0)$. Let $x_1 \in (c, x_0)$ be such that $u(x_1) = \phi(x_1)$ and $u(x) > \phi(x)$ at some points in $(x_1, x_0)$. Then for $\eta > 0$ sufficiently small the solution $v(x)$ of (1.1) with $v(x_1) = u(x_1)$, $v(x_0) = u(x_1)$, $v''(x_1) = \phi''(x_1) + \eta$ satisfies $v(c) < \phi(c)$, $v(d) > \phi(d)$, $u(x_0) < v(x_0) < \phi(x_0)$, and $v(x) > \phi(x)$ at some points in $(x_1, x_0)$. Let $x_2 \in (x_1, x_0)$ be such that $v(x_2) = \phi(x_2)$ and $v(x) > \phi(x)$ at some points in $(x_1, x_0)$. Let $\phi_u$ be the solution of the initial value problem

\[
y'' = f(x, y, y', y'') + \phi''(x) - f(x, \phi(x), \phi'(x), \phi''(x)) - 1/n,
\]

\[
y^{(i)}(x_2) = \phi^{(i)}(x_2), \quad i = 0, 1, 2,
\]
where \( n \) is a positive integer. Then for \( n \) sufficiently large \( \phi_n \) is defined on \([c, d]\), \( \phi_n \) is a strict lower solution of (1.1) on \([c, d]\), and \( \lim \phi_n = \phi \) uniformly on \([c, d]\). Thus for sufficiently large \( n \) \( \nu(c) < \phi_n(c) \), \( \nu(x_2) = \phi_n(x_2) \), \( \nu(d) > \phi_n(d) \), \( \nu(x) > \phi_n(x) \) at some points in \((c, x_2)\), and \( \nu(x) < \phi_n(x) \) at some points in \((x_2, d)\). We conclude that for sufficiently large \( n \) the conclusion of Theorem 4.1 is contradicted. Hence \( \phi \) is a subfunction on \( I \).

**Theorem 4.4.** Assume that \( f \) and \( \phi \) satisfy the hypotheses of Theorem 4.3. Then \( \phi \) is a two point subfunction with respect to solutions of (1.1) on \( I \).

**Proof.** This follows directly from Theorem 4.3 and Theorem 3.4.

### 5. Generalizations

**Remark 5.1.** Theorems 3.3, 3.4, 3.5, 4.1, 4.2, 4.3, and 4.4 are true if the last sentence in the statement of each theorem is changed by replacing the term *with respect to solutions of (1.1)* by *with respect to strict upper solutions of (1.1)*. In each case the proof is similar so is not presented again. The details for the second order case are found in [4].

**Remark 5.2.** Any result in this paper, including Remark 5.1, whose hypotheses include condition \((C)\) is also a correct result if \((C)\) is replaced by \((C)'\) Every solution \( y \) of every initial value problem for (1.1) extends to a maximal interval of existence \((\alpha, \beta) \subset (a, b)\) and satisfies

\[
\lim \sup_{x \to \beta^-} |y(x)| = +\infty \quad \text{if} \quad \beta \neq b
\]

and

\[
\lim \sup_{x \to \alpha^+} |y(x)| = +\infty \quad \text{if} \quad \alpha \neq a.
\]

Again, the proofs are essentially the same as before so are omitted.

**Corollary 5.1.** If \( f \) satisfies conditions \((A)\) and \((B)\) but does not depend on \( z \) and \( w \) then all Theorems in Sections 3 and 4 as well as Remark 5.1 are valid without requiring \((C)\) to hold.

**Proof.** It is well known that such an \( f \) satisfies \((C)'\).

### References


