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Existence of periodic solutions for the Lotka–Volterra type systems

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Abstract

In this paper we prove the existence of nonstationary periodic solutions of delay Lotka–Volterra equations. In the proofs we use the S^1-degree due to Dylawerski et al. [G. Dylawerski, K. Geba, J. Jodel, W. Marzantowicz, An S^1-equivariant degree and the Fuller index, Ann. Polon. Math. 63 (1991) 243–280].

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1. Introduction

The aim of this paper is to prove the existence of nonstationary periodic solutions of autonomous delay differential equations of Lotka–Volterra type

$$\begin{cases} \dot u_1(t) = u_1(t)(r_1 - a_{11}u_1(t-\tau) - a_{12}u_2(t-\tau) - \cdots - a_{1n}u_n(t-\tau)), \\ \dot u_2(t) = u_2(t)(r_2 - a_{21}u_1(t-\tau) - a_{22}u_2(t-\tau) - \cdots - a_{2n}u_n(t-\tau)), \\ \quad \vdots \\ \dot u_n(t) = u_n(t)(r_n - a_{n1}u_1(t-\tau) - a_{n2}u_2(t-\tau) - \cdots - a_{nn}u_n(t-\tau)), \end{cases} \tag{1.1}$$

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where $n \geqslant 1$, $\tau > 0$, $r_1, \dots, r_n \in \mathbb{R}$, $a_{ij} \in \mathbb{R}$, for $i, j = 1, \dots, n$.

It is known that a broad class of problems in mathematical biology, economics and mechanics are described in the form above with initial conditions

$$\begin{cases} u_i(s) = \varphi_i(s), & s \in [-\tau, 0],\ \varphi_i(0) > 0, \\ \varphi_i \in C([-\tau, 0], \mathbb{R}), & i = 1, 2, \dots, n. \end{cases} \tag{1.2}$$

In case $n = 1$, problem (1.1) is known as delay logistic equation. The existence and multiplicity of solutions of delay logistic equation has been investigated by many authors (cf. Goparlsamy [3] and Hale [4] and references therein). To compare with the method employed here with that for delay logistic equation, we illustrate the proof for the existence of periodic solutions of the logistic equation

$$\dot{u}(t) = \alpha u(t)\big(1 - u(t - \tau)\big), \tag{1.3}$$

where $\alpha > 0$. For each initial function $\varphi \in C([-\tau, 0], \mathbb{R})$ with $\varphi(0) > 0$, one can find a solution $u(\varphi)$ of (1.3) with initial value $u(\varphi)(s) = \varphi(s)$, $s \in [-\tau, 0]$. We put

$$z(\varphi, \alpha) = \min\big\{t > 0\colon\ u(\varphi)(t) = 0,\ \dot{u}(\varphi)(t) > 0\big\}$$

and $[A(\alpha)\varphi](t) = u(\varphi)(t - z(\varphi, \alpha) - \tau)$ for $t > 0$. Then one can see that each fixed point u of $A(\alpha)$ is a periodic solution of (1.3). The existence of the nonstationary fixed points of $A(\alpha)$ is proved by combination of the Hopf bifurcation theorem and fixed point theorems (cf. Hale [4, Section 11.4]). For $\tau = 1$, it is known that $\alpha = \pi/2$ is the bifurcation point of solutions to (1.3) and for each $\alpha > \pi/2$, problem (1.3) has a nonstationary periodic solution.

On the other hand, it is natural to ask if there are multiple solutions of (1.3) for sufficiently large τ. The multiple existence of periodic solution of (1.3) for sufficiently large τ also follows from the Hopf bifurcation. In general, the methods employed for delay logistic equation are not valid for (1.1) with $n > 1$.

In this paper we work with the space of periodic functions instead of considering the initial value problem and make use of the S^1-degree, see [2], to prove the multiplicity of solutions of problem (1.1). Applications of the degree for equivariant maps to the study of periodic solutions of a van der Pol system one can find in [1,5].

To avoid unnecessary complexity, we restrict ourselves to the case $n = 2$, that is, we consider the coupled equations of the form

$$\begin{cases} \dot{u}(t) = u(t)(r_1 - a_{11}u(t - \tau) - a_{12}v(t - \tau)), \\ \dot{v}(t) = v(t)(r_2 - a_{21}u(t - \tau) - a_{22}v(t - \tau)). \end{cases} \tag{1.4}$$

Our argument does not depend on any specific property of $n = 2$. That is why our result is valid for $n \geqslant 2$ with modifications of assumptions for the case that $n \geqslant 2$.

We impose the following conditions on matrix $A = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]$:

(A0) $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2 > 0$, where $\left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right] = A^{-1}\left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right]$,

(A1) $\langle Ax, x\rangle > 0$ for all $x \in \mathbb{R}^2 \setminus \{0\}$,

(A2) a matrix $\left[\begin{smallmatrix} b_1a_{11} & b_1a_{12} \\ b_2a_{21} & b_2a_{22} \end{smallmatrix}\right]$ possesses two real eigenvalues $\mu_1, \mu_2 > 0$.

Remark 1.1. Notice that for an arbitrary $n \in \mathbb{N}$ assumptions (A0)–(A2) can be reformulated in the following way:

(A0) $a_{ij}, b_i > 0$ for $1 \leqslant i, j \leqslant n$,
(A1) $\langle Ax, x\rangle > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$,
(A2) a matrix $\operatorname{diag}(b_1, \ldots, b_n) \cdot A$ possesses only real, positive eigenvalues $\mu_1, \ldots, \mu_n$, each with multiplicity one.

We can now formulate the main result of this article.

Theorem 1.1. *Fix $\tau > 0$ such that $\min\{\frac{2\pi}{\mu_1}, \frac{2\pi}{\mu_2}\} < \tau < \infty$. Assume that there are $n_1, n_2 \in \mathbb{N} \cup \{0\}$ such that $n_1 \neq n_2$ and for $i = 1, 2$,*

$$\frac{\pi}{2} + 2n_i\pi < \mu_i\tau < \frac{\pi}{2} + 2(n_i + 1)\pi. \tag{1.5}$$

Under the above assumptions there is at least one nonstationary τ-periodic solution of (1.4).

After this introduction our paper is organized as follows.

For the convenience of the reader in Section 2 we have repeated the relevant material from [2] without proofs, thus making our exposition self-contained.

In Section 3 we have performed a functional setting for our problem. This section is of technical nature. Namely, applying transformation of functions and fixing the period we have obtained a parameterized problem (3.2) which is equivalent to the original problem (1.4). Next we have defined a Banach space E which is an infinite-dimensional representation of the group S^1, an open S^1-invariant subset $\Theta_0 \subset E$ and an S^1-equivariant compact operator $F : (E \times \mathbb{R}^+) \times [0, 1] \to E$, see formula (3.5), such that solutions of equation $F(((x_1, x_2), \lambda), 1) = (x_1, x_2)$ in $\Theta_0 \times \mathbb{R}^+$ are exactly periodic solutions of problem (3.2).

In Section 4 we have defined an open, bounded S^1-invariant subset $\Omega_{\lambda_1,\lambda_2} \subset \Theta_0 \times \mathbb{R}^+ \subset E \times \mathbb{R}^+$ such that the homotopy $Q - F(\cdot, \theta)$, defined by (3.5) does not vanish on $\partial\Omega_{\lambda_1,\lambda_2}$. This allow us to simplify computations of the S^1-degree of $Q - F(\cdot, 1)$ on $\Omega_{\lambda_1,\lambda_2}$, see Lemma 4.4.

In Section 5 we have proved Theorem 1.1.

2. S^1-degree

In this section we have compiled some basic facts on the S^1-degree defined in [2]. Let $S^1 = \{z \in \mathbb{C}\colon |z| = 1\} = \{e^{i\cdot\theta}\colon \theta \in [0, 2\pi)\}$ be the group with an action given by the multiplication of complex numbers. For any fixed $m \in \mathbb{N}$ we denote by $\mathbb{Z}_m$ a cyclic group of order m and define homomorphism $\rho_m : S^1 \to GL(2, \mathbb{R})$ as follows

$$\rho_m\left(e^{i\cdot\theta}\right) = \begin{bmatrix} \cos(m\theta) & -\sin(m\theta) \\ \sin(m\theta) & \cos(m\theta) \end{bmatrix}.$$

Let E be a Banach space which is an S^1-representation. We denote by $Q : E \times \mathbb{R} \to E$ the projection. For each closed subgroup H of S^1 and each S^1-invariant subset $\Omega \subset E$, we denote by Ω^H the subset of fixed points of the action of H on Ω. For given $a \in E$, $S^1_a = \{s \in S^1\colon s \cdot a = a\}$ is called the isotropy group of a and the set $S^1 \cdot a = \{s \cdot a\colon s \in S^1\}$ is called the orbit of a. Denote by Γ_0 the free abelian group generated by $\mathbb{N}$ and let $\Gamma = \mathbb{Z}_2 \oplus \Gamma_0$. Then $\gamma \in \Gamma$ means $\gamma = \{\gamma_r\}$, where $\gamma_0 \in \mathbb{Z}_2$ and $\gamma_r \in \mathbb{Z}$ for $r \in \mathbb{N}$.

Fix an open, bounded S^1-invariant subset $\Omega \subset E \times \mathbb{R}$ and continuous S^1-equivariant compact mapping $\Phi : cl(\Omega) \to E$ such that $(Q + \Phi)(\partial\Omega) \subset E \setminus \{0\}$. In this situation the S^1-degree $\mathrm{Deg}(Q + \Phi, \Omega) = \{\gamma_r\} \in \Gamma$, where $\gamma_0 = \deg_{S^1}(Q + \Phi, \Omega)$ and $\gamma_r = \deg_{\mathbb{Z}_r}(Q + \Phi, \Omega)$, $r \in \mathbb{N}$, has been defined in [2].

Theorem 2.1. (Cf. [2].) *Let E be a Banach space which is a representation of the group S^1, $\Omega_0, \Omega_1, \Omega_2 \subset \Omega$ be open bounded, S^1-invariant subsets of $E \times \mathbb{R}$. Assume that $\Phi : cl(\Omega) \to E$ is a compact S^1-equivariant mapping such that $(Q + \Phi)(\partial\Omega) \subset E \setminus \{0\}$. Then there exists a Γ-valued function* $\mathrm{Deg}(Q + \Phi, \Omega)$ *called the S^1-degree, satisfying the following properties*:

(a) *if* $\deg_H(Q + \Phi, \Omega) \neq 0$, *then* $(Q + \Phi)^{-1}(0) \cap \Omega^H \neq \emptyset$,
(b) *if* $(Q + \Phi)^{-1}(0) \cap \Omega \subset \Omega_0$, *then* $\mathrm{Deg}(Q + \Phi, \Omega) = \mathrm{Deg}(Q + \Phi, \Omega_0)$,
(c) *if* $\Omega_1 \cap \Omega_2 = \emptyset$ *and* $(Q + \Phi)^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$, *then*

$$\mathrm{Deg}(Q + \Phi, \Omega) = \mathrm{Deg}(Q + \Phi, \Omega_1) + \mathrm{Deg}(Q + \Phi, \Omega_2),$$

(d) *if* $h : cl(\Omega) \times [0,1] \to E$ *is an S^1-equivariant compact homotopy such that* $(Q + h)(\partial\Omega \times [0,1]) \subset E \setminus \{0\}$, *then* $\mathrm{Deg}(Q + h_0, \Omega) = \mathrm{Deg}(Q + h_1, \Omega)$.

Let E_0, E be real Banach spaces. We denote by $K(E_0, E)$ the set of compact operators $B : E_0 \to E$. Fix $B_0 \in K(E, E)$. A real number μ is called a characteristic value of B_0 if $\dim\ker(I - \mu B_0) > 0$. Suppose now that $A_0 = I - B_0$ is an invertible operator. Denote by $\{\mu_1 < \mu_2 < \cdots < \mu_p\}$ the set of all characteristic values of B_0 contained in $[0, 1]$. We set $\operatorname{sgn} A_0 = (-1)^d$, where $d = \sum_{i=1}^{p} \dim\ker(I - \mu_i B_0)$.

Fix $B_1 \in K(E \times \mathbb{R}, E)$ and define $A = Q + B_1 : E \times \mathbb{R} \to E$. Assume that A is surjective. Since A is a Fredholm operator of index 1 and A is surjective, $\dim\ker A = 1$. Fix $v \in \ker A \setminus \{0\}$ and define a linear functional $\xi : E \times \mathbb{R} \to E$ such that $\xi(v) = 1$. Finally, define an operator $A^{\sim} : E \times \mathbb{R} \to E \times \mathbb{R}$ by $A^{\sim}(w) = (Aw, \xi(w))$ and $\operatorname{sgn}(A, v) = \operatorname{sgn}(A^{\sim})$.

Assume additionally that $f = Q + \Phi \in C^1(cl(\Omega), E)$. Suppose that $a \in \Omega$ is such that there is $k \in \mathbb{N}$ such that $S^1_a = \mathbb{Z}_k$, $f^{-1}(0) \cap \Omega = S^1 \cdot a \approx S^1/\mathbb{Z}_k$ and $Df(a) : E \times \mathbb{R} \to E$ is surjective. Let $f^{\mathbb{Z}_k} : E^{\mathbb{Z}_k} \times \mathbb{R} \to E^{\mathbb{Z}_k}$ denotes the restriction of f. Then $Df^{\mathbb{Z}_k}(a) : E^{\mathbb{Z}_k} \times \mathbb{R} \to E^{\mathbb{Z}_k}$ is also surjective. We denote by v the tangent vector to the orbit $S^1 \cdot a$ at a. Notice that $v \in \ker Df^{\mathbb{Z}_k}(a)$.

This theorem ensures the nontriviality of the S^1-index of the nondegenerate S^1-orbit $S^1 \cdot a$.

Theorem 2.2. (Cf. [2].) *Under the above assumptions* $\deg_{\mathbb{Z}_k}(f, \Omega) = \operatorname{sgn}(Df^{\mathbb{Z}_k}(a), v)$. *Moreover,* $\deg_{\mathbb{Z}_{k'}}(f, \Omega) = 0$ *for every* $k' > k$.

3. Functional setting

Throughout the rest of this article we assume that assumptions (A0)–(A2) are fulfilled. Moreover, we fix $\tau > 0$ satisfying assumptions of Theorem 1.1.

In this section we convert problem (1.4) to an equivalent problem (3.2). Next we define spaces on which we will work and define a homotopy F of S^1-invariant compact mappings. The study of periodic solutions of problem (3.2) is equivalent to the study of fixed S^1-orbits of the operator $F(\cdot, 1)$.

By a transformation of functions (cf. [4]), one can see that problem (1.4) is equivalent to the problem

$$\begin{cases} \dot u(t) = -(a_{11}u(t-\tau) + a_{12}v(t-\tau))(b_1 + u(t)), \\ \dot v(t) = -(a_{21}u(t-\tau) + a_{22}v(t-\tau))(b_2 + v(t)). \end{cases} \tag{3.1}$$

Let $(u, v) \in C(\mathbb{R}, \mathbb{R}) \times C(\mathbb{R}, \mathbb{R})$ be a pair of periodic functions with period T. Then by putting $\lambda = \frac{T}{2\pi}$, $x_1(t) = u(\lambda t)$ and $x_2(t) = v(\lambda t)$ for $t \in \mathbb{R}$, we have that $x = (x_1, x_2)$ is a 2π-periodic solution of problem

$$\begin{cases} \dot x_1(t) = -\lambda(a_{11}x_1(t-\tau/\lambda) + a_{12}x_2(t-\tau/\lambda))(b_1 + x_1(t)), \\ \dot x_2(t) = -\lambda(a_{21}x_1(t-\tau/\lambda) + a_{22}x_2(t-\tau/\lambda))(b_2 + x_2(t)). \end{cases} \tag{3.2}$$

We will study the existence of 2π-periodic solutions of (3.2) for some $\lambda > 0$ instead of looking for periodic solutions of (3.1). We note that each function $u : [0, 2\pi] \to \mathbb{R}$ with $u(0) = u(2\pi)$ is extended to a 2π-periodic function on $\mathbb{R}$. Therefore we identify a 2π-periodic function u on $\mathbb{R}$ with a function on $[0, 2\pi]$ with $u(0) = u(2\pi)$.

Define a Banach space

$$\widehat{E} = \left\{ x \in C([0, 2\pi], \mathbb{R}) \colon \int_0^{2\pi} \dot x(t)^2\, dt < \infty,\ x(0) = x(2\pi) \text{ and } \int_0^{2\pi} x(t)\, dt = 0 \right\}$$

with a norm $\|\cdot\|$ given by $\|x\|^2 = \int_0^{2\pi} \dot x(t)^2 + x(t)^2\, dt$, for $x \in \widehat{E}$. Moreover, we put $|u|_\infty = \sup\{|u(t)|\colon t \in [0, 2\pi]\}$ for $u \in \widehat{E}$. Put $E = \widehat{E} \times \widehat{E}$ and define an action $\rho : S^1 \times E \to E$ of the group S^1 as follows

$$\rho\big(e^{i\phi}, \big(x_1(t), x_2(t)\big)\big) = \big(x_1(t+\phi), x_2(t+\phi)\big) \mod 2\pi. \tag{3.3}$$

Define an open S^1-invariant subset $\Theta_0 \subset E$ as follows

$$\Theta_0 = \big\{(x_1, x_2) \in E\colon -b_i < x_i(t) \text{ for } t \in [0, 2\pi],\ i = 1, 2\big\}.$$

Next we define a homotopy of S^1-equivariant mappings $F : (E \times \mathbb{R}^+) \times [0, 1] \to E$ associated to problem (3.2) such that if $((x_1, x_2), \lambda) \in \Theta_0 \times \mathbb{R}^+$ satisfies $F((x_1, x_2), \lambda, 1) = (x_1, x_2)$, then $((x_1, x_2), \lambda)$ is a solution of problem (3.2).

Let $\beta : E \to [0, 1]$ be a continuous mapping and $\mathcal{N} : E \times [0, 1] \to E$ be a mapping defined by

$$\mathcal{N}\big((x_1, x_2), \theta\big)(t) = \begin{pmatrix} -(a_{11}x_1(t-\tau/\lambda) + a_{12}x_2(t-\tau/\lambda))(b_1 + \theta x_1(t)) \\ -(a_{21}x_1(t-\tau/\lambda) + a_{22}x_2(t-\tau/\lambda))(b_2 + \theta x_2(t)) \end{pmatrix}$$

for $((x_1, x_2), \theta) \in E \times [0, 1]$. We put

$$c_1\big((x_1, x_2), \theta\big) = \begin{pmatrix} c_{11}((x_1, x_2), \theta) \\ c_{12}((x_1, x_2), \theta) \end{pmatrix} = -\frac{\lambda}{2\pi} \int_0^{2\pi} \beta(x_1, x_2)\mathcal{N}\big((x_1, x_2), \theta\big)(s)\, ds \quad \text{and}$$

$$c_2\big((x_1,x_2),\theta\big)=\begin{pmatrix} c_{21}((x_1,x_2),\theta)\\ c_{22}((x_1,x_2),\theta)\end{pmatrix}$$
$$=-\frac{\lambda}{2\pi}\int\limits_0^{2\pi}\int\limits_0^{t}\beta(x_1,x_2)\mathcal{N}\big((x_1,x_2),\theta\big)(s)\,ds\,dt-\pi c_1\big((x_1,x_2),\theta\big).$$

We define a mapping $F:(E\times\mathbb{R}^+)\times[0,1]\to C(\mathbb{R},\mathbb{R})\times C(\mathbb{R},\mathbb{R})$ by

$$F\big(\big((x_1,x_2),\lambda\big),\theta\big)(t)$$
$$=-\lambda\int\limits_0^{t}\beta(x_1,x_2)\mathcal{N}\big((x_1,x_2),\theta\big)(s)\,ds-tc_1\big((x_1,x_2),\theta\big)-c_2\big((x_1,x_2),\theta\big). \tag{3.4}$$

From the definitions of c_1 and c_2, we can see that for all $((x_1,x_2,\lambda),\theta)\in(E\times\mathbb{R}^+)\times[0,1]$

$$F\big(\big((x_1,x_2),\lambda\big),\theta\big)(0)=F\big(\big((x_1,x_2),\lambda\big),\theta\big)(2\pi)\quad\text{and}\quad\int\limits_0^{2\pi}F\big(\big((x_1,x_2),\lambda\big),\theta\big)(t)\,dt=0$$

holds.

Summing up, $F(((x_1,x_2),\lambda),\theta)\in E$ for all $(((x_1,x_2),\lambda),\theta)\in(E\times\mathbb{R}^+)\times[0,1]$. It is also easy to see that $F:(E\times\mathbb{R}^+)\times[0,1]\to E$ is an S^1-equivariant compact mapping.

From the definition of F, we find that $(((x_1,x_2),\lambda),\theta)\in(\Theta_0\times\mathbb{R}^+)\times[0,1]$ satisfies

$$F\big(\big((x_1,x_2),\lambda\big),\theta\big)=Q\big((x_1,x_2),\lambda\big) \tag{3.5}$$

if and only if

$$\begin{cases}\dot{x}_1(t)=-\lambda\beta(x_1,x_2)(a_{11}x_1(t-\tau/\lambda)+a_{12}x_2(t-\tau/\lambda))(b_1+\theta x_1(t))-c_{11}((x_1,x_2),\theta),\\ \dot{x}_2(t)=-\lambda\beta(x_1,x_2)(a_{21}x_1(t-\tau/\lambda)+a_{22}x_2(t-\tau/\lambda))(b_2+\theta x_2(t))-c_{21}((x_1,x_2),\theta).\end{cases} \tag{3.6}$$

We claim that system (3.6) is equivalent to

$$\begin{cases}\dot{x}_1(t)=-\lambda\beta(x_1,x_2)(a_{11}x_1(t-\tau/\lambda)+a_{12}x_2(t-\tau/\lambda))(b_1+\theta x_1(t)),\\ \dot{x}_2(t)=-\lambda\beta(x_1,x_2)(a_{21}x_1(t-\tau/\lambda)+a_{22}x_2(t-\tau/\lambda))(b_2+\theta x_2(t)).\end{cases} \tag{3.7}$$

What is left is to show that $c_{i1}((x_1,x_2),\theta)=0$ for $i=1,2$.

Fix $i\in\{1,2\}$ and notice that

$$\frac{d}{dt}\ln\big(b_i+x_i(t)\big)=\frac{\dot{x}_i(t)}{b_i+x_i(t)}$$
$$=-\lambda\beta(x_1,x_2)\big(a_{i1}x_1(t-\tau/\lambda)+a_{i2}x_2(t-\tau/\lambda)\big)-\frac{c_{i1}((x_1,x_2),\theta)}{b_i+x_i(t)}.$$

Thus

$$\ln\bigl(b_i + x_i(t)\bigr) - \ln\bigl(b_i + x_i(s)\bigr)$$
$$= -\lambda \int_s^t \beta(x_1, x_2)\bigl(a_{i1}x_1(w - \tau/\lambda) + a_{i2}x_2(w - \tau/\lambda)\bigr)\, dw - \int_s^t \frac{c_{i1}((x_1, x_2), \theta)}{b_i + x_i(w)}\, dw$$

for $s, t \in \mathbb{R}$ with $s < t$. Since

$$x_i(2\pi) = x_i(0), \qquad \int_0^{2\pi} x_1(t)\, dt = \int_0^{2\pi} x_2(t)\, dt = 0, \qquad \int_0^{2\pi} \frac{c_{i1}((x_1, x_2), \theta)}{b_i + x_i(t)}\, dt = 0.$$

Finally, condition $x_i(t) > -b_i$ for all $t \in [0, 2\pi]$, implies $c_{i1}((x_1, x_2), \theta) = 0$, which completes the proof.

We finish this section with the following lemma which yields a priori estimates for periodic solutions of problem (3.8).

Lemma 3.1.

(1) *For $\lambda_1, \lambda_2 \in \mathbb{R}^+$ with $\lambda_1 < \lambda_2$, there exist positive numbers m_0, $\{d_i\}_{1 \leqslant i \leqslant 4}$ such that for each $\lambda \in [\lambda_1, \lambda_2]$, $\alpha \in [0, 1]$ and $\tau \geqslant 1$, each solution $(x_1, x_2) \in \Theta_0$ of the following problem*

$$\begin{cases} \dot{x}_1(t) = -\alpha\lambda(a_{11}x_1(t - \tau/\lambda) + a_{12}x_2(t - \tau/\lambda))(b_1 + x_1(t)), \\ \dot{x}_2(t) = -\alpha\lambda(a_{21}x_1(t - \tau/\lambda) + a_{22}x_2(t - \tau/\lambda))(b_2 + x_2(t)), \end{cases} \tag{3.8}$$

satisfies $|\dot{x}_i|_\infty < m_0$, $|\ddot{x}_i|_\infty < m_0$ for $i = 1, 2$, and

$$-b_1 < -d_1 < x_1(t) < d_3, \qquad -b_2 < -d_2 < x_2(t) < d_4 \quad \text{on } [0, 2\pi];$$

(2) *there exists $\alpha_0 \in (0, 1)$ such that there is no nontrivial solution of (3.8) for $\alpha \in [0, \alpha_0]$.*

Proof. (1) Let $\lambda_1, \lambda_2 \in \mathbb{R}^+$ with $\lambda_1 < \lambda_2$ and $\lambda \in [\lambda_1, \lambda_2]$. Let $(x_1, x_2) \in \Theta_0$ be a solution of (3.8). Then for $i \in \{1, 2\}$ we find that

$$\ln\bigl(b_i + x_i(t)\bigr) - \ln\bigl(b_i + x_i(s)\bigr) = -\alpha\lambda \int_s^t \sum_{j=1}^2 a_{ij}x_j(w - \tau/\lambda)\, dw$$

for $t, s \in \mathbb{R}$ with $s \leqslant t$. Let $s \in \mathbb{R}$ such that $x_i(s) = 0$. Then we have

$$b_i + x_i(t) = b_i \exp\left(-\alpha\lambda \int_s^t \sum_{j=1}^2 a_{ij}x_j(w - \tau/\lambda)\, dw\right).$$

Then noting that $x_i(t) > -b_i$ for $i = 1, 2$, and taking into account assumption (A0) we obtain

$$x_1(t) < d_3 = b_1 \exp\bigl(2\pi\lambda(a_{11}b_1 + a_{12}b_2)\bigr) - b_1 \quad \text{for all } t \in [0, 2\pi]. \tag{3.9}$$

Similarly as above we obtain

$$x_2(t) < d_4 = b_2 \exp\bigl(2\pi\lambda(a_{21}b_1 + a_{22}b_2)\bigr) - b_2 \quad \text{for all } t \in [0, 2\pi]. \tag{3.10}$$

On the other hand, for all $t \in [0, 2\pi]$ we have

$$-b_1 < -d_1 = b_1 \exp\bigl(-2\pi\lambda(a_{11}d_3 + a_{12}d_4)\bigr) - b_1 < x_1(t) \quad \text{and}$$
$$-b_2 < -d_2 = b_2 \exp\bigl(-2\pi\lambda(a_{21}d_3 + a_{22}d_4)\bigr) - b_2 < x_2(t).$$

From (3.8) and the inequalities above we find that

$$\bigl|\dot{x}_i(t)\bigr| \leqslant \max_{i=1,2} C\bigl(|a_{i1}|d_3 + |a_{i2}|d_4\bigr)(b_i + d_{i+2}) \quad \text{for } t \in [0, 1] \text{ and } i = 1, 2. \tag{3.11}$$

We also have by differentiating the both sides of (3.8) and using the inequalities above that $\{|\ddot{x}_i(t)|\colon t \in [0, 2\pi],\ i = 1, 2\}$ is bounded, which completes the proof of (1).

(2) Suppose that there exists a sequence $\{\alpha_n\} \subset \mathbb{R}^+$ and $\{(x_{1n}, x_{2n})\} \subset E$ such that $\lim_{n\to\infty} \alpha_n = 0$ and each (x_{1n}, x_{2n}) is a solution of (3.8) with $\alpha = \alpha_n$. Then (3.9) and (3.10) holds with x_1 and x_2 replaced by x_{1n} and x_{2n}, respectively. Then we have that $\lim_{n\to\infty} |x_{in}|_\infty = 0$, $i = 1, 2$. By subtracting subsequences, we may assume, without any loss of generality, that $|x_{1n}|_\infty \geqslant |x_{2n}|_\infty$ for all $n \geqslant 1$. We put $u_{in}(t) = x_{in}/|x_{1n}|_\infty$ for $i = 1, 2$. Then we have that $u_{in} \in H$ for $n \geqslant 1$ and $i = 1, 2$. We also have $|u_{1n}|_\infty = 1$ for all $n \geqslant 1$. Then it follows that for n sufficiently large $|\dot{u}_{1n}(t)| \leqslant 2b_1\alpha_n\lambda(a_{11} + a_{12})$ for all $t \in [0, 2\pi]$. That is $\lim_{n\to\infty} |\dot{u}_{1n}|_\infty = 0$. This contradicts to the fact that $|u_{1n}|_\infty = 1$ for all $n \geqslant 1$, which completes the proof of (2). □

4. Homotopies of admissible S^1-equivariant mappings

The aim of this section is to define an open, bounded S^1-invariant subset $\Omega_{\lambda_1,\lambda_2} \subset \Theta_0 \times \mathbb{R}^+ \subset E \times \mathbb{R}^+$ such that the homotopy $Q - F(\cdot, \theta)$, defined by (3.5), does not vanish on $\partial\Omega_{\lambda_1,\lambda_2}$. We underline that solutions of equation $Q((x_1, x_2), \lambda) = F(((x_1, x_2), \lambda), 1)$ in $\Omega_{\lambda_1,\lambda_2}$ are exactly the periodic solutions of problem (3.2) in $\Omega_{\lambda_1,\lambda_2}$.

We finish this section with Lemma 4.4, where we reduce the computation of the S^1-degree of $Q - F(\cdot, 1)$ on $\Omega_{\lambda_1,\lambda_2}$ to the computation of the S^1-degree of $Q - F(\cdot, 0)$ on $\Omega_{\lambda_1,\lambda_2}$.

We first consider the following eigenvalue problem associated with problem (3.2)

$$\begin{cases} \dot{u}(t) = -\gamma\lambda b_1(a_{11}u(t - \tau/\lambda) + a_{12}v(t - \tau/\lambda)), \\ \dot{v}(t) = -\gamma\lambda b_2(a_{21}u(t - \tau/\lambda) + a_{22}v(t - \tau/\lambda)), \end{cases} \tag{4.1}$$

where $\lambda, \tau > 0$, $\gamma \in \mathbb{R}$ and $(u, v) \in E$. By assumption (1.5) $\mu_1 \neq \mu_2$. Hence from assumption (A2) it follows that there exists a nondegenerate matrix P such that

$$P\begin{bmatrix} a_{11}b_1 & a_{12}b_1 \\ a_{21}b_2 & a_{22}b_2 \end{bmatrix}P^{-1} = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}. \tag{4.2}$$

Then linear problem (4.1) is transformed into the form

$$\begin{cases} \dot{u_1}(t) = -\gamma\lambda\mu_1 u_1(t-\tau/\lambda), \\ \dot{u_2}(t) = -\gamma\lambda\mu_2 u_2(t-\tau/\lambda), \end{cases} \tag{4.3}$$

by putting $(u_1, u_2) = P^{-1}(u, v)$.

Let $i = 1, 2$. We put $u_i(t) = \sum_{k=1}^{\infty}(c_k \cos kt + s_k \sin kt)$, where $\{c_k\}, \{s_k\} \subset \mathbb{R}$. Then $\dot{u}_i(t) = \sum_{k=1}^{\infty}((ks_k)\cos kt + (-kc_k)\sin kt)$ and

$$\begin{aligned} &-\gamma\lambda\mu_i u(t-\tau') \\ &\quad = -\gamma\lambda\mu_i \sum_{k=1}^{\infty}\bigl\{c_k(\cos kt\cos k\tau' + \sin k\tau' \sin kt) + s_k(\cos k\tau' \sin kt - \sin k\tau' \cos kt)\bigr\} \\ &\quad = -\gamma\lambda\mu_i \sum_{k=1}^{\infty}\bigl\{(c_k \cos k\tau' - s_k \sin k\tau')\cos kt + (c_k \sin k\tau' + s_k \cos k\tau')\sin kt\bigr\}, \end{aligned}$$

where $\tau' = \tau/\lambda$. If u_1 is a nontrivial solution of (4.3), then

$$\begin{aligned} ks_k &= -\gamma\lambda\mu_1(c_k \cos k\tau' - s_k \sin k\tau'), \\ -kc_k &= -\gamma\lambda\mu_1(c_k \sin k\tau' + s_k \cos k\tau') \end{aligned}$$

for all $k \in \mathbb{N}$. That is we obtain the following system of linear equations

$$c_k \cos k\tau' - s_k\left(\sin k\tau' - \frac{k}{\gamma\lambda\mu_1}\right) = 0, \qquad c_k\left(\sin k\tau' - \frac{k}{\gamma\lambda\mu_1}\right) + s_k \cos k\tau' = 0$$

for all $k \in \mathbb{N}$. Then $\cos^2 k\tau' + (\sin k\tau' - \frac{k}{\gamma\lambda\mu_1})^2 = 0$ and therefore we find that

$$\frac{k}{\gamma\lambda\mu_1} = 1, \qquad \frac{k\tau}{\lambda} = \frac{\pi}{2} + 2n\pi, \quad \text{for some } n \in \mathbb{N} \cup \{0\}. \tag{4.4}$$

If u_2 is a nontrivial solution of (4.3), then by the same argument as above we obtain that

$$\frac{k}{\gamma\lambda\mu_2} = 1, \qquad \frac{k\tau}{\lambda} = \frac{\pi}{2} + 2n\pi, \quad \text{for some } n \in \mathbb{N} \cup \{0\}. \tag{4.5}$$

Consequently, we have that the eigenvalue γ of problem (4.3) is of the form

$$\gamma = \frac{1}{\mu_i \tau}\left(\frac{\pi}{2} + 2n\pi\right), \quad i = 1, 2 \text{ and } n \in \mathbb{N}. \tag{4.6}$$

Based on the observation above we obtain the following lemma.

Lemma 4.1. *Let $\lambda_1, \lambda_2 \in \mathbb{R}^+$ with $\lambda_1 < \lambda_2$. Then the origin is an isolated solution of* (3.2)*, i.e., there exists $m_1 > 0$ such that if $((x_1, x_2), \lambda) \in \Theta_0 \times [\lambda_1, \lambda_2]$ is a nontrivial solution of* (3.2) *then $(x_1, x_2) \notin \{(x_1, x_2) \in \Theta_0\colon\ \|x_1\| \leqslant m_1,\ \|x_2\| \leqslant m_1\}$.*

Proof. Let $\lambda_1, \lambda_2 \in \mathbb{R}^+$ with $\lambda_1 < \lambda_2$. Suppose that there exists a sequence $\{((x_{1n}, x_{2n}), \lambda_n)\} \subset E \times \mathbb{R}^+$ such that each $((x_{1n}, x_{2n}), \lambda_n)$ is a solution of (3.2) and

$$\lim_{n\to\infty} \|x_{1n}\| = \lim_{n\to\infty} \|x_{2n}\| = 0.$$

We may assume that $\lim_{n\to\infty} \lambda_n = \lambda_0 \in [\lambda_1, \lambda_2]$ and $|x_{2n}|_\infty \leqslant |x_{1n}|_\infty$ for all $n \geqslant 1$. We put $u_{in}(t) = x_{in}/|x_{1n}|_\infty$ for each $n \geqslant 1$ and $i = 1, 2$. Then we have

$$\begin{cases} \dot{u}_{1n}(t) = -\lambda_n (a_{11} u_{1n}(t - \tau/\lambda_n) + a_{12} u_{2n}(t - \tau/\lambda_n))(b_1 + x_{1n}(t)), \\ \dot{u}_{2n}(t) = -\lambda_n (a_{21} u_{1n}(t - \tau/\lambda_n) + a_{22} u_{2n}(t - \tau/\lambda_n))(b_2 + x_{2n}(t)). \end{cases}$$

Then one can see that $\sup\{\|\dot{u}_{in}\|\colon n \geqslant 1,\ i = 1, 2\} < \infty$. By differentiating the equalities above, we also have that $\sup\{\|\ddot{u}_{in}\|\colon n \geqslant 1,\ i = 1, 2\} < \infty$. Therefore we may assume that for each i, $u_{in} \to u_i$ and $\dot{u}_{in} \to \dot{u}_i$ strongly in E. Then we have

$$\begin{cases} \dot{u}_1(t) = -\lambda_0 b_1 (a_{11} u_1(t - \tau/\lambda_0) + a_{12} u_2(t - \tau/\lambda_0)), \\ \dot{u}_2(t) = -\lambda_0 b_2 (a_{21} u_1(t - \tau/\lambda_0) + a_{22} u_2(t - \tau/\lambda_0)). \end{cases} \tag{4.7}$$

That is, (4.1) holds with $\gamma = 1$. By assumption (1.5), we have that (4.6) does not hold with $\gamma = 1$. Therefore problem (4.7) has no nontrivial solution. Then $u_1 \equiv 0$. This contradicts the definition of u_1. □

Now fix $\lambda_1, \lambda_2 \in \mathbb{R}^+$ with $\lambda_1 < \lambda_2$ and $\alpha_0, m_0, m_1, \{d_i\}_{1\leqslant i\leqslant 4}$ be the positive numbers satisfying the assertion of Lemmas 3.1 and 4.1. We may assume without any loss of generality that $d_{i+2} > (b_i + d_i)/2$ for $i = 1, 2$. Define open bounded S^1-invariant subsets

$$\Theta_M = \left\{(x_1, x_2) \in E\colon -\frac{b_i + d_i}{2} < x_i(t) < 2d_{i+2},\ \text{for } i = 1, 2,\ t \in [0, 2\pi]\right\},$$
$$\widetilde{\Theta}_M = \big\{(x_1, x_2) \in E\colon -d_i < x_i(t) < d_{i+2},\ \text{for } i = 1, 2,\ t \in [0, 2\pi]\big\},$$

and closed S^1-invariant subset as follows

$$\Theta_{m_1} = \big\{(x_1, x_2) \in E\colon \|x_i\| \leqslant m_1/2,\ i = 1, 2\big\}.$$

Since $b_i > d_i$, $i = 1, 2$, and $m_1 > 0$ can be chosen sufficiently small

$$\Theta_{m_1} \subsetneq \widetilde{\Theta}_M \subsetneq \Theta_M \subsetneq \Theta_0.$$

Moreover, define open bounded S^1-invariant subsets in the following way

$$\Theta_1 = \left\{(x_1, x_2) \in E\colon \int_0^{2\pi} \dot{x}_i(t)^2\, dt < 2\pi m_0^2,\ i = 1, 2\right\}, \qquad \widetilde{\Theta} = \big(\widetilde{\Theta}_M \cap \Theta_1\big) \setminus \Theta_{m_1},$$
$$\Theta = (\Theta_M \cap \Theta_1) \setminus \Theta_{m_1}, \tag{4.8}$$

and notice that $\widetilde{\Theta}_M \subset cl(\widetilde{\Theta}_M) \subset \Theta$. Then we can choose $\delta_0 > 0$ such that $\operatorname{dist}^2(\widetilde{\Theta}_M, \partial\Theta) \geqslant \delta_0$. Let $\xi : [0, +\infty) \to (0, 1]$ be a smooth function such that

$$\xi(t) = \begin{cases} \alpha_0 & \text{for } t = 0, \\ \text{strictly increasing} & \text{for } 0 < t < \delta_0, \\ 1 & \text{for } t \geqslant \delta_0. \end{cases} \tag{4.9}$$

We put that

$$\beta(x_1, x_2) = \xi\big(\mathrm{dist}^2\big((x_1, x_2), \partial\Theta\big)\big) \quad \text{for } (x_1, x_2) \in E. \tag{4.10}$$

Then $\beta \in C^1(E; \mathbb{R})$ and we have

$$\beta(x_1, x_2) = \begin{cases} 1 & \text{for } (x_1, x_2) \in \Theta_{m_1}, \\ \alpha_0 & \text{for } (x_1, x_2) \in cl(E \setminus (\Theta_M \cap \Theta_1)). \end{cases} \tag{4.11}$$

Put $\Omega_{\lambda_1,\lambda_2} = \Theta \times (\lambda_1, \lambda_2)$ and

$$\mathcal{S} = \big\{\big(\big((x_1, x_2), \lambda\big), \theta\big) \in \Omega_{\lambda_1,\lambda_2} \times [0, 1]\colon\ F\big(\big((x_1, x_2), \lambda\big), \theta\big) = Q\big((x_1, x_2), \lambda\big)\big\}.$$

Lemma 4.2. *Under the above assumptions* $\mathcal{S} \cap ((\partial\Theta \times (\lambda_1, \lambda_2)) \times [0, 1]) = \emptyset$.

Proof. Suppose that $(((x_1, x_2), \lambda), \theta) \in \mathcal{S}$. Then (3.6) holds for $(((x_1, x_2), \lambda), \theta)$. Multiplying (3.6) by θ, and denoting $u_i(t) = \theta x_i(t)$ for $i = 1, 2$, we find that

$$\begin{cases} \dot{u}_1(t) = -\lambda\beta(x_1, x_2)(a_{11}u_1(t - \tau/\lambda) + a_{12}u_2(t - \tau/\lambda))(b_1 + u_1(t)), \\ \dot{u}_2(t) = -\lambda\beta(x_1, x_2)(a_{21}u_1(t - \tau/\lambda) + a_{22}u_2(t - \tau/\lambda))(b_2 + u_2(t)), \end{cases}$$

holds. By Lemma 3.1 we obtain $\int_0^{2\pi} \dot{x}_i(t)^2\, dt < 2\pi m_0^2$, $i = 1, 2$. Then we have $(x_1, x_2) \notin \partial\Theta_1$.

If $(x_1, x_2) \in \partial\Theta_M$, then we have that $\beta(x_1, x_2) = \alpha_0$. Then by (2) of Lemma 3.1, we have that $u_1 \equiv u_2 \equiv 0$. This contradicts to $(x_1, x_2) \in cl(\Theta)$. If $(x_1, x_2) \in \partial\Theta_{m_1}$, then $\beta(x_1, x_2) = 1$ and $\|u_i\| < m_1/2$, for $i = 1, 2$. Then by Lemma 4.1, we have that $x_1 \equiv x_2 \equiv 0$. Thus we have that $\mathcal{S} \cap \partial\Omega_{\lambda_1,\lambda_2} = \emptyset$, which completes the proof. □

The following result is known. For completeness, we give a proof.

Lemma 4.3. *Suppose that* $\tau = 0$. *Then problem* (3.6) *does not have nonstationary periodic solution* $((x_1, x_2), \lambda) \in E \times \mathbb{R}^+$ *for any* $\theta \in [0, 1]$.

Proof. Let $\theta \in [0, 1]$ and $((x_1, x_2), \lambda) \in \Theta_0 \times \mathbb{R}^+$ satisfy (3.6). We first consider the case that $\theta > 0$. Since $\tau = 0$, problem (3.6) reduces to the problem

$$\begin{cases} \dot{x}_1(t) = -\lambda(a_{11}x_1(t) + a_{12}x_2(t))(b_1 + \theta x_1(t)), \\ \dot{x}_2(t) = -\lambda(a_{21}x_1(t) + a_{22}x_2(t))(b_2 + \theta x_2(t)). \end{cases} \tag{4.12}$$

We integrate the both sides of (4.12) from 0 to 2π. Then by the periodicity, we have

$$\begin{cases} a_{11}\int_0^{2\pi} x_1(t)^2\, dt + a_{12}\int_0^{2\pi} x_1(t)x_2(t)\, dt = 0, \\ a_{21}\int_0^{2\pi} x_1(t)x_2(t)\, dt + a_{22}\int_0^{2\pi} x_2(t)^2\, dt = 0. \end{cases}$$

Then one can see that $x_1 = x_2 \equiv 0$ from condition (A1). We next consider the case that $\theta = 0$. In this case we multiply Eqs. (4.12) by x_i and integrate over $[0, 2\pi]$. Then we have the equalities above. This completes the proof. □

Lemma 4.4. *Suppose that* $\lambda_1 = \frac{\tau}{2j_1\pi} < \lambda_2 = \frac{\tau}{2j_2\pi}$, *where* $j_1, j_2 \in \mathbb{N}$. *Then*

$$\mathrm{Deg}\big(Q - F(\cdot, 0), \Omega_{\lambda_1,\lambda_2}\big) = \mathrm{Deg}\big(Q - F(\cdot, 1), \Omega_{\lambda_1,\lambda_2}\big).$$

Proof. To prove the assertion, it is sufficient to show that there exists no solution of (3.6) in $\partial\Omega_{\lambda_1,\lambda_2} = \partial(\Theta \times (\lambda_1, \lambda_2)) = cl(\Theta) \times \{\lambda_1, \lambda_2\} \cup \partial\Theta \times (\lambda_1, \lambda_2)$. We first see that there exists no solution on $cl(\Theta) \times \{\lambda_1, \lambda_2\}$. From the definitions of λ_1, λ_2, we have that problem (3.6) is equivalent to (4.12) with $\lambda = \lambda_1$ or $\lambda = \lambda_2$. Then by Lemma 4.3, we find that $x_1 = x_2 = 0$. This contradicts to the assumption that $(x_1, x_2) \in cl(\Theta)$. We also have by Lemma 4.2 that there exists no solution of (3.6) in $\partial\Theta \times (\lambda_1, \lambda_2)$, which completes the proof. □

5. Proof of Theorem 1.1

Throughout this section we assume that $\lambda_1 = \frac{\tau}{2j_1\pi} < \lambda_2 = \frac{\tau}{2j_2\pi}$, where $j_1, j_2 \in \mathbb{N}$ and put $\Omega_{\lambda_1,\lambda_2} = \Theta \times (\lambda_1, \lambda_2)$. From Theorem 2.1 and Lemma 4.4 it follows that to finish the proof of Theorem 1.1, it is sufficient to show that $\mathrm{Deg}(Q - F(\cdot, 0), \Omega_{\lambda_1,\lambda_2}) \neq \Theta \in \Gamma$. But the mapping $F(\cdot, 0)$ is still too complicated to calculate the S^1-degree. Therefore we will provide another homotopy G of S^1-equivariant compact mappings such that $F(\cdot, 0) = G(\cdot, 0)$ and the S^1-degree of $Q - G(\cdot, 1)$ on $\Omega_{\lambda_1,\lambda_2}$ is easy to compute.

We fix a C^1-mapping $\sigma : [\lambda_1, \lambda_2] \to [\lambda_1, \lambda_2]$ such that σ is increasing on $[\lambda_1, \lambda_2]$ with $\sigma(\lambda_1) = \lambda_1$ and $\sigma(\lambda_2) = \lambda_2$, and

$$\sigma(\lambda_{k,n}) = \lambda_{k,n} \quad \text{and} \quad \dot{\sigma}(\lambda_{k,n}) = 0 \quad \text{for each } \lambda_{k,n} = \frac{k\tau}{\frac{\pi}{2} + 2n\pi} \in [\lambda_1, \lambda_2],\ k, n \in \mathbb{N}.$$

We now define a homotopy of S^1-equivariant mappings $G : \Omega_{\lambda_1,\lambda_2} \times [0, 1] \to E$ by

$$G\big(\big((x_1, x_2), \lambda\big), \theta\big) = -\big(\theta\sigma(\lambda) + (1 - \theta)\lambda\big) \int_0^t \beta(x_1, x_2)\mathcal{N}\big((x_1, x_2), 0\big)\, ds. \tag{5.1}$$

By definition of $\mathcal{N}(\cdot, 0)$, we have $G(((x_1, x_2), \lambda), \theta) \in E$ for $(((x_1, x_2), \lambda), \theta) \in (E \times \mathbb{R}^+) \times [0, 1]$. If $(((x_1, x_2), \lambda), \theta) \in \Omega_{\lambda_1,\lambda_2} \times [0, 1]$ satisfies $Q((x_1, x_2), \lambda) = G(((x_1, x_2), \lambda), \theta)$ then

$$\begin{cases} \dot{x}_1(t) = -(\theta\sigma(\lambda) + (1 - \theta)\lambda)b_1\beta(x_1, x_2)(a_{11}x_1(t - \tau/\lambda) + a_{12}x_2(t - \tau/\lambda)), \\ \dot{x}_2(t) = -(\theta\sigma(\lambda) + (1 - \theta)\lambda)b_2\beta(x_1, x_2)(a_{21}x_1(t - \tau/\lambda) + a_{22}x_2(t - \tau/\lambda)). \end{cases} \tag{5.2}$$

Lemma 5.1. *Under the above assumptions*:

$$\mathrm{Deg}\big(Q - F(\cdot, 0), \Omega_{\lambda_1,\lambda_2}\big) = \mathrm{Deg}\big(Q - G(\cdot, 1), \Omega_{\lambda_1,\lambda_2}\big).$$

Proof. By the same argument as in the proof of Lemma 4.4, we see that

$$\mathrm{Deg}\big(Q - G(\cdot,0), \Omega_{\lambda_1,\lambda_2}\big) = \mathrm{Deg}\big(Q - G(\cdot,1), \Omega_{\lambda_1,\lambda_2}\big).$$

Then since $G(\cdot,0) = F(\cdot,0)$, we have by Lemma 4.4 that the assertion holds. □

For $n,m \in \mathbb{N}$ define $\Phi(n,m) = \{j \in \mathbb{N}\colon [\frac{n}{j}] < [\frac{m}{j}] = \frac{m}{j}\}$. Notice that if $n < m$ then $\Phi(n,m) \neq \emptyset$.

The following lemma plays crucial role in our article.

Lemma 5.2. *Let assumptions of Theorem* 1.1 *be fulfilled. If* $n_1 < n_2$, $j \in \Phi(n_1,n_2)$ *and* $\lambda_1 = \frac{\tau}{2(j+1)\pi}$, $\lambda_2 = \frac{\tau}{2j\pi}$ *then* $\mathrm{Deg}(Q - G(\cdot,1), \Omega_{\lambda_1,\lambda_2}) \neq \Theta \in \Gamma$.

Proof. Before we prove this lemma, we outline the main steps of the proof. Namely, we will prove that $(Q - G(\cdot,1))^{-1}(0) \cap \Omega_{\lambda_1\lambda_2}$ consists of a finite number of nondegenerate orbits $S^1 \cdot a_1, \ldots, S^1 \cdot a_p$. Since these orbits are nondegenerate, there are open bounded S^1-invariant subsets $U_i \subset cl(U_i) \subset \Omega_{\lambda_1\lambda_2}$, $i = 1,\ldots,p$, such that $(Q - G(\cdot,1))^{-1}(0) \cap U_i = S^1 \cdot a_i$, $i = 1,\ldots,p$. Moreover, we will prove that there are $k_0 \in \mathbb{N}$ and $1 \leqslant i_0 \leqslant p$ such that $S^1_{a_{i_0}} = \mathbb{Z}_{k_0}$ and $S^1_{a_i} \neq \mathbb{Z}_{k_0}$ for every $i \neq i_0$.

By Theorem 2.1 we obtain

$$\mathrm{Deg}\big(Q - G(\cdot,1), \Omega_{\lambda_1,\lambda_2}\big) = \mathrm{Deg}\big(Q - G(\cdot,1), U_1\big) + \cdots + \mathrm{Deg}\big(Q - G(\cdot,1), U_p\big) \in \Gamma.$$

From the above and Theorem 2.2 we obtain that

$$\begin{aligned}\deg_{\mathbb{Z}_{k_0}}\big(Q - G(\cdot,1), \Omega_{\lambda_1,\lambda_2}\big) &= \deg_{\mathbb{Z}_{k_0}}\big(Q - G(\cdot,1), U_1\big) + \cdots + \deg_{\mathbb{Z}_{k_0}}\big(Q - G(\cdot,1), U_p\big)\\ &= \deg_{\mathbb{Z}_{k_0}}\big(Q - G(\cdot,1), U_{i_0}\big) \neq 0 \in \mathbb{Z}.\end{aligned}$$

Let us begin the proof. First of all notice that since $\mu_1 \neq \mu_2$ applying change of coordinates (4.2) to system (5.2) we obtain the following equivalent system:

$$\begin{cases} \dot{x}_1(t) = -\sigma(\lambda)\beta(P^{-1}(x_1,x_2))\mu_1 x_1(t - \tau/\lambda),\\ \dot{x}_2(t) = -\sigma(\lambda)\beta(P^{-1}(x_1,x_2))\mu_2 x_2(t - \tau/\lambda).\end{cases} \tag{5.3}$$

Notice that system (5.3) does not have solutions on $\partial(P\Omega_{\lambda_1,\lambda_2}) = \partial(P\Theta \times (\lambda_1,\lambda_2))$. Since $\beta(P^{-1}(x_1,x_2)) = \alpha_0$ for any $(x_1,x_2) \in P(cl(E \setminus (\Theta_M \cap \Theta_1)))$, we find that (5.3) does not have solutions on $P(E \setminus (\Theta_M \cap \Theta_1)) \times (\lambda_1,\lambda_2)$.

Therefore we can choose $R \gg r > 0$ such that

$$\begin{aligned}&\mathrm{Deg}\big(Q - PG(\cdot,1)P^{-1}, P\Theta \times (\lambda_1,\lambda_2)\big)\\ &\quad = \mathrm{Deg}\big(Q - PG(\cdot,1)P^{-1}, \big(D_R(E) \setminus cl\big(D_r(E)\big) \times (\lambda_1,\lambda_2)\big)\big),\end{aligned}$$

where

$$P\big(cl(\Theta_M \cap \Theta_1)\big) \subset D_R(E) = \big\{x \in E\colon \|x\| < R\big\},$$
$$cl\big(D_r(E)\big) = \big\{x \in E\colon \|x\| \leqslant r\big\} \subset P(\Theta_{m_1}).$$

Here we replace β by a function for which the calculation of degree is easier.

Let $\tilde{\xi} \in C^\infty([0,+\infty),[0,1])$ with $\tilde{\xi}(t) = 1$ for $0 \leqslant t \leqslant \sqrt{r}$, $\tilde{\xi}(t) = \alpha_0$ for $t \in [\sqrt{R},+\infty)$ and $\tilde{\xi}$ is strictly monotone decreasing on $[\sqrt{r},\sqrt{R}\,]$. Define $\tilde{\beta}: E \to [\alpha_0, 1]$ as follows

$$\tilde{\beta}(x_1,x_2) := \tilde{\xi}\big(\big\|(x_1,x_2)\big\|^2\big). \tag{5.4}$$

We denote by $\widetilde{G}(\cdot,1)$ the mapping $G(\cdot,1)$ with β replaced by $\tilde{\beta}$. Since maps $\beta, \tilde{\beta}$ coincide on $\partial(D_R(E) \setminus cl(D_r(E)))$, $\widetilde{G}(\cdot,1)|_{\partial(D_R(E)\setminus cl(D_r(E)))} = G(\cdot,1)|_{\partial(D_R(E)\setminus cl(D_r(E)))}$ and by the homotopy invariance of the S^1-degree we have

$$\begin{aligned}&\operatorname{Deg}\big(Q - PG(\cdot,1)P^{-1}, P\Theta \times (\lambda_1,\lambda_2)\big)\\ &\quad= \operatorname{Deg}\big(Q - P\widetilde{G}(\cdot,1)P^{-1}, \big(D_R(E) \setminus cl\big(D_r(E)\big) \times (\lambda_1,\lambda_2)\big)\big).\end{aligned}$$

For the simplicity of notation we will denote $\widetilde{G}(\cdot,1)$, $\tilde{\beta}$ and $\tilde{\xi}$ by $G(\cdot,1)$, β and ξ, respectively.

If $a(t) = (a_1(t), a_2(t))$ satisfies (5.3), then we have by (4.6) that $\lambda = \lambda_{k,n} = \frac{k\tau}{\pi/2+2n\pi}$ for some $k,n \in \mathbb{N}$ and $a_i(t) \in \operatorname{span}\{\cos kt, \sin kt\}$ for $i = 1,2$. Then by the definition of σ, we find that $\sigma(\lambda_{k,n}) = \lambda_{k,n} = \frac{k\tau}{\pi/2+2n\pi}$ for some $k,n \geqslant 1$.

Now suppose that $a_1 \not\equiv 0$. Then since $\sigma(\lambda_{k,n})\beta(a_1,a_2)\mu_1 = k$ and $\tau\mu_1 \neq \pi/2 + 2n\pi$ for $n \geqslant 0$, we find that $\beta(a_1,a_2) < 1$. Then taking into account (1.5) we obtain

$$k < \lambda_{k,n}\mu_1 = \frac{k\tau\mu_1}{\frac{\pi}{2} + 2n\pi} < k\frac{\frac{\pi}{2} + 2(n_1+1)\pi}{\frac{\pi}{2} + 2n\pi}.$$

Therefore we have $n \leqslant n_1$. On the other hand, we have by the definition that

$$\frac{\tau}{2(j+1)\pi} \leqslant \lambda_{k,n} = \frac{k\tau}{(\frac{\pi}{2} + 2n\pi)} \leqslant \frac{\tau}{2j\pi}, \tag{5.5}$$

which is equivalent to $kj \leqslant n < k(j+1)$.

Therefore $1 \leqslant k \leqslant [n_1/j]$. Then noting that $\mu_1 \neq \mu_2$, we have that

$$\big(a(t),\lambda\big) = \big(\big(a_1(t),a_2(t)\big),\lambda\big) = \big((c_{1,k}\cos kt, 0),\lambda_{k,n}\big) \quad \text{for some } 1 \leqslant k \leqslant [n_1/j],\ 1 \leqslant n \leqslant n_1,$$

where $c_{1,k} > 0$ is such that $\beta(c_{1,k}\cos kt, 0)\lambda_{k,n}\mu_1 = k$.

Similarly, we have that if $a_2 \not\equiv 0$,

$$\big(a(t),\lambda\big) = \big(\big(a_1(t),a_2(t)\big),\lambda\big) = \big((0, c_{2,k}\cos kt),\lambda_{k,n}\big) \quad \text{for some } 1 \leqslant k \leqslant [n_2/j],\ 1 \leqslant n \leqslant n_2,$$

where $c_{2,k} > 0$ is such that $\beta(0, c_{2,k}\cos kt)\lambda_{k,n}\mu_2 = k$.

It is clear, that the map $s \to \beta(su)$ is decreasing for any $u \in D_R(E) \setminus cl(D_r(E))$. Then since $\beta(a_1,a_2) < 1$, the map $s \to \beta(sa_1, sa_2)$ is strictly decreasing in $[1-\varepsilon, 1+\varepsilon]$.

This implies that each $\{(\rho(e^{i\theta}, (a_1(t),a_2(t))),\lambda)\colon \theta \in [0,2\pi)\}$ is an isolated orbit satisfying (5.3).

Now fix $(a_0(t), \lambda_{k,n}) = ((a_1(t), a_2(t)), \lambda_{k,n}) = ((c_{1,k}\cos kt, 0), \lambda_{k,n})$, where $1 \leqslant k \leqslant [n_1/j]$ and $1 \leqslant n < n_1$. Then $(\dot a_0(t), 0) = ((\dot a_1(t), \dot a_2(t)), 0) = ((-c_{1,k}k\sin kt, 0), 0)$ is the tangent vector to the orbit $S^1 \cdot (a_0, \lambda_{k,n})$ at $(a_0, \lambda_{k,n})$.

Summing up, we have proved that $(Q - G(\cdot, 1))^{-1}(0)$ consists of a finite number of S^1-orbits. Below we prove that these orbits are nondegenerate.

For simplicity of notation we put $x = (x_1(t), x_2(t))$. Then

$$f(x,\lambda) = \begin{pmatrix} f_1(x,\lambda) \\ f_2(x,\lambda) \end{pmatrix} = \begin{pmatrix} \int_0^t -\sigma(\lambda)\beta(x)\mu_1 x_1(s - \frac{\tau}{\lambda})\,ds \\ \int_0^t -\sigma(\lambda)\beta(x)\mu_2 x_2(s - \frac{\tau}{\lambda})\,ds \end{pmatrix},$$
$$D_x f = \begin{pmatrix} D_x f_1 \\ D_x f_2 \end{pmatrix}, \qquad D_\lambda f = \begin{pmatrix} D_\lambda f_1 \\ D_\lambda f_2 \end{pmatrix}. \tag{5.6}$$

Then we obtain

$$\begin{aligned} &D_x f(a_0, \lambda_{k,n})(v) \\ &= -\int_0^t \begin{pmatrix} kv_1(s - \frac{\tau}{\lambda_{k,n}}) + 2\lambda_{k,n}\mu_1 a_1(s - \frac{\tau}{\lambda_{k,n}})\xi'(a_0)\langle a_0, v\rangle \\ k\frac{\mu_2}{\mu_1} v_2(s - \frac{\tau}{\lambda_{k,n}}) \end{pmatrix} ds \\ &= -\int_0^t \begin{pmatrix} kv_1(s - \frac{\tau}{\lambda_{k,n}}) + 2\lambda_{k,n}\mu_1 a_1(s - \frac{\pi}{2k})\xi'(a_0)\langle a_1, v_1\rangle \\ k\frac{\mu_2}{\mu_1} v_2(s - \frac{\tau}{\lambda_{k,n}}) \end{pmatrix} ds \\ &= \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \end{aligned} \tag{5.7}$$

where $\xi'(a_0) < 0$. On the other hand,

$$D_\lambda f(x,\lambda) = \begin{pmatrix} \int_0^t -\beta(x)\mu_1(\sigma'(\lambda)x_1(s - \frac{\tau}{\lambda}) + \frac{\sigma(\lambda)\tau}{\lambda^2}\dot x_1(s - \frac{\tau}{\lambda}))\,ds \\ \int_0^t -\beta(x)\mu_2(\sigma'(\lambda)x_2(s - \frac{\tau}{\lambda}) + \frac{\sigma(\lambda)\tau}{\lambda^2}\dot x_2(s - \frac{\tau}{\lambda}))\,ds \end{pmatrix},$$

and noting that $\sigma(\lambda_{k,n}) = \lambda_{k,n}$, $\sigma'(\lambda_{k,n}) = 0$ and that $a_0 = (a_1, 0)$ we obtain

$$\begin{aligned} D_\lambda f(a_0, \lambda_{k,n}) &= \begin{pmatrix} -\int_0^t \beta(a_0)\mu_1\lambda_{k,n}\frac{\tau}{\lambda_{k,n}^2}\dot a_1(s - \frac{\tau}{\lambda_{k,n}})\,ds \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -\int_0^t \frac{k\tau}{\lambda_{k,n}^2}\dot a_1(s - \frac{\pi}{2k})\,ds \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{k\tau}{\lambda_{k,n}^2} a_1(t - \frac{\pi}{2k}) \\ 0 \end{pmatrix} = \begin{pmatrix} T_3 \\ 0 \end{pmatrix}. \end{aligned} \tag{5.8}$$

Let us consider the following eigenvalue problem $v = \mu D_x f(a_0, \lambda_{k,n})v$, i.e.,

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} v = \mu \begin{pmatrix} T_1 v_1 \\ T_2 v_2 \end{pmatrix}$$

for $v = (v_1, v_2) \in E$, which is equivalent to the following system

$$\begin{cases} \dot{v}_1(t) = -\mu(kv_1(t - \frac{\tau}{\lambda_{k,n}}) + 2\lambda_{k,n}\mu_1 a_1(t - \frac{\pi}{2k})\xi'(a_0)\langle a_1, v_1\rangle), \\ \dot{v}_2(t) = -\mu\frac{k\mu_2}{\mu_1}v_2(t - \frac{\tau}{\lambda_{k,n}}). \end{cases} \tag{5.9}$$

Since $\langle a_1, \dot{a}_1\rangle = 0$, it is easy to verify that $\mu = 1$ is the eigenvalue with corresponding eigenvector $(\dot{a}_1, 0)$.

Summing up, we obtain

$$\begin{aligned} Q - Df(a_0, \lambda_{k,n}) &= Q - \big(D_x f(a_0, \lambda_{k,n}), D_\lambda f(a_0, \lambda_{k,n})\big) = Q - T \\ &= \begin{pmatrix} Id & 0 & 0 \\ 0 & Id & 0 \end{pmatrix} - \begin{pmatrix} T_1 & 0 & T_3 \\ 0 & T_2 & 0 \end{pmatrix} : E \times \mathbb{R} \to E \end{aligned}$$

is a surjection such that $\ker(Q - T) = \operatorname{span}\{((\dot{a}_1, 0), 0)\}$. Notice that we have just proved that S^1-orbits of $(Q - G(\cdot, 1))^{-1}(0)$ are nondegenerate. In other words the assumptions of Theorem 2.2 are satisfied.

Since $(a_0(t), \lambda_{k,n})$ is an isolated nondegenerate solution of (5.3) and $S^1_{a_0} = \mathbb{Z}_k$, applying Theorem 2.2 we obtain

$$\deg_{\mathbb{Z}_k}(Q - f, \Omega) = \pm 1, \qquad \deg_{\mathbb{Z}_{k'}}(Q - f, \Omega) = 0 \quad \text{for } k' > k \tag{5.10}$$

for an open, bounded S^1-invariant subset $\Omega \subset cl(\Omega) \subset \Omega_{\lambda_1,\lambda_2}$ such that $(Q - f)^{-1}(0) \cap \Omega = S^1 \cdot a_0 \times \{\lambda_{k,n}\}$.

The same computation one can perform for $(a_0(t), \lambda) = ((a_1(t), a_2(t)), \lambda) = ((0, c_{2,k}\cos kt), \lambda_{k,n})$ for some $1 \leqslant k, n \leqslant n_2$, satisfying (5.5).

Summing up, we have proved that $(Q - f)^{-1}(0) \cap \Omega_{\lambda_1\lambda_2} = S^1 \cdot a_1 \cup \cdots \cup S^1 \cdot a_p$, i.e., it consist of a finite number of nondegenerate S^1-orbits $S^1 \cdot a_1, \ldots, S^1 \cdot a_p$, each with nontrivial S^1-index, see formula (5.10). Since these orbits are nondegenerate, there are open bounded S^1-invariant subsets $U_i \subset cl(U_i) \subset \Omega_{\lambda_1\lambda_2}$, $i = 1, \ldots, p$, such that $(Q - G(\cdot, 1))^{-1}(0) \cap U_i = S^1 \cdot a_i$, $i = 1, \ldots, p$. And consequently by the properties of S^1-degree we obtain

$$\operatorname{Deg}(Q - f, \Omega_{\lambda_1,\lambda_2}) = \operatorname{Deg}(Q - f, U_1) + \cdots + \operatorname{Deg}(Q - f, U_p) \in \Gamma.$$

Notice that for $k_0 = [\frac{n_2}{j}]$ only $n = n_2$ satisfies (5.5). Therefore, there is exactly one solution of (5.3) in $\Omega_{\lambda_1,\lambda_2}$ of the form $((0, c_{2,k_0}\cos k_0 t), \lambda_{k_0,n_2})$. Moreover, other solutions of (5.3) are of the form $((c_{1,k}\cos kt, 0), \lambda_{k,n})$ or $((0, c_{2,k}\cos kt), \lambda_{k,n})$, where $k < k_0$. In other words there is exactly one orbit with the isotropy group $\mathbb{Z}_{k_0}$.

Finally, combining Theorem 2.2 with (5.10) we obtain $\deg_{\mathbb{Z}_{k_0}}(Q - f, \Omega_{\lambda_1,\lambda_2}) \neq 0$, which completes the proof. □

Proof of Theorem 1.1. Without loss of generality, we can assume that $n_1 < n_2$. Fix $j \in \Phi(n_1, n_2)$ and define $\lambda_1 = \frac{\tau}{2(j+1)\pi}$, $\lambda_2 = \frac{\tau}{2j\pi}$, $\Omega_{\lambda_1\lambda_2} = \Theta \times (\lambda_1, \lambda_2)$, where $\Theta \subset E$ is an open bounded S^1-invariant subset defined by (4.8). In other words $\Omega_{\lambda_1\lambda_2}$ is a Cartesian product of an "annulus" Θ and an open interval (λ_1, λ_2).

To complete the proof it is enough to show that $(Q - F(\cdot, 1))^{-1}(0) \cap \Omega_{\lambda_1\lambda_2} \neq \emptyset$, where the operator F is defined by formula (3.4). By Theorem 2.1 it is enough to show that either $(Q - F(\cdot, 1))^{-1}(0) \cap \partial\Omega_{\lambda_1\lambda_2} \neq \emptyset$, or $\operatorname{Deg}(Q - F(\cdot, 1), \Omega_{\lambda_1,\lambda_2}) \neq \Theta \in \Gamma$.

By Lemma 4.4 we obtain that $(Q - F)^{-1}(0) \cap (\partial\Omega_{\lambda_1\lambda_2} \times [0,1]) = \emptyset$. Therefore by the homotopy property of the S^1-degree we obtain that

$$\mathrm{Deg}\big(Q - F(\cdot,1), \Omega_{\lambda_1,\lambda_2}\big) = \mathrm{Deg}\big(Q - F(\cdot,0), \Omega_{\lambda_1,\lambda_2}\big).$$

By Theorem 2.1, what is left is to show that $\mathrm{Deg}(Q - F(\cdot,0), \Omega_{\lambda_1,\lambda_2}) \neq \Theta \in \Gamma$.

From Lemma 5.1 it follows that

$$\mathrm{Deg}\big(Q - F(\cdot,0), \Omega_{\lambda_1,\lambda_2}\big) = \mathrm{Deg}\big(Q - G(\cdot,1), \Omega_{\lambda_1,\lambda_2}\big).$$

Finally, by Lemma 5.2 we obtain $\mathrm{Deg}(Q - G(\cdot,1), \Omega_{\lambda_1,\lambda_2}) \neq \Theta \in \Gamma$. Notice that we have just proved that $\mathrm{Deg}(Q - F(\cdot,1), \Omega_{\lambda_1,\lambda_2}) \neq \Theta \in \Gamma$. The rest of the proof is a direct consequence of Theorem 2.1. □

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