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Geometric Hyperplanes of Non-embeddable Grassmannians

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1. INTRODUCTION

Many results show that geometric hyperplanes of embeddable Lie incidence geometries arise from a universal embedding as a preimage of the intersection of the set of embedded points with a projective hyperplane. Heretofore, except for the isolated results for the two non-embeddable polar spaces in [1], there has been little or no information concerning geometric hyperplanes of non-embeddable Lie incidence geometries. Surprisingly, the possibilities for geometric hyperplanes are quite restrictive in the non-embeddable case. A corollary of the main result of this paper classifies all geometric hyperplanes of the non-embeddable Grassmann spaces (the points of which are all k-dimensional subspaces of a vector space over a non-commutative division ring) up to isomorphism. Together with the results of [2], this gives a complete classification of geometric hyperplanes of Grassmannians. The main result classifies geometric hyperplanes of arbitrary Grassmannians of k-spaces which satisfy a condition $(*)$ which happens to hold in the non-embeddable case.

2. NOTATION AND STATEMENT OF RESULTS

A *subspace* of a point–line geometry $(\mathscr{P}, \mathscr{L})$ is a subset X of the set of points $\mathscr{P}$ such that every line of $\mathscr{L}$ incident with at least two points of X has all its incident points in X. A proper subspace is called a *geometric hyperplane* of $(\mathscr{P}, \mathscr{L})$ iff every line has at least one of its incident points in X. It is an easy exercise that a geometric hyperplane H of a point line geometry $(\mathscr{P}', \mathscr{L}')$ is a maximal subspace iff $\mathscr{P}' - H$ has a connected collinearity graph. Let V be a left vector space of possibly infinite dimension over a division ring D. A *Grassman space* $\mathscr{G}(V, k)$ is a point–line geometry $(\mathscr{P}, \mathscr{L})$ the points of which are all k-dimensional subspaces of a vector space V (k is a positive integer) and the lines of which are the $(k-1, k+1)$-dimensional flags (A, B), the points incident with (A, B) being the set of k-spaces W such that $A \leqslant W \leqslant B$. In the case that V has finite dimension $n+1$, so $V \simeq D^{(n+1)}$, it is customary to denote $\mathscr{G}(V, k)$ by its Lie incidence notation, $A_{n,k}(D)$. We shall maintain that convention here.

Any Grassman space $(\mathscr{P}, \mathscr{L})$ possesses a fundamental property:

(2.1) *If (x, y) is a pair of points at distance 2 from each other in $(\mathscr{P}, \mathscr{L})$, then the convex closure $\langle x, y\rangle$ of the set $\{x, y\}$ is a non-degenerate polar space of rank 3.*

These subspaces $\langle x, y\rangle$ are called *symplecta.* If S is a symplecton containing the point x, the set $x^{\perp} \cap S = \{s \in S \mid s \text{ is collinear with } x\}$ is a geometric hyperplane of S. A geometric hyperplane of this type—consisting, as it does, of a point of S together with the point shadow of all lines on it—is called a *star.*

Now let $(\mathscr{P}, \mathscr{L})$ be any connected point–line geometry satisfying property (2.1) in a non-trivial way: that is, at least one distance 2 pair exists. Then we have

(2.2) *No point is collinear with all remaining points, and every line lies in some symplecton.*

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Since a distance 2 pair exists, there must exist a symplecton $S = \langle x, y \rangle$. If p were a point with $p^\perp = \mathcal{P}$, then p would lie in $x^\perp \cap y^\perp$ and hence in S against its non-degeneracy. So no such point can exist.

Let L be any line. If L is not contained in any symplecton, then by (2.1), $L^\perp$ is a singular subspace and $x^\perp$ is contained in $L^\perp$ for all points x incident with L. But as $x^\perp \neq \mathcal{P}$ and $(\mathcal{P}, \mathcal{L})$ is connected, there exists a point z at distance 2 from x, and a symplecton $S = \langle z, x \rangle$. But then $S \cap x^\perp \subseteq L^\perp$, a clique, against the fact that, in a non-degenerate polar space S of rank at least 2, a star $x^\perp \cap S$ is never a clique. Thus all assertions in (2.2) hold.

Again suppose that $(\mathcal{P}, \mathcal{L})$ satisfies property (2.1). We wish to consider a subset H of the set of points satisfying condition:

$(*)$ *For each symplecton S, either $S \subseteq H$ or $H \cap S$ is a star.*

Then we observe

(2.3) *Suppose $(\mathcal{P}, \mathcal{L})$ satisfies property* (2.1) *and is not a singular space: that is, the collinearity graph on $\mathcal{P}$ is not a complete graph. If a subset H of $\mathcal{P}$ satisfies condition $(*)$, then either $H = \mathcal{P}$, or H is a geometric hyperplane of $(\mathcal{P}, \mathcal{L})$.*

To see this, consider any line L and let S be a symplecton on L (S exists by (2.2)(ii)). Then as $S \cap H = S$ or is a geometric hyperplane of S, we see that $L \cap (H \cap S)$ is not empty and that if it contains at least two points, then $L \subseteq H \cap S$. Thus $L \cap H$ is non-empty, and if it contains two points, $L \subseteq H$. But this is the assertion that either $H = \mathcal{P}$ or is a geometric hyperplane of $(\mathcal{P}, \mathcal{L})$.

Our main result classifies all subsets H of the Grassmann space $\mathcal{G}(V, k)$ which satisfy $(*)$. Our arguments, as well as the precise statement of the conclusion, are best suited to the affine setting, where all discussion takes place among subspaces of a vector space V. For this reason we restate condition $(*)$ for Grassmann spaces in this affine setting:

$(*k)$ *For some positive integer, k $\mathcal{H}$ is a non-empty collection of proper k-subspaces of a vector space V such that one of the following occurs:*

(i) $1 < k < \dim V - 1$, *and for every flag (A, B), where A and B are respectively $(k-2)$- and $(k+2)$-spaces, the collection $\mathcal{H}(A, B)$ of k-spaces of $\mathcal{H}$ incident with both A and B either comprise all k-spaces incident with A and B, or else there exists a particular k-space R incident with A and B such that $\mathcal{H}(A, B)$ is all k-spaces incident with A and B which intersect R in at least a hyperplane of R*;

(ii) $k = 1$ *and $\mathcal{H}$ satisfies the weaker condition that for any 2-space A, either one or all 1-subspaces of A belong to $\mathcal{H}$*;

(iii) $k = \dim V - 1$ *is finite, and for each subspace B of codimension 2, either one or all hyperplanes containing B belong to $\mathcal{H}$.*

Clearly, in cases (ii) and (iii), $\mathcal{G}(k, V)$ is an ordinary projective space and $\mathcal{H}$ is simply a projective hyperplane of it. Thus case (i), is the case of interest. We can now state the following:

MAIN THEOREM. *Let $\mathcal{H}$ be a collection of k-subspaces of a vector space V satisfying condition $(*k)$. Then either:*
(i) $\mathcal{H}$ consists of all k-spaces of V; or
(ii) there exists a subspace R of codimension k in V such that $\mathcal{H}$ is precisely the collection of all k-subspaces of V which meet R non-trivially.

Conversely, it is easy to verify that if R is a subspace of codimension k in V, then the collection $\mathcal{H}_R$, of all k-subspaces of V which intersect R non-trivially, is a collection of

k-spaces satisfying condition $(*k)$. Since it does not comprise all k-subspaces of V, by (2.3) and (2.2)(i), $\mathscr{H}_R$ is a geometric hyperplane of $\mathscr{G}(V, k)$. Moreover, since the complement of $\mathscr{H}_R$ is easily shown to be connected, $\mathscr{H}_R$ is a maximal subspace of $\mathscr{G}(V, k)$.

This theorem yields the following:

COROLLARY. *Let $\mathscr{G}(V, k)$ be the Grassmann space of k-dimensional subspaces of a (possibly infinite dimensional) vector space V over a non-commutative division ring D. Then any geometric hyperplane of $\mathscr{G}(V, k)$ has the form $\mathscr{H}_R$ for some subspace R of codimension k in V.*

PROOF. Suppose that V is a vector space over a non-commutative division ring D and that $\mathscr{H}$ is a geometric hyperplane of the Grassmann space $\mathscr{G}(V, k)$, where k is a (finite) positive integer. By the Main Theorem, we need only show that $\mathscr{H}$ satisfies $(*k)$. The symplecta $S = \langle x, y\rangle$ which arise from property (2.1) are of the type $A_{3,2}(D)$, and it is well known that when D is non-commutative, the only type of geometric hyperplane of S is a star $s^{\perp} \cap S$, for some Grassmann point s in S (see Proposition 5.1 of [1]). Thus $(*k)$ holds and the Corollary is immediate from the Main Theorem. □

3. PROOF OF THE MAIN THEOREM

3.1. *Radicals for Grassmann Subspaces*

A collection $\mathscr{S}$ of k-subspaces of a vector space V is called a *Grassmann subspace* if, for any $(k-1, k+1)$-dimensional flag (A, B), if at least two of the k-spaces incident with both A and B belong to $\mathscr{S}$, then all of them do. (This, of course, just says that $\mathscr{S}$ is a subspace of $\mathscr{G}(V, k)$.)

Now suppose that $\mathscr{S}$ is a Grassmann subspace of $A_{n,k}(D)$. For subspaces $U \leqslant W$ of V, we say that *U is radical in W* (with respect to $\mathscr{S}$) if every k-subspace of W which meets U non-trivially is a member of $\mathscr{S}$.

LEMMA A. *If $\mathscr{S}$ is a Grassmann subspace of $\mathscr{G}(V, k) = A_{n,k}(D)$, then each subspace W has a unique maximal radial subspace.*

PROOF. It is sufficient to prove for U, Z radical in W with $Z \cap U$ a hyperplane of Z, that $\langle U, Z\rangle$ is also radical in W.

Let the k-space X of W meet $\langle U, Z\rangle$ non-trivially. Set $B = X \cap \langle U, Z\rangle$; $C = X \cap U$ a hyperplane of B and D a complement to B in X. Choose a point (that is, a 1-space) a with $B = \langle a, C\rangle$. Then there exist points $u \in U$ and $z \in Z$ with $a \in \langle u, z\rangle$. Now the k-space $\langle u, C, D\rangle$ of W meets U in u and so is in $\mathscr{S}$. The k-space $\langle z, C, D\rangle$ of W meets Z in z, and so is in $\mathscr{S}$. These k-spaces meet in the $(k-1)$-space $\langle C, D\rangle$; so the complete Grassmann line of $A_{n,k}(D)$ on them lies in $\mathscr{S}$, since $\mathscr{S}$ is a Grassmann subspace. In particular, $X = \langle a, C, D\rangle$ is in $\mathscr{S}$. Thus $\langle U, Z\rangle$ is radical, as desired. □

The maximal radical subspace of W in Lemma A will be called the *radical of* W, and it will be denoted $R(W)$. Thus $R(W)$ is the largest subspace of W such that every k-space X in W with $X \cap R(W)$ non-trivial lies in $\mathscr{S}$. Clearly, if $\dim W < k$, then $R(W) = W$. In all cases, either $R(W) = W$ or the codimension of $R(W)$ in W *is* at least k.

3.2. *The Finite-dimensional Case*

We wish to apply Lemma A when $\mathscr{S} = \mathscr{H}$ is a collection of k-subspaces of a finite-dimensional space V satisfying $(*k)$. (Note that by (2.3) $\mathscr{H}$ *is* a Grassmann subspace in this case.) Specifically, in this section we wish to prove the following:

THEOREM 1. *Let $\mathscr{H}$ be a geometric hyperplane of $A_{n,k}(D)$ which satisfies condition $(*k)$. Then there is an $(n+1-k)$-space R in V with $\mathscr{H} = \mathscr{H}_R$.*

Theorem 1 states that under assumption $(*k)$, the radical $R(V)$ has codimension exactly k; that is, dimension $n+1-k$. Once it is known that $R(V)$ has codimension k, we have $\mathscr{H} \geqslant \mathscr{H}_R$, hence $\mathscr{H} = \mathscr{H}_R$ as both are geometric hyperplanes of $A_{n,k}(D)$, and $\mathscr{H}_R$ is a maximal subspace.

The reason for first proving the Main Theorem in the finite-dimensional case as Theorem 1 is that this allows us to exploit an obvious duality. Thus we fix a division ring D and set $V = D^{(n+1)}$ (as a left vector space) and set $V^* = (D^{op})^{(n+1)}$, where D^{op} is the opposite ring, and V^* is viewed as a set of right operators on V. Then there is a pairing $\langle\ ,\ \rangle$ on $V \times V^*$ given by $\langle v, \lambda\rangle = v\lambda$. For any subspace $W \leqslant V$, set

$$W^\perp = \{\lambda \in V^* \mid w\lambda = 0 \text{ for all } w \in W\}.$$

For $\Delta \leqslant V^*$, set

$$^\perp\Delta = \{v \in V \mid v\delta = 0 \text{ for all } \delta \in \Delta\}.$$

Then there is an inclusion-reversing isomorphism of the lattice of projective subspaces of $\mathbb{P}(V)$ with its opposite lattice $\mathbb{P}(V^*)$; namely, the mapping

$$(\perp_R; \mathbb{P}(V) \to \mathbb{P}(V^*)$$

given by $W \to W^\perp$. Similarly, the inverse mapping

$$(\perp_L: \mathbb{P}(V^*) \to \mathbb{P}(V)$$

is given by $\Delta \to {}^\perp\Delta$.

If W has dimension k, then $W^\perp$ has dimension $k' = n+1-k$, so $\perp$ induces an isomorphism of Grassmann spaces $A_{n,k}(D) \simeq A_{n,k'}(D^{op})$. If $\mathscr{H}$ satisfies the following:

$(*)$ *For each symplecton S, $\mathscr{H} \cap S$ is either all of S or is a star of S.*

then $\mathscr{H}^\perp = \{\mathscr{H}^\perp \mid H \in \mathscr{H}\}$ satisfies $(*)$ relative to the Grassmann space $A_{n,k'}(D^{op})$ as well. Specifically, if $\mathscr{H}$ is a collection of k-subspaces of V satisfying condition $(*k)$, then $\mathscr{H}^\perp$ satisfies condition $(*k')$ relative to V^*.

We now begin a proof of Theorem 1 by induction on n. (Note that we do not employ any induction on k as we want to be free to use duality.) Notice that for $k=1$ or $k=n$, condition $(*k)$ is rather degenerate, and the theorem is well known. Similarly, by considering $A_{n,1}(D^{op}) \simeq A_{n,n}(D)$, the theorem is immediate for $k=n$ as well. So, from now on we assume $1<k<n$. In particular, the initial cases $n=1, 2$ of the induction are immediate.

By induction we have the following:

(3.1) *For each hyperplane W of V, either:*
(i) $R(W) = W$; or
(ii) $R(W)$ has codimension k in W and dimension $n-k$, and any k-space X of W lies in $\mathscr{H}$ iff $X \cap R(W) \neq 0$.

LEMMA *B*. *Let W_1 and W_2 be hyperplanes of V with $R(W_1) \neq W_1$. Then either:*
(i) $R(W_1) \leqslant W_2$ *and either* $R(W_2) \leqslant W_1$ *or* $R(W_2) = W_2$; *or*
(ii) $R(W_1) \cap W_2 = W_1 \cap R(W_2) = R(W_1) \cap R(W_2)$ *and* $R(W_2) \neq W_2$.

PROOF. Suppose that $R(W_1)$ is not a subspace of W_2, so that $R(W_1) \cap W_2$ is an $(n-k-1)$-space. Assume by way of contradiction that $R(W_1) \cap W_2 \neq R(W_2) \cap W_1$. Then in the $(n-1)$-space, $W_1 \cap W_2$, there is a k-space X meeting $R(W_1) \cap W_2$ trivially and $W_1 \cap R(W_2)$ non-trivially. Then as $X \leqslant W_1$ and $X \cap R(W_1) = 0$, X is not a member of $\mathscr{H}$. But on the other hand, as $X \leqslant W_2$ and meets $R(W_2)$ non-trivially, X is in $\mathscr{H}$, a contradiction. Thus if $R(W_1)$ is not contained in W_2 we have

$$R(W_1) \cap W_2 = W_1 \cap R(W_2) = R(W_1) \cap R(W_2),$$

which is an $(n-k-1)$-space. Since this subspace contains a hyperplane of $R(W_2)$ of dimension at least $n-k$, we see that $W_2 \neq R(W_2)$, and $R(W_2)$ is not contained in W_1.

Reversing the roles of W_1 and W_2, we see that if $R(W_1) \leqslant W_2$, then either $R(W_2) = W_2$ or else $R(W_2) \leqslant W_1$, as stated in (i).

LEMMA *C*. *Theorem* 1 *holds for* $\mathscr{H}$ *if* $k < n-1$.

PROOF. Let Y be a k-space not in $\mathscr{H}$ and let U be a hyperplane of V containing Y. Then $R(U) \neq U$, so $\dim R(U) = n-k > 0$. We can then choose a second hyperplane W with $R(U)$ not contained in W. By Lemma B, $R(W)$ has dimension $n-k$, and $R := \langle R(U), R(W) \rangle$ has dimension $n-k+1$. We claim that $\mathscr{H} \subseteq \mathscr{H}_R = \{X \mid \dim X = k, X \cap R \neq 0\}$.

Let X be a k-space intersecting R trivially. Since the dimension $n-k-1$ of $R(U) \cap R(W)$ is not 0 (as $k < n-1$, by hypothesis), it is possible to choose a hyperplane Z containing X which does not contain $R(U) \cap R(W)$. Now two applications of Lemma B show that $R(Z) \neq Z$ and that both $R(Z) \cap U = R(Z) \cap R(U)$ and $R(Z) \cap W = R(Z) \cap R(W)$ are hyperplanes of $R(Z)$. We now note that these two $(n-k-1)$-spaces $R(Z) \cap R(U)$ and $R(Z) \cap R(W)$ cannot be equal, for otherwise they would equal the $(n-k-1)$-space $R(U) \cap R(W)$ which is not contained in Z. Therefore, we have the following:

$$R(Z) = \langle R(Z) \cap R(U), R(Z) \cap R(W) \rangle \leqslant \langle R(U), R(W) \rangle = R.$$

Since the k-space X meets R trivially, it certainly meets its subspace $R(Z)$ trivially. But now, as $X \leqslant Z$, this implies $X \notin \mathscr{H}$. We have shown, then, that if X meets R trivially, $X \notin \mathscr{H}$. The contrapositive of this is the desired assertion that $\mathscr{H} \subseteq \mathscr{H}_R$.

But now $\mathscr{H} \subseteq \mathscr{H}_R$ implies $\mathscr{H} = \mathscr{H}_R$. For if false, then $\mathscr{H} \subset \mathscr{H}_R$ and since $\mathscr{H}$ and $\mathscr{H}_R$ are both hyperplanes, this means that

$$\mathscr{H}_R - \mathscr{H} \text{ is a union of connected components of } \mathscr{P} - \mathscr{H}.$$

Now let C be any hyperplane of V containing R such that $R(C) \neq C$. Then $R(C)$ has codimension k in C and R has codimension $k-1$ in C. It follows that there exists a k-space X in C meeting R at a 1-space X_0 and meeting $R(C)$ trivially. Then $X \in \mathscr{H}_R - \mathscr{H}$. Let X_1 be a complementary subspace to X_0 in X. Then X_1 is a $(k-1)$-subspace of C meeting R and $R(C)$ trivially. Then there exists a k-space Z of V containing X_1 and meeting R trivially. Then Z is a point of $\mathscr{P} - \mathscr{H}_R$ collinear with the point X of $\mathscr{H}_R - \mathscr{H}$. This contradicts the assertion presented above.

This completes the proof of Lemma C. □

We can now complete the proof of Theorem 1. By Lemma C, the theorem holds if $k<n-1$. Similarly, the theorem holds for $\mathscr{H}^{\perp}$ in $A_{n,k'}(D^{op})\simeq A_{n,k}(D)$ if $k'<n-1$. But as $k'=n+1-k$, this means that the theorem holds for $\mathscr{H}$ in $A_{n,k}(D)$ if $k'=n+1-k<n-1$; i.e. if $k>2$. But for the only case left open $k=2$ and $n=3$, so the conclusion of the theorem is precisely the assumption $(*k)$.

REMARK. The condition $(*k)$ or something close to it is necessary. Even if $\mathscr{H}$ is a subspace of $\mathscr{G}(V, k)$ for which one is allowed unlimited use of Lemmas A and B, one sees that the case $(n, k)=(3, 2)$ is a genuine exception. For here, $\mathscr{H}$ can be the collection of all 2-subspaces of $V=D^{(4)}$, D a field, which are isotropic with respect to a non-degenerate symplectic form; and in that case $\mathscr{H}$ does not have the form $\mathscr{H}_R$.

3.3. *The Infinite-dimensional Case*

Here we prove the Main Theorem in Section 2 as a corollary to Theorem 1. Here we suppose V to be an infinite-dimensional left vector space over a division ring D. Let $\mathscr{H}$ be a collection of k-subspaces of V (k is a positive integer) satisfying condition $(*k)$. If $\mathscr{H}$ is not all of $\mathscr{P}$, the collection of all k-subspaces of V, we claim that there exists a subspace R of codimension k in V such that $\mathscr{H}=\mathscr{H}_R$. By Theorem 1, for each finite-dimensional subspace W in V, $R(W)=W$ or $R(W)$ has codimension k in W, and all k-subspaces of $\mathscr{H}$ in W meet $R(W)$ non-trivially.

Moreover, we have the following:

LEMMA D. *If* $R(W_1)<W_1<W_2$ *are proper containments, where* dim W_2 *is finite, then* $R(W_1)<R(W_2)<W_2$.

PROOF. Since $R(W_1)\neq W_1$, there is a k-subspace of W_1 (hence a k-subspace of W_2) not in $\mathscr{H}$, so $R(W_2)\neq W_2$. But by the definition of the radical $R(W_2)$, we see that:

$$R(W_2)\cap W_1\leqslant R(W_1). \tag{3.2}$$

But, as Theorem 1 shows that $R(W_2)$ has codimension k in W_2, we see that $R(W_2)\cap W_1$ has codimension at most k in W_1. On the other hand, (3.2) implies that this codimension is at least k since, by Theorem 1 again, $R(W_1)$ has codimension exactly k in W_1. Thus $R(W_2)\cap W_1=R(W_1)$, so the conclusion follows and the proof is complete. □

Our candidate for $R(V)$ is the set R of all vectors v which lie in $R(W)\neq W$ for some finite-dimensional subspace W (necessarily of dimension at least $k+1$).

Clearly R is a subspace, for if u and v are vectors in R, there exist finite-dimensional subspaces W_u and W_v such that $u\in R(W_u)<W_u$ and $v\in R(W_v)<W_v$. Then by two applications of Lemma D, the 2-space $\langle u, v\rangle$ is contained in $R(\langle W_u, W_v\rangle)$, and the latter is proper in $\langle W_u, W_v\rangle$ since it contains a k-space (for example, one from W_u) which is not in $\mathscr{H}$. Thus $\langle u, v\rangle\subseteq R$.

Suppose that $X\in\mathscr{H}$. By the hypothesis of our claim, there is a k-space Y not in $\mathscr{H}$. Then, as $\dim(\langle X, Y\rangle)<2k+1$ and $R(\langle X, Y\rangle)\neq\langle X, Y\rangle$, we have $R(\langle X, Y\rangle)\subseteq R$. But by Theorem 1 applied to $\langle X, Y\rangle$, we have $X\cap R(\langle X, Y\rangle)\neq 0$, so $X\in\mathscr{H}_R$. Thus $\mathscr{H}\subseteq\mathscr{H}_R$. On the other hand, if $X\in\mathscr{H}_R$, then X contains a non-zero vector x in R. By definition of R, there exists a finite-dimensional subspace W_x such that $x\in R(W_x)<W_x$. Then, by Lemma D, x is a vector in $R(\langle X, W_x\rangle)$. Since X is a k-subspace of $\langle X, W_x\rangle$ which meets its radical non-trivially, $X\in\mathscr{H}$. Thus, also, $\mathscr{H}_R\subseteq\mathscr{H}$. Since $\mathscr{H}=\mathscr{H}_R$, the proof is complete.

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