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On the limit distributions of continuous-state branching processes with immigration

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Abstract

We consider the class of continuous-state branching processes with immigration (CBI-processes), introduced by Kawazu and Watanabe (1971) [10] and their limit distributions as time tends to infinity. We determine the Lévy–Khintchine triplet of the limit distribution and give an explicit description in terms of the characteristic triplet of the Lévy subordinator and the scale function of the spectrally positive Lévy process, which describe the immigration resp. branching mechanism of the CBI-process. This representation allows us to describe the support of the limit distribution and characterize its absolute continuity and asymptotic behavior at the boundary of the support, generalizing several known results on self-decomposable distributions.
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1. Introduction

Continuous-state branching processes with immigration (CBI-processes) have been introduced by Kawazu and Watanabe [10] as scaling limits of discrete single-type branching processes with immigration. In [10] the authors show that in general a CBI-process has a

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representation in terms of the Laplace exponents $F(u)$ and $R(u)$ of two independent Lévy processes: a Lévy subordinator X^F and a spectrally positive Lévy process X^R, which can be interpreted as immigration and branching mechanism of the CBI-process respectively. Following [5] a CBI-process can in fact be represented as the unique strong solution of a non-linear SDE driven by X^F and X^R.

For discrete branching processes with immigration, limit distributions have been studied already in [6,7] and some results on the existence of limit distributions of a CBI-process were published by Pinsky [18], albeit without proofs. Recently, proofs for the results of [18] have appeared in [15]. The main result of [18] states that under an integral condition on the ratio $F(u)/R(u)$ a limit distribution exists and can be described in terms of its Laplace exponent (cf. Theorem 2.6). The contribution of this article is to build on the results of [18] in order to give a finer description of the limit distribution: We show that it is infinitely divisible, give a representation of its Lévy–Khintchine triplet (Theorem 3.1) and then use this new representation to obtain results on smoothness, support and other properties of the limit distribution. From this main result, several other representations of the Lévy–Khintchine triplet are then derived. The most concise representation is given by Eq. (3.17), which states that the Lévy measure of the limit distribution has a density of the form $x \mapsto k(x)/x$ and the corresponding k-function $k : (0, \infty) \to \mathbb{R}_{\geqslant 0}$ is given by the formula

$$k = -\mathcal{A}_{\check{X}^F} W,$$

where $\mathcal{A}_{\check{X}^F}$ is the generator of the modified Lévy subordinator $\check{X}^F$ and W is the *scale function* that corresponds to the spectrally one-sided Lévy process X^R (see Remark 3.4 for the precise statement of this factorization). In Section 4 we derive further properties of the limit distribution. In particular, we characterize the support of the limit distribution, show its absolute continuity and describe its boundary behavior at the left endpoint of the support. Furthermore we prove that the class of limit distributions of CBI-processes is strictly larger than the class of self-decomposable distributions on $\mathbb{R}_{\geqslant 0}$ and is strictly contained in the class of all infinitely divisible distributions on $\mathbb{R}_{\geqslant 0}$.

Most of our results can also be regarded as extensions of known results on limit distributions of Ornstein–Uhlenbeck-type (OU-type), see e.g. [9,20,19], to the class of CBI-processes. The knowledge of the Lévy–Khintchine triplet for stationary distributions of OU-type processes has been applied to statistical estimation of the underlying process in [17]. We suggest that in further research our results may be used for extensions of this methodology to CBI-processes.

2. Preliminaries

2.1. Continuous-state branching processes with immigration

Let X be a continuous-state branching process with immigration. Following [10], such a process is defined as a stochastically continuous Markov process with state space $[0, \infty]$, whose Laplace exponent is affine in the state variable, i.e. there exist functions $\phi(t, u)$ and $\psi(t, u)$ such that

$$-\log \mathbb{E}^x \left[e^{-uX_t} \right] = \phi(t, u) + x\psi(t, u), \quad \text{for all } t \geq 0,\ u \geq 0,\ x \in \mathbb{R}_{\geqslant 0}, \tag{2.1}$$

where, as usual for the theory of Markov processes, $\mathbb{E}^x$ denotes expectation, conditional on $X_0 = x$. Since we are interested in the limit behavior of the process X as $t \uparrow \infty$, we further assume that X is *conservative*, i.e. that X_t is a proper random variable for each $t \geq 0$ with state space $\mathbb{R}_{\geqslant 0} := [0, \infty)$. The following theorem is proved in [10].

Theorem 2.1 (*[10]*). *Let $(X_t)_{t\geq 0}$ be a conservative CBI-process. Then the functions $\phi(t,u)$ and $\psi(t,u)$ in (2.1) are differentiable in t with derivatives*

$$F(u) = \frac{\partial}{\partial t}\phi(t,u)\Big|_{t=0}, \qquad R(u) = \frac{\partial}{\partial t}\psi(t,u)\Big|_{t=0} \tag{2.2}$$

and F, R are of Levy–Khintchine form

$$F(u) = bu - \int_{(0,\infty)} (\mathrm{e}^{-u\xi} - 1) m(\mathrm{d}\xi), \tag{2.3}$$

$$R(u) = -\alpha u^2 + \beta u - \int_{(0,\infty)} (\mathrm{e}^{-u\xi} - 1 + u\xi I_{(0,1]}(\xi))\mu(\mathrm{d}\xi), \tag{2.4}$$

where $\alpha, b \in \mathbb{R}_{\geqslant 0}$, $\beta \in \mathbb{R}$, $I_{(0,1]}$ is the indicator function of the interval $(0,1]$ and m, μ are Lévy measures on $(0,\infty)$, with m satisfying $\int_{(0,\infty)} (x \wedge 1)\, m(\mathrm{d}x) < \infty$, and R satisfying[1]

$$\int_{0+} \frac{1}{R^*(s)} \mathrm{d}s = \infty \quad \text{where } R^*(u) = \max(R(u), 0). \tag{2.5}$$

Moreover $\phi(t,u)$, $\psi(t,u)$ take values in $\mathbb{R}_{\geqslant 0}$ and satisfy the ordinary differential equations

$$\begin{aligned} &\frac{\partial}{\partial t}\phi(t,u) = F(\psi(t,u)), \quad \phi(0,u) = 0, \\ &\frac{\partial}{\partial t}\psi(t,u) = R(\psi(t,u)), \quad \psi(0,u) = u. \end{aligned} \tag{2.6}$$

Remark 2.2. The Eqs. (2.6) are often called generalized Riccati equations, since they are classical Riccati differential equations, when $m = \mu = 0$.

We call (F, R) the *functional characteristics* of the CBI-process X. Furthermore the article of [10] contains the following converse result: for any functions F and R defined by (2.3) and (2.4) respectively, which satisfy condition (2.5) and the restrictions on the parameters α, b, β and the Lévy measure m stated in Theorem 2.1, there exists a unique conservative CBI-process with functional characteristics (F, R). In this sense the pair (F, R) truly characterizes the process X. Clearly, $F(u)$ is the Laplace exponent of a Lévy subordinator X^F, and $R(u)$ is the Laplace exponent of a Lévy process X^R without negative jumps. Thus, we also have a one-to-one correspondence between (conservative) CBI-processes and pairs of Lévy processes (X^F, X^R), of which the first is a subordinator, and the second a process without negative jumps that satisfies condition (2.5). In the case of a CBI-process without immigration (i.e. a CB-process), which corresponds to $F = 0$, a pathwise transformation of X^R to X and vice versa was given by Lamperti [14], and is often referred to as 'Lamperti transform'. Recently, a pathwise correspondence between the pair (X^F, X^R) and the CBI-process X has been constructed by Caballero, Pérez Garmendia, and Uribe Bravo [2].

The following properties of $F(u)$ and $R(u)$ can be easily derived from the representations (2.3) and (2.4) and the parameter conditions stated in Theorem 2.1.

[1] The notation $\int_{0+}$ denotes an integral over an arbitrarily small right neighborhood of 0.

Lemma 2.3. *The functions $F(u)$ and $R(u)$ are concave and continuous on $\mathbb{R}_{\geqslant 0}$ and infinitely differentiable in $(0, \infty)$. At $u = 0$ they satisfy $F(0) = R(0) = 0$, the right derivatives $F'_+(0)$ and $R'_+(0)$ exist in $(-\infty, +\infty]$ and satisfy $F'_+(0) = \lim_{u\downarrow 0} F'(u)$ and $R'_+(0) = \lim_{u\downarrow 0} R'(u)$.*

We will also need the following result, which can be found e.g. in [12, Ch. 8.1]

Lemma 2.4. *For the function $R(u)$ exactly one of the following holds:*

(i) $R'_+(0) > 0$ *and there exists a* $u_0 > 0$ *such that* $R(u_0) = 0$;
(ii) $R \equiv 0$;
(iii) $R'_+(0) \le 0$ *and* $R(u) < 0$ *for all* $u > 0$.

Remark 2.5. In case (i) $R(u)$ is called a supercritical branching mechanism, while case (iii) can be further distinguished into critical ($R'_+(0) = 0$) and subcritical branching ($R'_+(0) < 0$).

In what follows we will be interested in the limit distribution and the invariant distribution of $(X_t)_{t\ge 0}$. We write $P_t f(x) = \mathbb{E}^x[f(X_t)]$ for all $x \in \mathbb{R}_{\geqslant 0}$ and denote by $(P_t)_{t\ge 0}$ the transition semigroup associated to the Markov process X. We say that L is the *limit distribution* of the process $X = (X_t)_{t\ge 0}$ if X_t converges in distribution to L under all $\mathbb{P}^x$ for any starting value $x \in \mathbb{R}_{\geqslant 0}$ of X. We call L an *invariant (or stationary) distribution* of $X = (X_t)_{t\ge 0}$, if

$$\int_{[0,\infty)} P_t f(x) \mathrm{d}L(x) = \int_{[0,\infty)} f(x) \mathrm{d}L(x),$$

for any $t \ge 0$ and bounded measurable $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$. Finally we denote the Laplace exponent of L by

$$l(u) = -\log \int_{[0,\infty)} e^{-ux} \mathrm{d}L(x), \quad (u \ge 0).$$

2.2. Limit distributions of CBI-processes

Theorem 2.6 and Corollary 2.8 concern the existence of a limit distribution of a CBI-process and have been announced in a similar form but without proof in [18]. A proof has recently appeared in [15, Theorem 3.20, Corollary 3.21]; the only difference to the result given here is that we drop a mild moment condition assumed in [15, Eq. (3.1)f] and that we include stationary distributions in the statement of our result. Some weaker results on the existence of a limit distribution of a CBI-process have also appeared in [11]. We give a self-contained proof of the theorem and its corollary in the appendix of the article.

Theorem 2.6 (*[18,15]*)**.** *Let $(X_t)_{t\ge 0}$ be a CBI-process on $\mathbb{R}_{\geqslant 0}$. Then the following statements are equivalent:*

(a) $(X_t)_{t\ge 0}$ *converges to a limit distribution L as* $t \to \infty$;
(b) $(X_t)_{t\ge 0}$ *has the unique invariant distribution L;*
(c) *It holds that* $R'_+(0) \le 0$ *and*

$$-\int_0^u \frac{F(s)}{R(s)} \mathrm{d}s < \infty \tag{2.7}$$

for some $u > 0$.

Moreover the limit distribution L has the following properties:

(i) *L is infinitely divisible;*
(ii) *the Laplace exponent $l(u) = -\log \int_{[0,\infty)} e^{-ux} \mathrm{d}L(x)$ of L is given by*

$$l(u) = -\int_0^u \frac{F(s)}{R(s)} \mathrm{d}s \quad (u \geq 0). \tag{2.8}$$

Remark 2.7. Note that the existence of the right derivative R'_+ at 0 that appears in statement (c) is guaranteed by Lemma 2.3.

Corollary 2.8. *If $R'_+(0) < 0$ then the integral condition (2.7) is equivalent to the log-moment condition*

$$\int_{\xi>1} \log \xi m(\mathrm{d}\xi) < \infty. \tag{2.9}$$

2.3. Results on Ornstein–Uhlenbeck-type processes

A subclass of CBI-processes, whose limit distributions have been studied extensively in the literature is the class of $\mathbb{R}_{\geqslant 0}$-valued Ornstein–Uhlenbeck-type (OU-type) processes. We briefly discuss some of the known results on OU-type processes, that will be generalized by our results in the next section. Let $\lambda > 0$ and Z be a Lévy subordinator with drift $b \in \mathbb{R}_{\geqslant 0}$ and Lévy measure $m(\mathrm{d}\xi)$. An $\mathbb{R}_{\geqslant 0}$-valued OU-type process X is the strong solution of the SDE

$$dX_t = -\lambda X_t dt + dZ_t, \quad X_0 \in \mathbb{R}_{\geqslant 0}, \tag{2.10}$$

which is given by $X_t = X_0 e^{-\lambda t} + \int_0^t e^{\lambda(s-t)} dZ_s$. This is the classical Ornstein–Uhlenbeck process, where the Brownian motion has been replaced by an increasing Lévy process. It follows from elementary calculations that an $\mathbb{R}_{\geqslant 0}$-valued OU-type process is a CBI-process with $R(u) = -\lambda u$. In terms of the two Lévy processes X^F, X^R, this corresponds to the case that $X^F = Z$, and X^R is the degenerate Lévy process $X^R_t = -\lambda t$. For OU-type processes analogues of Theorem 2.6 and the log-moment condition of Corollary 2.8 already appeared in [4].

An interesting characterization of the limit distributions of OU-type processes is given in terms of self-decomposability: Recall that a random variable Y has a *self-decomposable distribution* if for every $c \in [0, 1]$ there exists a random variable Y_c, independent of Y, such that

$$Y \stackrel{d}{=} cY + Y_c. \tag{2.11}$$

Self-decomposable distributions are a subclass of infinitely divisible distributions, and exhibit in many aspects an increased degree of regularity. It is known for example, that every non-degenerate self-decomposable distribution is absolutely continuous (cf. [19, 27.8]) and unimodal (cf. [21] or [19, Chapter 53]), neither of which holds for general infinitely divisible distributions. As we are working with non-negative processes, we focus on self-decomposable distributions on the half-line $\mathbb{R}_{\geqslant 0}$, and we denote this class by SD_+. The connection to OU-type processes is made by the following result.

Theorem 2.9 *([9,20])*. *Let X be an OU-type process on $\mathbb{R}_{\geqslant 0}$ and suppose that $m(\mathrm{d}\xi)$ satisfies the log-moment condition $\int_{\xi>1} \log \xi \, m(\mathrm{d}\xi) < \infty$. Then X converges to a limit distribution L which* *is self-decomposable. Conversely, for every self-decomposable distribution L with support* $\mathbb{R}_{\geqslant 0}$ *there exists a unique subordinator Z with drift* $b \in \mathbb{R}_{\geqslant 0}$ *and a Lévy measure* $m(\mathrm{d}\xi)$*, satisfying* $\int_{\xi > 1} \log \xi \, m(\mathrm{d}\xi) < \infty$*, such that L is obtained as the limit distribution of the corresponding OU-type process.*

Since a self-decomposable distribution is infinitely divisible, its Laplace exponent has a Lévy–Khintchine decomposition. The following characterization is due to Paul Lévy and can be found in [19, Corollary 15.11]: an infinitely divisible distribution L on $\mathbb{R}_{\geqslant 0}$ is self-decomposable, if and only if its Laplace exponent is of the form

$$-\log \int_{[0,\infty)} e^{-ux} \mathrm{d}L(x) = \gamma u - \int_0^\infty (\mathrm{e}^{-ux} - 1) \frac{k(x)}{x} \mathrm{d}x, \tag{2.12}$$

where $\gamma \geq 0$ and k is a decreasing function on $\mathbb{R}_{\geqslant 0}$. The parameters γ and k are related to the Lévy subordinator Z by

$$\gamma = \frac{b}{\lambda} \quad \text{and} \quad k(x) = \frac{1}{\lambda} m(x, \infty) \quad \text{for } x > 0. \tag{2.13}$$

Following [19] we call k the k-function of the self-decomposable distribution L. Many properties of L, such as smoothness of its density, can be characterized through k. In fact, several subclasses of SD_+ have been defined, based on more restrictive assumptions on k. For example, the class of self-decomposable distributions whose k-function is completely monotone, is known as the Thorin class, and arises in the study of mixtures of Gamma distributions; see [8] for an excellent survey. In our main result, Theorem 3.1 we give analogues of the formulas (2.12) and (2.13) for the limit distribution of a CBI-process. As it turns out, a representation as in (2.12) still holds, with the class of decreasing k-functions replaced by a more general family. However, we do not obtain a structural characterization of the CBI limit distributions that replaces self-decomposability. Identifying such a structural condition (if there is any) constitutes an interesting question that is left open by our results.

3. Lévy–Khintchine decomposition of the limit distribution

Let X^F and X^R be the Lévy processes that correspond to the Laplace exponents F and R given in (2.3) and (2.4) respectively. As remarked in Section 2.1, X^F is a subordinator and X^R is a Lévy process with no negative jumps. From Theorem 2.6 it follows that whenever a limit distribution exists, then $\mathbb{E}[X_1^R] = R'_+(0) \leq 0$ and $R \not\equiv 0$, such that X^R is not a subordinator, but a true *spectrally positive Lévy process* in the sense of [1]. The fluctuation theory of spectrally one-sided Lévy processes has been studied extensively. The convention used in much of the literature is to study a spectrally negative process. In our setting such a process is given by the dual $\widehat{X}^R = -X^R$ and its Laplace exponent is $\log \mathbb{E}[\mathrm{e}^{u\widehat{X}^R}] = -R(u)$ for $u \geq 0$. A central result in the fluctuation theory of spectrally one-sided Lévy processes (see [1, Theorem 8, Ch VII]) states that for each function R of the form (2.4) there exists a unique function $W : \mathbb{R} \to [0, \infty)$, known as the *scale function* of $\widehat{X}^R$, which is increasing and continuous on the interval $[0, \infty)$ with Laplace transform

$$\int_0^\infty \mathrm{e}^{-ux} W(x) \mathrm{d}x = -\frac{1}{R(u)}, \quad \text{for } u > 0 \tag{3.1}$$

and identically zero on the negative half-line ($W(x) = 0$ for all $x < 0$). Note that this equality implies that

$$W(x) = o(e^{\epsilon x}) \quad \text{as } x \to \infty \text{ for any } \epsilon > 0, \tag{3.2}$$

i.e. that W has sub-exponential growth, a fact that will be needed subsequently. Furthermore the scale function W has the representation

$$\frac{W(x)}{W(y)} = \exp\left\{-\int_x^y n(\overline{\varepsilon} \geq z)\mathrm{d}z\right\} \quad \text{for any } 0 < x < y, \tag{3.3}$$

where n is the Itô excursion measure on the set

$$\begin{aligned}\mathcal{E} = \{&\varepsilon \in D(\mathbb{R}) : \exists \zeta_\varepsilon \in (0, \infty] \text{ s.t.} \\ &\varepsilon(t) = 0 \text{ if } \zeta_\varepsilon \leq t < \infty, \varepsilon(0) \geq 0, \varepsilon(t) > 0 \forall t \in (0, \zeta_\varepsilon)\},\end{aligned} \tag{3.4}$$

with $D(\mathbb{R})$ the Skorokhod space. The measure n is the intensity measure of the Poisson point process of excursions from the supremum of $\widehat{X}^R$ and $\{\overline{\varepsilon} \geq z\} \subset \mathcal{E}$ denotes the set of excursions of height $\overline{\varepsilon} = \sup_{t<\zeta_\varepsilon} \varepsilon(t)$ at least $z > 0$ (see [1] for details on the Itô excursion theory in the context of Lévy processes). The representation (3.3) implies that, on the interval $(0, \infty)$, the scale function W is strictly positive, absolutely continuous, log-concave with right- and left-derivative given by $W'_+(x) = n(\overline{\varepsilon} > x)W(x)$ and $W'_-(x) = n(\overline{\varepsilon} \geq x)W(x)$ respectively. Furthermore at $x = 0$ the right-derivative $W'_+(0)$ exists in $[0, \infty]$.

Using the scale function W associated to $\widehat{X}^R$ we can formulate our main result on the Lévy–Khintchine decomposition of the limit distribution of a CBI-process.

Theorem 3.1. *Let X be a CBI-process with functional characteristics (F, R) given in (2.3) and (2.4) and assume that X converges to a limit distribution L. Let W be the scale function associated to the dual $\widehat{X}^R$ of the spectrally positive Lévy process X^R and let (b, m) be the drift and Lévy measure of the subordinator X^F. Then L is infinitely divisible, and its Laplace exponent has the Lévy–Khintchine decomposition*

$$-\log \int_0^\infty e^{-ux}\mathrm{d}L(x) = u\gamma - \int_{(0,\infty)} (\mathrm{e}^{-xu} - 1)\frac{k(x)}{x}\,\mathrm{d}x, \tag{3.5}$$

where $\gamma \geq 0$ and $k : (0, \infty) \to \mathbb{R}_{\geqslant 0}$ are given by

$$\gamma = bW(0), \tag{3.6}$$

$$k(x) = b\, W'_+(x) + \int_{(0,\infty)} [W(x) - W(x - \xi)]\, m(\mathrm{d}\xi). \tag{3.7}$$

Remark 3.2. If X is an OU-type process, then R is of the form $R(u) = -\lambda u$ with $\lambda > 0$. Since the scale function in this case takes the form $W(x) = \frac{1}{\lambda} I_{[0,\infty)}(x)$, Eqs. (3.6) and (3.7) reduce to $\gamma = \frac{b}{\lambda}$ and $k(x) = \frac{1}{\lambda} m(x, \infty)$. This is precisely the known result for the $\mathbb{R}_{\geqslant 0}$-valued OU-type process stated in (2.13).

Before proving this result, we state a corollary that connects the limit distribution in Theorem 3.1 with the excursion measure n associated to the Poisson point process of excursions away from the supremum of the dual of the branching mechanism X^R. To state it, we introduce the effective drift λ_0 of $\widehat{X}^R$, which is defined as

$$\lambda_0 = \begin{cases} \int_{(0,1]} \xi \mu(\mathrm{d}\xi) - \beta, & \text{if } X^R \text{ has bounded variation,} \\ +\infty, & \text{if } X^R \text{ has unbounded variation.} \end{cases} \tag{3.8}$$

Note that $\lambda_0 > 0$ must hold if $R'_+(0) \le 0$.

Corollary 3.3. *Let the assumptions of Theorem 3.1 hold. Then*

$$\gamma = \frac{b}{\lambda_0} \quad \textit{and} \tag{3.9}$$

$$k(x) = W(x)\Big(b\, n(\overline{\varepsilon} > x) + \int_{(0,\infty)} \left(1 - \mathbf{1}_{\{\xi \le x\}} \exp\left(-\int_{x-\xi}^{x} n(\overline{\varepsilon} \ge z)\mathrm{d}z\right)\right) m(\mathrm{d}\xi)\Big) \tag{3.10}$$

where λ_0 is the effective drift of $\widehat{X}^R$, n is the Itô excursion measure corresponding to the Poisson point process of excursions from the supremum of $\widehat{X}^R$.

Proof of Theorem 3.1. For every $t > 0$, the distribution of X_t is infinitely divisible and supported on $\mathbb{R}_{\geqslant 0}$. Hence the same is true of the limit distribution L. The Laplace exponent of L can by Theorem 2.6 be expressed as

$$-\log \int_{[0,\infty)} e^{-ux} \mathrm{d}L(x) = -\int_0^u \frac{F(s)}{R(s)} ds = du - \int_{(0,\infty)} (\mathrm{e}^{-ux} - 1)\nu(\mathrm{d}x), \quad u \ge 0, \tag{3.11}$$

for $d \ge 0$ and some Lévy measure $\nu(\mathrm{d}x)$ satisfying $\int_{(0,1)} x\nu(\mathrm{d}x) < \infty$. Moreover, it is clear from Theorem 2.6 that $R'_+(0) \le 0$ and $R \not\equiv 0$. Thus, Lemma 2.4 implies that the quotient F/R is continuous at any $u > 0$. Since the elementary inequality $|\mathrm{e}^{-xh} - 1|/h < x$ holds for all $x, h > 0$, the dominated convergence theorem and the fundamental theorem of calculus applied to (3.11) yield the identity

$$-\frac{F(u)}{R(u)} = d + \int_{(0,\infty)} \mathrm{e}^{-ux} x\nu(\mathrm{d}x) \quad \text{for all } u > 0. \tag{3.12}$$

Any twice-differentiable function f that tends to zero as $|x| \to \infty$, i.e. $f \in C_0^2(\mathbb{R})$, is in the domain of the generator $\mathcal{A}_{X^F}$ of the subordinator X^F and the following formula holds

$$\mathcal{A}_{X^F} f(x) = bf'(x) + \int_{(0,\infty)} [f(x+\xi) - f(x)]\, m(\mathrm{d}\xi) \quad \text{for } x \in \mathbb{R}. \tag{3.13}$$

Fix $u > 0$ and let $f_u \in C_0^2(\mathbb{R})$ be a function that satisfies $f_u(x) = \mathrm{e}^{-ux}$ for all $x \ge 0$. Applying (3.13) to f_u yields $\mathcal{A}_{X^F} f_u(x) = F(u) f_u(x)$ for all $u > 0, x \ge 0$. Multiplying by $-W(x)$ and integrating from 0 to ∞ gives the following identity for all $u > 0$:

$$-\int_0^\infty W(x)\mathcal{A}_{X^F} f_u(x)\,\mathrm{d}x = b\int_0^\infty u\mathrm{e}^{-ux} W(x)\,\mathrm{d}x - \int_0^\infty W(x) \int_{(0,\infty)} [\mathrm{e}^{-u(x+\xi)} - \mathrm{e}^{-ux}]\, m(\mathrm{d}\xi)\mathrm{d}x. \tag{3.14}$$

Note that $\mathcal{A}_{X^F} f_u(x) \sim e^{-ux}$ for $x \to \infty$, which guarantees that the integrals are finite in light of Eq. (3.2). Since W is increasing and absolutely continuous, integration by parts gives

$$-\int_0^\infty W(x)\mathcal{A}_{X^F} f_u(x)\,\mathrm{d}x = bW(0) + b\int_0^\infty \mathrm{e}^{-ux} W'_+(x)\,\mathrm{d}x + \int_0^\infty \mathrm{e}^{-ux}\int_{(0,\infty)} [W(x) - W(x-\xi)]m(\mathrm{d}\xi)\,\mathrm{d}x, \quad (3.15)$$

for all $u > 0$. The second integral on the right-hand side of (3.15) is a consequence of the following steps: (i) note that the corresponding integrand in (3.14) does not change sign on the domain of integration, (ii) approximate the Lévy measure m by a sequence of measures $(m_n)_{n\in\mathbb{N}}$ with finite mass, (iii) apply Fubini's theorem to obtain the formula for each m_n, (iv) take the limit by applying the monotone convergence theorem.

On the other hand, combining the identity $\mathcal{A}_{X^F} f_u(x) = F(u)e^{-ux}$ for all $u > 0, x \in \mathbb{R}_{\geqslant 0}$ with (3.1) and (3.11) yields

$$-\int_0^\infty W(x)\mathcal{A}_{X^F} f_u(x)\mathrm{d}x = -\int_0^\infty W(x)F(u)e^{-ux}\mathrm{d}x = -\frac{F(u)}{R(u)} = d + \int_{(0,\infty)} e^{-ux} x\nu(\mathrm{d}x), \quad (3.16)$$

which in turn must equal the right hand side of (3.15). In the limit as $u \to \infty$, the equality of the expressions in (3.15) and (3.16) yields $d = bW(0)$. Subtracting this term we arrive at the equality

$$\int_0^\infty \mathrm{e}^{-ux}\left[bW'_+(x) + \int_{(0,\infty)} [W(x) - W(x-\xi)]\,m(\mathrm{d}\xi)\right]\mathrm{d}x = \int_{(0,\infty)} e^{-ux} x\nu(\mathrm{d}x).$$

Both sides are Laplace transforms of Borel measures on $(0,\infty)$, and we conclude from the equality of transforms the equality of the measures

$$\left[bW'_+(x) + \int_{(0,\infty)} [W(x) - W(x-\xi)]\,m(\mathrm{d}\xi)\right]\mathrm{d}x = x\nu(\mathrm{d}x)$$

for all $x > 0$. In particular it follows that $\nu(\mathrm{d}x)$ has a density $k(x)/x$ with respect to the Lebesgue measure, and that $k(x)$ is given by

$$k(x) = bW'_+(x) + W(x)m(x,\infty) + \int_{(0,x]} [W(x) - W(x-\xi)]$$

almost everywhere, which concludes the proof. □

Proof of Corollary 3.3. Following [12, Lemma 8.6], $W_+(0) > 0$ if and only if X^R has finite variation, and is equal to $1/\lambda_0$ in this case, with λ_0 defined by (3.8). If X^R has infinite variation, then $W(0) = 0$ and $\lambda_0 = \infty$, and hence Eq. (3.9) holds. Substituting the representation (3.3) of W in terms of the measure n into (3.7) yields the second Eq. (3.10). □

Remark 3.4. The formula for the k-function in (3.7) looks very much like the Feller generator of the subordinator X^F applied to the scale function W of $\widehat{X}^R$. However, the Feller generator is only defined on a subset (i.e. its domain) of the Banach space of continuous functions that tend to 0 at infinity, $C_0(\mathbb{R})$. Any function in the domain of the generator of X^F must be in $C_0(\mathbb{R})$ and differentiable; sufficient conditions for the differentiability of W are given in [3]. However, the scale function W, which is non-decreasing on $\mathbb{R}_{\geqslant 0}$, is not in $C_0(\mathbb{R})$ and thus never in the domain of the Feller generator of X^F. To remedy this problem, consider that by (3.2) and (3.3), both W and W'_+ are elements of the weighted L_1-space defined by

$$L_1^h(0,\infty) := \left\{ f \in L_1^{\text{loc}}(0,\infty) : \int_0^\infty |f(x)|h(x)\mathrm{d}x < \infty \right\},$$

where $h : (0,\infty) \to (0,\infty)$ is a continuous bounded function with $\lim_{x\downarrow 0} h(x) = 0$ and $h(x) \sim e^{-cx}$ as $x \to \infty$ for some $c > 0$. The semigroup $(\check{P}_t)_{t\geq 0}$ of the Markov process $\check{X}_t^{F,x} = (x - X_t^F)I_{\{X_t^F \leq x\}} + \partial I_{\{X_t^F > x\}}$ (i.e. the dual, started at $x > 0$, of X^F, sent to a killing state ∂ upon the first passage into $(-\infty, 0)$) acts on $L_1^h(0,\infty)$ by $\check{P}_t f(x) = \mathbb{E}[f(\check{X}_t^{F,x})] = \mathbb{E}[f(x - X_t^F)\mathbf{1}_{\{X_t^F < x\}}]$, for each $f \in L_1^h(0,\infty)$ where we take $f(\partial) = 0$. It can be shown that the $L_1^h(0,\infty)$-semigroup $(\check{P}_t)_{t\geq 0}$ is strongly continuous with a generator $\mathcal{A}_{\check{X}^F}$ and, if $R'_+(0) \leq 0$, then the scale function W associated to R is in the domain of $\mathcal{A}_{\check{X}^F}$. Furthermore the k-function k in Theorem 3.1 can be written as

$$k = -\mathcal{A}_{\check{X}^F} W. \tag{3.17}$$

The proof of these facts is straightforward but technical and rather lengthy and hence omitted.

4. Further properties of the limit distribution

As discussed in Section 2.3, self-decomposable distributions, which arise as limit distributions of $\mathbb{R}_{\geqslant 0}$-valued OU-type processes are in many aspects more regular than general infinitely divisible distributions. For self-decomposable distributions precise results are known about their support, absolute continuity and behavior at the boundary of their support. Using the notation of Section 2.3, and excluding the degenerate case of a distribution concentrated in a single point, the following holds true when L is self-decomposable:

(i) the support of L is $[b/\lambda, \infty)$;
(ii) the distribution of L is absolutely continuous;
(iii) the asymptotic behavior of the density of L at b/λ is determined by $c = \lim_{x\downarrow 0} k(x)$.

We refer the reader to [19, Theorems 15.10, 24.10, 27.13 and 53.6]. The goal in this section is to show analogous results for the limit distributions L arising from general CBI-processes, i.e. to characterize the support, the continuity properties and the asymptotic behavior at the boundary of the support of L. We start by isolating the degenerate cases. A Lévy process, such as X^F or X^R, is called degenerate, if is deterministic, or equivalently if its Laplace exponent is of the form $u \mapsto \lambda u$ for some $\lambda \in \mathbb{R}$. For a CBI-process X we draw a finer distinction.

Definition 4.1. A CBI-process X is *degenerate of the first kind*, if it is deterministic for all starting values $X_0 = x \in \mathbb{R}_{\geqslant 0}$. X is *degenerate of the second kind*, if it is deterministic when started at $X_0 = 0$.

Remark 4.2. Clearly degeneracy of the first kind implies degeneracy of the second kind. From Theorem 2.1 the following can be easily deduced: a CBI-process is degenerate of the first kind if and only if both X^F and X^R are degenerate. In this case $F(u) = bu$ and $R(u) = \beta u$, and X is the deterministic process given by

$$X_t = X_0 e^{\beta t} + \frac{b}{\beta}(e^{\beta t} - 1). \tag{4.1}$$

A CBI-process X is degenerate of the second kind, but not of the first, if and only if $X^F = 0$ and X^R is non-degenerate. In this case it is a CB-process, i.e. a continuous-state branching process *without* immigration.

The following proposition describes the support of the limit L in the degenerate cases.

Proposition 4.3. *Let X be a CBI-process, let L be its limit distribution and let k be the function defined in Theorem 3.1. If X is degenerate of the first kind, then* $\operatorname{supp} L = \{-b/\beta\}$. *If X is degenerate of the second kind but not of the first kind, then* $\operatorname{supp} L = \{0\}$. *Moreover, the following statements are equivalent:*

(a) *the support of L is concentrated at a single point;*
(b) *X is degenerate (of either kind);*
(c) *there exists a sequence $x_i \downarrow 0$ such that $k(x_i) = 0$ for all $i \in \mathbb{N}$;*
(d) *$k(x) = 0$ for all $x > 0$.*

Proof. Suppose that X is degenerate of the first kind. Then the limit L is concentrated at $-b/\beta$ by (4.1). Suppose next, that X is degenerate of the second, but not the first kind. Then $F = 0$ and by Theorem 3.1 the Laplace exponent of L is 0. It follows that L is concentrated at 0 in this case.

We proceed to show the second part of the proposition. It is obvious that (d) implies (c). To show that (c) implies (b), note that the inequality

$$k(x) \geq W(x)(b\,n(\overline{\varepsilon} > x) + m(x, \infty))$$

holds for all $x > 0$ by Eq. (3.10). Since $W(x) > 0$ for any $x > 0$, assumption (c) implies that

$$bn(\overline{\varepsilon} > x_i) + m(x_i, \infty) = 0 \quad \text{for all } x_i,\, i \in \mathbb{N}.$$

We can conclude that $m \equiv 0$ and hence $X_t^F = bt$ for all $t \geq 0$. Furthermore we see that either $b = 0$ or $n \equiv 0$. If $b = 0$, then $F = 0$ and hence, by Remark 4.2, X is a degenerate CBI-process of the second kind. On the other hand, if the Itô excursion measure n is zero, then the representation in (3.3) implies that the scale function W is constant. In this case it follows from (3.1) that $R(u) = \beta u$ for some $\beta < 0$, or equivalently that $X_t^R = \beta t$ for all $t \geq 0$ and hence that X is degenerate of the first kind.

The fact that (b) implies (a) follows from the first part of the proposition. It remains to show that (a) implies (d); this is a consequence of the fact that L is infinitely divisible with support in $\mathbb{R}_{\geqslant 0}$, and that the support of an infinitely divisible distribution in $\mathbb{R}_{\geqslant 0}$ is concentrated at a single point if and only if its Lévy measure is trivial (cf. [19, Theorem 24.3, Corollary 24.4]). □

The next result describes the support of the limit L in the non-degenerate case.

Proposition 4.4. *Let X be a non-degenerate CBI-process and let L be its limit distribution. Then*

$$\operatorname{supp} L = [b/\lambda_0, \infty),$$

where λ_0 is the effective drift of $\widehat{X}^R$, defined in (3.8). In particular $\operatorname{supp} L = \mathbb{R}_{\geqslant 0}$ *if and only if $b = 0$ or the paths of X^R have infinite variation.*

Proof. From Proposition 4.3(c) we know that there is some $\delta > 0$, such that the k-function of L is non-zero on $(0, \delta)$. For any $h \in (0, \delta)$, define L_h as the infinitely divisible distribution with Laplace exponent $\int_{(h,\infty)}(e^{-xu} - 1)\frac{k(x)}{x}\mathrm{d}x$. Each L_h is a compound Poisson distribution, with Lévy measure $\nu_h(\mathrm{d}\xi) = \frac{k(x)}{x}\mathbf{1}_{(h,\infty)}(x)$. Since k is non-zero on $(0, \delta)$

$$(h, \delta) \subset \operatorname{supp} \nu_h \subset (h, \infty). \tag{4.2}$$

From Theorem 3.1 we deduce that as $h \to 0$ the distributions L_h converge to $L(\gamma + .)$, i.e. to L shifted to the left by γ. For the supports, this implies that

$$\operatorname{supp} L = \{\gamma\} + \overline{\lim_{h \downarrow 0} \operatorname{supp} L_h} \tag{4.3}$$

where the limit denotes an increasing union of sets and '+' denotes pointwise addition of sets. Using (4.2) and the fact that L_h is a compound Poisson distribution it follows that

$$\{0\} \cup \bigcup_{n=1}^{\infty} (nh, n\delta) \subset \operatorname{supp} L_h \subset \{0\} \cup (h, \infty),$$

by Sato [19, Theorem 24.5]. Let $h \downarrow 0$ and apply (4.3) to obtain

$$\bigcup_{n=1}^{\infty} [\gamma, \gamma + n\delta] \subset \operatorname{supp} L \subset [\gamma, \infty),$$

and we conclude that $\operatorname{supp} L = [\gamma, \infty)$. By Corollary 3.3 $\gamma = b/\lambda_0$, which completes the proof. □

Proposition 4.5. *Let X be a CBI-process and let λ_0 be as in (3.8). Then the limit distribution L is either absolutely continuous on $\mathbb{R}_{\geqslant 0}$ or absolutely continuous on $\mathbb{R}_{\geqslant 0} \setminus \{b/\lambda_0\}$ with an atom at $\{b/\lambda_0\}$, according to whether*

$$\int_0^1 \frac{k(x)}{x} \mathrm{d}x = \infty \quad \textit{or} \quad \int_0^1 \frac{k(x)}{x} \mathrm{d}x < \infty. \tag{4.4}$$

Proof. If X is degenerate, then the assertion follows immediately from Proposition 4.3. In this case $k(x) = 0$ for all $x > 0$, the integral in (4.4) is always finite and the distribution of L consists of a single atom at b/λ_0.

It remains to treat the non-degenerate case. Assume first that the integral in (4.4) takes a finite value. Then also the total mass $\nu(0, \infty)$ of the Lévy measure $\nu(\mathrm{d}x) = \frac{k(x)}{x}\mathrm{d}x$ is finite, and $L - \gamma$ has compound Poisson distribution. By Sato [19, Remark 27.3] this implies that for any Borel-set $A \subset \mathbb{R}_{\geqslant 0}$

$$\int_{A+\gamma} \mathrm{d}L(x) = e^{-t\nu(0,\infty)} \sum_{j=0}^{\infty} \frac{t^j}{j!} \nu^{*j}(A), \tag{4.5}$$

where $\nu^{*j}(\mathrm{d}x)$ is the j-th convolution power of ν, and it is understood that ν^{*0} is the Dirac measure at 0. Since $\nu(\mathrm{d}x)$ is absolutely continuous – it has density $\frac{k(x)}{x}$ – also the convolution powers $\nu^{*j}(\mathrm{d}x)$ are absolutely continuous for $j \geq 1$. The first summand ν^{*0} however has an atom at 0. It follows by (4.5) that $L - \gamma$ is absolutely continuous on $(0, \infty)$ with an atom at 0, and we have shown the claim for the case $\int_0^1 \frac{k(x)}{x}\mathrm{d}x < \infty$.

Assume that $\int_0^1 \frac{k(x)}{x}\mathrm{d}x = \infty$. Then the Lévy measure $\nu(\mathrm{d}x) = \frac{k(x)}{x}\mathrm{d}x$ of L has infinite total mass, and [19, Theorem 27.7] implies that L has a distribution that is absolutely continuous, which completes the proof. □

So far, we know that the left endpoint of the support of L is $\gamma = b/\lambda_0$, and that the distribution of L may or may not have an atom at this point. In case that there is no atom, the following proposition yields an even finer description of the behavior of the distribution close to γ.

Proposition 4.6. *Let X be a CBI-process satisfying the assumptions of Theorem 3.1, and let L be its limit distribution. Suppose that $c = \lim_{x \downarrow 0} k(x)$ is in $(0, \infty)$, and define*

$$K(x) = \exp\left(\int_x^1 (c - k(y)) \frac{dy}{y}\right). \tag{4.6}$$

Then $K(x)$ is slowly varying at 0 and L satisfies

$$L(x) \sim \frac{\kappa}{\Gamma(c)} (x - \gamma)^{c-1} K(x - \gamma) \quad \text{as } x \downarrow \gamma, \tag{4.7}$$

where $\gamma = b/\lambda_0$ and

$$\kappa = \exp\left(c \int_0^1 (e^{-x} - 1) \frac{\mathrm{d}x}{x} + c \int_1^\infty e^{-x} \frac{\mathrm{d}x}{x} - \int_1^\infty k(x) \frac{\mathrm{d}x}{x}\right).$$

Proof. Note that the inequality $c > 0$ and Proposition 4.5 imply that L is absolutely continuous. Its support is by Proposition 4.4 equal to $[\gamma, \infty)$ and $L(\gamma) = 0$. The proof of [19, Theorem 53.6] for self-decomposable distributions can now be applied without change. □

Recall that ID_+ and SD_+ denote the classes of infinitely divisible and self-decomposable distributions on $\mathbb{R}_{\geqslant 0}$ respectively. Let CLIM be the class of distributions on $\mathbb{R}_{\geqslant 0}$ that arise as limit distributions of CBI-processes.

Proposition 4.7. *The class* CLIM *is contained strictly between the self-decomposable and the infinitely divisible distributions on $\mathbb{R}_{\geqslant 0}$, i.e.*

$$\mathrm{SD}_+ \subsetneq \mathrm{CLIM} \subsetneq \mathrm{ID}_+.$$

Proof. The inclusion $\mathrm{CLIM} \subset \mathrm{ID}_+$ follows from Theorem 2.6, and the inclusion $\mathrm{SD}_+ \subset \mathrm{CLIM}$ from Theorem 2.9 and the fact that each $\mathbb{R}_{\geqslant 0}$-valued OU-type process (see (2.10)) is a CBI-process with $R'_+(0) = -\lambda < 0$. The strictness of the inclusions can be deduced from the following facts:

- all distributions in SD_+ are either degenerate or absolutely continuous (cf. [19, Theorem 27.13]);
- all distributions in CLIM are absolutely continuous on $\mathbb{R}_{\geqslant 0} \setminus \{b/\lambda_0\}$, but some concentrate non-zero mass at $\{b/\lambda_0\}$ (cf. Propositions 4.4 and 4.5);
- the class ID_+ contains singular distributions (cf. [19, Theorem 27.19]). □

For a more direct proof of the fact that SD_+ is *strictly* included in CLIM we exhibit an example of a distribution that is in CLIM but not in SD_+.

Example 4.8 (*CBI-Process with Non Self-Decomposable Limit Distribution*)**.** In this example we consider the class of CBI-processes X given by a general subordinator X^F and spectrally positive process X^R equal to a Brownian motion with strictly negative drift. The Laplace exponent of X^R is $R(u) = -\alpha u^2 + \beta u$ with $\alpha > 0$, $\beta < 0$. It is easy to check using (3.1) that the scale function of the dual $\widehat{X}^R$ and its derivative are

$$W(x) = [\exp(x\beta/\alpha) - 1]/\beta \quad \text{and} \quad W'(x) = \exp(x\beta/\alpha)/\alpha. \tag{4.8}$$

Theorem 3.1 implies that the characteristics of the limit distribution L are given by $\gamma = 0$ and

$$k(x) = e^{x\beta/\alpha}\left[\frac{b}{\alpha} + \frac{1}{\beta}\left(m(x,\infty) + \int_{(0,x]}(1 - e^{-\xi\beta/\alpha})\, m(d\xi)\right)\right] - m(x,\infty)/\beta, \tag{4.9}$$

where $b \in \mathbb{R}_{\geqslant 0}$ is the drift and m the Lévy measure of the subordinator X^F. Assuming in addition that X^F is a compound Poisson process with exponential jumps and setting parameters equal to

$$m(x,\infty) = e^{-x}, \qquad b = 0, \qquad \alpha = 1/2, \qquad \beta = -1,$$

formula (4.9) reduces to $k(x) = 2(e^{-x} - e^{-2x})$. Since this k-function is not decreasing, the corresponding distribution L, which is in CLIM, cannot be in SD_+.

Proposition 4.9 gives sufficient conditions for a distribution in CLIM to be self-decomposable.

Proposition 4.9. *Let X be a CBI-process and let L be its limit distribution. Each of the following conditions is sufficient for L to be self-decomposable:*

(a) $\mu = 0$ *and* $\alpha = 0$*,*
(b) $\mu = 0$ *and* $m = 0$*,*
(c) $m = 0$ *and W is concave on* $(0,\infty)$.

Conversely, if $m = 0$ and L is self-decomposable, then W must be concave on $(0,\infty)$.

Remark 4.10. The monotonicity of the derivative of the scale function, which arises in Proposition 4.9, also plays a role in other applications of scale functions (e.g. control theory [16]; conjugate Bernstein functions and one-sided Lévy processes [13]).

Proof. The first two conditions are rather trivial. In the first case X is an OU-type process, and self-decomposability follows from the classical results of [9,20] that we state as Theorem 2.9. In the second case X has no jumps, and hence is a Feller diffusion. This process is well-studied, and its limit distribution is known explicitly. It is a shifted gamma distribution, which is always self-decomposable. It remains to show (c) and the converse assertion. Assume that $m = 0$. By Theorem 3.1 we have $k(x) = bW'_+(x)$ in this case. An infinitely divisible distribution is self-decomposable if and only if it can be written as in (2.12) with decreasing k-function. Clearly k is decreasing if and only if W'_+ is, or equivalently if W is concave on $\mathbb{R}_{\geqslant 0}$. □

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Appendix. Additional proofs for Section 2

Proof of Theorem 2.6. We first show that (c) is equivalent to (a) and that L has to satisfy (i) and (ii). Consider the three alternatives for the behavior of $R(u)$ that are outlined in Lemma 2.4. Through the Riccati Eqs. (2.6) they imply the following behavior of $\psi(t,u)$: If $R'_+(0) > 0$ then

$\lim_{t\to\infty} \psi(t,u) = u_0$ for all $t, u > 0$, if $R \equiv 0$ then $\psi(t,u) = u$ for all $u \geq 0$, and if $R'_+(0) \leq 0$ but $R \not\equiv 0$ then $\lim_{t\to\infty} \psi(t,u) = 0$. Moreover,

$$\lim_{t\uparrow\infty} -\log \mathbb{E}[e^{-uX_t}] = \lim_{t\uparrow\infty} (\phi(t,u) + x\psi(t,u))$$
$$= \int_0^\infty F(\psi(r,u))\,\mathrm{d}r + x \cdot \lim_{t\to\infty} \psi(t,u). \tag{A.1}$$

We see that if $R'_+(0) > 0$ or $R \equiv 0$ the right-hand side diverges for $u > 0$, and hence no limit distribution exists in these cases. In case that $R'_+(0) \leq 0$ and $R \not\equiv 0$, the transformation $s = \psi(r,u)$ yields that

$$\lim_{t\uparrow\infty} -\log \mathbb{E}[e^{-uX_t}] = \int_0^u \frac{F(s)}{R(s)}\mathrm{d}s. \tag{A.2}$$

This integral is finite, if and only if condition (2.7) holds. If it is finite then Lévy's continuity theorem for Laplace transforms guarantees the existence of, and convergence to, the limit distribution L with Laplace exponent given by (2.8). It is also clear that L must be infinitely divisible, since it is the limit of infinitely divisible distributions. If on the other hand the integral in (A.2) is infinite for some $u \in \mathbb{R}_{\geqslant 0}$, then there is no pointwise convergence of Laplace transforms, and hence also no weak convergence of X_t as $t \to \infty$.

To complete the proof it remains to show that any limit distribution is also invariant and vice versa, i.e. that (b) is equivalent to (a). Assume that $\widetilde{L}$ is an invariant distribution of $(X_t)_{t\geq 0}$, and has Laplace exponent $\widetilde{l}(u) = -\log \int_{[0,\infty)} e^{-ux}\mathrm{d}\widetilde{L}(x)$. Denote $f_u(x) = e^{-ux}$ and note that the invariance of $\widetilde{L}$ implies

$$\int_{[0,\infty)} f_u(x)\mathrm{d}\widetilde{L}(x) = \int_{[0,\infty)} P_t f_u(x)\mathrm{d}\widetilde{L}(x) = e^{-\phi(t,u)} \int_{[0,\infty)} e^{-x\psi(t,u)}\mathrm{d}\widetilde{L}(x) \tag{A.3}$$

for all $t, u \geq 0$. This can be rewritten as $\widetilde{l}(u) = \phi(t,u) + \widetilde{l}(\psi(t,u))$. Taking derivatives with respect to t and evaluating at $t = 0$ this becomes $0 = F(u) + \widetilde{l}'(u)R(u)$. Since $\widetilde{l}(u)$ is continuous on $\mathbb{R}_{\geqslant 0}$ with $\widetilde{l}(0) = 0$, the above equation can be integrated to yield $\widetilde{l}(u) = -\int_0^u \frac{F(s)}{R(s)}\mathrm{d}s$. By the first part of the proof this implies that a limit distribution L exists, with Laplace exponent $l(u)$ coinciding with $\widetilde{l}(u)$. We conclude that also the probability laws L and $\widetilde{L}$ on $\mathbb{R}_{\geqslant 0}$ coincide, i.e. $L = \widetilde{L}$. Conversely, assume that a limit distribution L exists. To show that L is also invariant, note that (2.1), (2.6) and (2.8) imply

$$\int_{[0,\infty)} P_t f_u(x)\mathrm{d}L(x) = \exp(-\phi(t,u) - l(\psi(t,u)))$$
$$= \exp\left(-\int_0^t F(\psi(r,u))\mathrm{d}r + \int_0^{\psi(t,u)} \frac{F(s)}{R(s)}\mathrm{d}s\right)$$
$$= \exp\left(\int_0^u \frac{F(s)}{R(s)}\mathrm{d}s\right) = \int_{[0,\infty)} f_u(x)\mathrm{d}L(x).$$

This completes the proof. □

Proof of Corollary 2.8. Assume that $\int_{[1,\infty)} \log\xi\, m(\mathrm{d}\xi) < \infty$. From the concavity of $R(u)$, Lemma 2.4 and the fact that $F(u) \geq 0$ for all $u \geq 0$ we obtain that

$$0 \le -\int_0^u \frac{F(s)}{R(s)}\,\mathrm{d}s \le -\frac{1}{R'(0)}\int_0^u \frac{F(s)}{s}\,\mathrm{d}s$$
$$= -\frac{1}{R'(0)}\left(bu + \int_0^u \int_{(0,\infty)} \frac{1-e^{-s\xi}}{s} m(\mathrm{d}\xi)\mathrm{d}s\right). \tag{A.4}$$

In order to show that this upper bound is finite, it is enough to show that the double integral on the right takes a finite value. Since the integrand is positive, the integrals can be exchanged by the Tonelli–Fubini theorem. Defining the function $M(\xi) = \int_0^u \frac{1-e^{-s\xi}}{s}\mathrm{d}s$, we can write

$$\int_0^u \int_{(0,\infty)} \frac{1-e^{-s\xi}}{s} m(\mathrm{d}\xi)\mathrm{d}s = \int_{(0,\infty)} M(\xi)\, m(\mathrm{d}\xi).$$

An application of L'Hôpital's formula reveals the following boundary behavior of $M(\xi)$:

$$\lim_{\xi\to 0} \frac{M(\xi)}{\xi} = u \quad \text{and} \quad \lim_{\xi\to\infty} \frac{M(\xi)}{\log \xi} = 1. \tag{A.5}$$

Choosing suitable constants $C_1, C_2 > 0$ we can bound $M(\xi)$ from above by $C_1\xi$ on $(0, 1)$ and by $C_2 \log \xi$ on $[1, \infty)$. Note that $m(\mathrm{d}\xi)$ integrates the function $\xi \mapsto C_1\xi$ on $(0, 1)$ by Theorem 2.1, and integrates the function $\xi \mapsto C_2 \log \xi$ on $[1, \infty)$ by assumption. Hence

$$\int_{(0,\infty)} M(\xi) m(\mathrm{d}\xi) \le C_1 \int_{(0,1)} \xi\, m(\mathrm{d}\xi) + C_2 \int_{[1,\infty)} \log \xi\, m(d\xi) < \infty,$$

and we have shown that the upper bound in (A.4) is finite and that (2.7) holds true.

Suppose now that $\int_{\xi>1} \log \xi\, m(\mathrm{d}\xi) = \infty$. Since $R'_+(0) < 0$ we can find $\epsilon, \delta > 0$ such that $R'_+(0) + \epsilon < 0$ and $R(u) \ge (R'_+(0) - \epsilon)u$ for all $u \in (0, \delta)$. Hence,

$$-\int_0^u \frac{F(s)}{R(s)}\,\mathrm{d}s \ge \frac{1}{\epsilon - R'_+(0)} \int_0^u \frac{F(s)}{s}\,\mathrm{d}s$$
$$= \frac{1}{\epsilon - R'_+(0)}\left(bu + \int_0^u \int_{(0,\infty)} \frac{1-e^{-s\xi}}{s} m(\mathrm{d}\xi)\mathrm{d}s\right), \tag{A.6}$$

for all $u \in (0, \delta)$. Exchanging integrals by the Tonelli—Fubini theorem and using the function $M(\xi)$ defined above we get

$$\int_0^u \int_{(0,\infty)} \frac{1-e^{-s\xi}}{s} m(\mathrm{d}\xi)ds = \int_{(0,\infty)} M(\xi) m(\mathrm{d}\xi) \ge C'_2 \int_{[1,\infty)} \log \xi\, m(\mathrm{d}\xi) = \infty,$$

where $C'_2 > 0$ is a finite constant which exists by the second limit in (A.5). This shows that the right hand side of (A.6) is infinite and hence that (2.7) cannot hold true. □

References

[1] Jean Bertoin, Lévy Processes, Cambridge University Press, 1996.

[2] Ma. Emilia Caballero, José Luis Pérez Garmendia, Gerònimo Uribe Bravo, A Lamperti type representation of continuous-state branching processes with immigration, 2010. arXiv:1012.2346.

[3] T. Chan, A.E. Kyprianou, M. Savov, Smoothness of scale functions for spectrally negative Lévy processes, Probability Theory and Related Fields 150 (3–4) (2011) 691–708.

[4] E. Çinlar, M. Pinsky, A stochastic integral in storage theory, Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 17 (1971) 227–240.

[5] D.A. Dawson, Zenghu Li, Skew convolution semigroups and affine Markov processes, The Annals of Probability 34 (3) (2006) 1103–1142.
[6] C.R. Heathcote, A branching process allowing immigration, Journal of the Royal Statistical Society B 27 (1) (1965) 138–143.
[7] C.R. Heathcote, Corrections and comments on the paper a branching process allowing immigration, Journal of the Royal Statistical Society B 28 (1) (1966) 213–217.
[8] Lancelot F. James, Bernard Roynette, Mark Yor, Generalized gamma convolutions, Dirichlet means, Thorin measures, with explicit examples, Probability Surveys 5 (2008) 346–415.
[9] Z.J. Jurek, W. Vervaat, An integral representation for self-decomposable Banach space valued random variables, Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 62 (1983) 247–262.
[10] Kiyoshi Kawazu, Shinzo Watanabe, Branching processes with immigration and related limit theorems, Theory of Probability and its Applications XVI (1) (1971) 36–54.
[11] Martin Keller-Ressel, Thomas Steiner, Yield curve shapes and the asymptotic short rate distribution in affine one-factor models, Finance and Stochastics 12 (2) (2008) 149–172.
[12] Andreas E. Kyprianou, Introductory Lectures on Fluctuations of Lévy Processes with Applications, Springer, 2006.
[13] A.E. Kyprianou, V. Rivero, Conjugate and complete scale functions for spectrally negative Lévy processes, Electronic Journal of Probability 13 (57) (2008) 1672–1701.
[14] John Lamperti, Continuous-state branching processes, Bulletin of the AMS 73 (1967) 382–386.
[15] Zenghu Li, Measure-valued branching Markov processes, in: Probability and its Applications, Springer, 2011.
[16] Ronnie Loeffen, On optimality of the barrier strategy in de Finetti's dividend problem for spectrally negative Lévy processes, Annals of Applied Probability 18 (5) (2008) 1669–1680.
[17] H. Masuda, N. Yoshida, Asymptotic expansion for Barndorff–Nielsen and Shephard's stochastic volatility model, Stochastic Processes and their Applications 115 (2005) 1167–1186.
[18] Mark Pinsky, Limit theorems for continuous state branching processes with immigration, Bulletin of the AMS 78 (2) (1972) 242–244.
[19] Ken-Iti Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press, 1999.
[20] Ken-iti Sato, M. Yamazato, Operator-selfdecomposable distributions as limit distributions of processes of Ornstein–Uhlenbeck type, Stochastic Processes and Applications 17 (1984) 73–100.
[21] M. Yamazato, Unimodality of infinitely divisible distribution functions of class L, Annals of Probability 6 (1978) 523–531.