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Total Vertex-Irregularity Labelings for Subdivision of Several Classes of Trees

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Abstract

Motivated by the notion of the irregularity strength of a graph introduced by Chartrand et al.[3] in 1988 and various kind of other total labelings, Baca *et al.*[1] introduced the total vertex irregularity strength of a graph.
In 2010, Nurdin, Baskoro, Salman and Gaos[5] determined the total vertex irregularity strength for various types of trees, namely complete $k-$ary trees, a subdivision of stars, and subdivision of particular types of caterpillars. In other paper[6], they conjectured that the total vertex irregularity strength of any tree T is only determined by the number of vertices of degree 1, 2, and 3 in T. In this paper, we attempt to verify this conjecture by considering a subdivision of several types of trees, namely caterpillars, firecrackers, and amalgamation of stars.

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1. Introduction

The following problem was proposed by Chartrand, Jacobson, Lehel, Oellermann, Ruiz and Saba[3]. Assign positive integer labels to the edges of a simple connected graph of order at least 3 in such a way that the graph becomes irregular, i.e., the weights (label sums) at each vertex are distinct. The minimum value of the largest label over all such irregular assignments is known as the *irregularity strength* of such a graph. Finding the irregularity strength of a graph seems to be rather hard[2] even for simple graphs. Later, Baca, Jendrol, Miller and Ryan[1] introduced the *total vertex irregularity strength* of a graph as follows. Let $G(V, E)$ be a simple graph. For a labeling $\lambda : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, k\}$ the *weight* of a vertex x is defined as $wt(x) = \lambda(x) + \sum_{xz \in EG} \lambda(xz)$. The mapping λ is called a *vertex irregular total k-labeling* if for every pair of distinct two vertices x and y we have $wt(x) \neq wt(y)$. The minimum k for which the graph G has a vertex irregular total k-labeling is called the *total vertex irregularity strength* of G and is denoted by $tvs(G)$. Baca *et al.*[1] proved that $tvs(C_n) = \lceil \frac{n+2}{3} \rceil$, $n \geq 2$, $tvs(K_n) = 2$ for any $n \geq 3$,

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$tvs(K_{1,n}) = \lceil \frac{n+1}{2} \rceil$, and $tvs(C_n \times P_2) = \lceil \frac{2n+3}{4} \rceil$. If T is a tree with m pendant vertices and no vertex of degree 2, they proved that $\lceil \frac{t+1}{2} \rceil \leq tvs(T) \leq m$. They also proved that if G is a (p, q) graph with minimum degree δ and maximum degree Δ, then $\lceil \frac{p+\lambda}{\Delta+1} \rceil \leq tvs(G) \leq p + \Delta - 2\delta + 1$.

Nurdin, Baskoro, Salman and Gaos [5,6] determined the total vertex irregularity strength of trees containing vertices of degree 2, namely a subdivision of a star and a subdivision of a particular caterpillar. They also improved some of the bounds given in [1] and showed that $tvs(P_n) = \lceil \frac{n+1}{3} \rceil$. In [4] Nurdin, Baskoro, Salman and Gaos proved that for $t \geq 2$, $tvs(tP_n) = \lceil \frac{nt+1}{3} \rceil$ for $n \geq 4$, and $tvs(tP_n) = t + 1$ for $n = 2, 3$. In [7] Nurdin *et al.* proved that for a quadtree Q_d with $d \geq 2$, $tvs(Q_d) = 2^{2d-1} + 1$. They also proved that for banana tree (B_n, t), $tvs(B_n, t) = \lceil \frac{n(t-1)}{2} \rceil + 1$, $n \geq 3$ and $t \geq 3$.

In [4] Nurdin, Baskoro, Salman and Gaos proved that the total vertex irregularity strength of the complete k-ary tree ($k \geq 2$) with depth $d \geq 1$ is $\lceil \frac{k^d+1}{2} \rceil$ and the total vertex irregularity strength of the subdivision of $K_{1,n}$ for $n \geq 3$ is $\lceil \frac{n+1}{3} \rceil$. Let G be a special caterpillar obtained by taking a path P_m and m copies of P_n denoted by $P_{n,1}, P_{n,2}, \ldots, P_{n,m}$ where $m \geq 2, n \geq 2$, and then joining the i-th vertex of P_m to an end vertex of the path $P_{n,i}$. Then, they showed that $tvs(G) = \lceil \frac{mn+3}{3} \rceil$.

For any general tree T with maximum degree Δ, Nurdin *et al.* [6] showed that $tvs(T) \geq max\{t_1, t_2, \ldots, t_\Delta\}$ where $t_i = \lceil (1 + \sum_{j=1}^{i} n_j)/(i + 1) \rceil$ and n_i be the number of vertices degree $i \in [1, \Delta]$. Furthermore, they also conjectured that the total vertex irregularity strength of any tree T is only determined by the number of its vertices of degrees 1, 2 and 3. Precisely, they conjectured that for any tree T we have $tvs(T) = \max\{t_1, t_2, t_3\}$. Recently, Susilawati, E. T. Baskoro and R. Simanjuntak [8] proved that in any tree T with maximum degree 4, there is $i \in \{1, 2, 3\}$ such that $t_i \geq t_4$. As a consequence, tvs of such a tree T is at least $\max\{t_1, t_2, t_3\}$. They [8] also gave some condition for trees with maximum degree 4 whose $tvs(T) = \max\{t_1, t_2, t_3\}$. In this paper, we study the correctness of this conjecture by considering a subdivision of several types of trees, namely caterpillars, firecrackers and amalgamation of stars.

2. Main Results

Let T be a tree with p vertices and q edges. Let n_i be the number of vertices degree i. Baca *et al* in [1] proved that

$$n_1 = 2 + \sum_{i \geq 2} (i - 2)n_i.$$

Theorem 1. *[6] Let T be a tree with maximum degree Δ. Let n_i be the number of vertices of degree i, then*

$$tvs(T) \geq max\{t_1, t_2, \ldots, t_\Delta\},$$

where $t_i = \lceil (1 + \sum_{j=1}^{i} n_j)/(i + 1) \rceil$ and n_i be the number of vertices degree $i \in [1, \Delta]$.

2.1. Subdivision of caterpillars

In this subsection, we determine the total vertex irregularity strength of a subdivision graph of a non-homogeneous caterpillar. For integers $m, k_1, k_2, \ldots, k_m \geq 2$, define a *caterpillar* $C_m^{\{k_1,k_2,\ldots,k_m\}}$ as a graph obtained by attaching k_i vertices to each vertex of c_i of the path P_m, for $i \in [1, m]$. The path P_m in $C_m^{\{k_1,k_2,\ldots,k_m\}}$ is called the *backbone* of the caterpillar. All vertices degree one are called *pendant vertices*. All pendant edges adjacent to c_i are labeled by $c_{11}, c_{12}, \ldots, c_{ij}$ where $1 \leq j \leq |k_i|$. Edges c_ic_{ij} and c_ic_{i+1} are called *pendant edges* and *backbone edges*, respectively. If $k_1 = k_2 = \ldots = k_m = r$, then the caterpillar is called to be *homogeneous*, and it is denoted by C_m^r. Otherwise, the caterpillar is called to be *non-homogeneous*. Let $G = (V, E)$ be a connected graph and $e \in E(G)$. The *subdivision* of a graph G on the edge e in k times is a graph obtained from the graph G by replacing edge $e = uv$ with a path $(u, x_1, x_2, \cdots, x_k, v)$ on $k + 2$ vertices. The vertices x_i are called *subdivision vertices*. Now, denote by $Sub(C_m^{\{k_1,k_2,k_3,\ldots,k_m\}}, \{w_1, w_2, w_3, \ldots, w_{m-1}\})$ the subdivision of a non-homogeneous caterpillar on all the backbone edges in $w_1, w_2, \cdots$, or w_{m-1} times, respectively. Denote by x_i all the subdivision vertices. See Figure 1 as examples.

Theorem 2. *Let $T \simeq Sub(C_m^{\{k_1,k_2,\ldots,k_m\}}, \{w_1, w_2, w_3, \ldots, w_{m-1}\})$ where $m \geq 2$ and $w_k, k_i \geq 1$ for $1 \leq k \leq m - 1$ and $1 \leq i \leq m$. The number of vertices degree one and two are $n_1 = \sum_{i=1}^{m} k_i$ and $n_2 = \sum_{k=1}^{m-1} w_k$, respectively. For $n_2 \geq n_1$ and $n_{j+1} \leq \frac{1}{2}(n_j - 1)$ for $j \geq 2$, then $tvs(T) \leq t_2$.*

Proof. Define a labeling algorithm λ as follows.

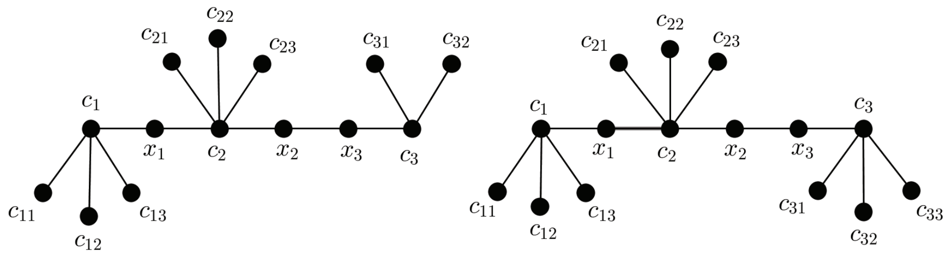

Fig. 1. Graf $T \simeq Sub(C_3^{\{3,3,2\}}, \{1,2\})$ dan $T \simeq Sub(C_3^{\{3,3,3\}}, \{1,2\})$

1. Label all pendant vertices and pendant edges by the following steps.

   (a). Let $V_1' = \{c_1, c_2, \ldots, c_m\}$ be a set of backbone vertices in P_m where $d(c_i) \geq d(c_{i+1})$. Let $V_1 = \{c_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq k_i\}$ be a set of pendant vertices in T.
   (b). Label t_2 first pendant vertices in V_1 by 1.
   (c). Label $(n_1 - t_2)$ remaining vertices in V_1 by $2, 3, \ldots, (n_1 - t_2 + 1)$.
   (d). Let $E_1 = \{e_{ij} \mid e_{ij}$ edge incident to $c_{ij}, \ \forall i, j\}$ be a set of pendant edges. Label t_2 edges in E_1 by $1, 2, \ldots, t_2$ and label $(n_1 - t_2)$ edges in E_1 by t_2.

2. Let $E_2 = \{x_n x_{n+1} \mid n \in [1, w_i - 1], i \in [1, m-1]\}$. Define $k_1 = n_1$ and $k_i = k_1 + \sum_{r=1}^{i-1} w_r - 1$ for each $2 \leq i \leq m-1$. Now, for $1 \leq i \leq m-1$ and $1 \leq n \leq w_i - 1$, define $\lambda(x_n x_{n+1}) = \lceil \frac{1+n+k_i}{3} \rceil$.
3. Next, define $\lambda(e) = t_2$, for $e \in E \setminus (E_1 + E_2)$.
4. Label all vertices in $V \setminus V_1$ by the following steps.
   Denote all vertices in $V \setminus V_1$ by $y_1, y_2, \ldots, y_N$, where $N = \sum_{i=1}^{m-1} w_i + m$, such that $s(y_1) \leq s(y_2) \leq \ldots \leq s(y_N)$ with $s(y) = \sum_{yz \in E(T)} \lambda(yz)$, which can be considered as the temporary weight of y_i in T.
   Define $\lambda(y_1)$ recursively as follows.

$$\lambda(y_1) = n_1 + 2 - s(y_1), \text{ which implies } wt(y_1) = \lambda(y_1) + s(y_1).$$

   For $2 \leq i \leq N$

$$\lambda(y_i) = max\{1, wt(y_{i-1}) + 1 - s(y_i)\}.$$

We conclude that:

1. λ is a mapping from $V(T) \cup E(T)$ into $\{1, 2, \ldots, t_2\}$,
2. The weights of pendant vertices are $2, 3, 4, \ldots, n_1 + 1$, respectively.
3. The weight of all remaining vertices forms $n_1 + 2 = wt(y_1) < wt(y_2) < \ldots < wt(y_N)$ where $N = \sum_{i=1}^{m-1} w_i + m$.

Then, $tvs(T) \leq t_2$. □

2.2. Subdivision of fire crackers

In this subsection, we determined the total vertex irregularity strength of subdivision graph of a non-homogeneous fire crackers. For integers $m, w_1, w_2, \ldots, w_m \geq 3$. Let $G_1, G_2, \ldots, G_m$ be a family of disjoint stars and w_i be the number of vertices degree one in G_i. Let a_i be a pendant vertex of G_i, $1 \leq i \leq m$, define a *fire crackers* $F_m^{\{w_1, w_2, \ldots, w_m\}}$ as a tree graph which contains all the m stars and a path joining $a_1, a_2, \ldots, a_m$.

The vertices in path P_m in $F_m^{\{w_1,w_2,\ldots,w_m\}}$ is called the *backbone vertices* of the fire cracker and labeled by $x_1, x_2, \ldots, x_m$. All vertices degree one are called *pendant vertices*. All vertices incident to pendant vertices are called *central vertices* and labeled by $c_1, c_2, \ldots, c_m$. All pendant vertices attached to c_i where $1 \le i \le m$ are labeled by $a_{11}, a_{12}, \ldots, a_{iw_m}$. Edges $c_i a_{iw_i}$ and $x_k x_{k+1}$ where $1 \le k + m$ are called *pendant edges* and *backbone edges*, respectively. If $w_1 = w_2 = \ldots = w_m = w$, then the fire cracker is called to be *homogeneous*, and it is denoted by F_m^w. Otherwise, the fire cracker is called to be *non-homogeneous*. Let $G = (V, E)$ be a connected graph and $e \in E(G)$. The *subdivision* of a graph G on the edge e in k times is a graph obtained from the graph G by replacing edge $e = uv$ with a path $(u, x_1, x_2, \ldots, x_k, v)$ on $k + 2$ vertices. Now, denoted by $T \backsimeq Sub(F_m^{\{w_1,w_2,\ldots,w_m\}}, \{q_1, q_2, \ldots, q_{m-1}\})$ the subdivision of a non-homogeneous fire cracker on all the backbone edges in $q_1, q_2, \ldots,$ or q_{m-1} times, respectively. See Figure 2 for illustration.

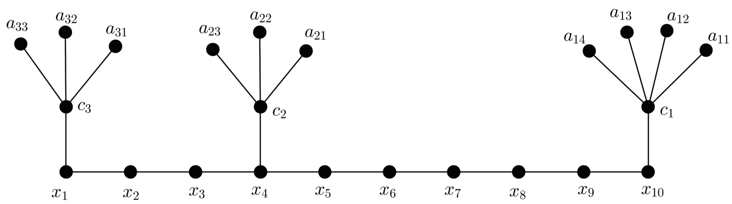

Fig. 2. $T \backsimeq Sub(F_3^{\{3,3,4\}}, \{2, 5\})$

Theorem 3. *Let $T \simeq Sub(F_m^{\{w_1,w_2,\ldots,w_m\}}, \{q_1, q_2, \ldots, q_{m-1}\})$ with $m, w_i \ge 3$ for $1 \le i \le m$ and $q_j \ge 2$ for $1 \le j \le m-1$. The number of vertices degree one and two are $n_1 = \sum_{i=1}^m w_i$ and $n_2 = \sum_{j=1}^{m-1} q_j + 2$, respectively. For $n_2 \ge n_1$ and $n_{k+1} \le \frac{1}{2}(n_k - 1)$ for $k \ge 2$, then $tvs(T) \le t_2$.*

Proof. Define a labeling algorithm ϕ as follows.

1. Label all pendant vertices and pendant edges by the following steps.

   (a). Let $V_1 = \{c_1, c_2, \ldots, c_l\}$ be a set of central vertices where $d(c_i) \ge d(c_{i+1})$. Let $V_1' = \{a_{ij} \mid 1 \le i \le m, 1 \le j \le w_i, \forall\, i\}$ be a set of pendant vertices in T.
   (b). Label t_2 first pendant vertices in V_1' by 1.
   (c). Label $(n_1 - t_2)$ remaining vertices in V_1 by $2, 3, \ldots, n_1 - t_2 + 1$.
   (d). Let $E_1 = \{e_{ij} \mid e_{ij}$ edges incident to $a_{ij}, 1 \le i \le m$ and $1 \le j \le w_i\}$ be a set of pendant edges. Label t_2 edges in E_1 by $1, 2, \ldots, t_2$ and label $(n_1 - t_2)$ edges in E_1 by t_2.

2. Let $E_2 = \{x_i x_{i+1} \mid 1 \le i \le \sum_{j=1}^{m-1} q_j + m - 1\}$ be a set of backbone edges in P_m. Define $\phi(x_i x_{i+1}) = \lceil \frac{i+1+n_1}{3} \rceil$.
3. Next, define $\phi = t_2$ for $e \in E \backslash (E_1 + E_2)$.
4. Label all vertices in $V \setminus V_1$ by the following steps. Denote vertices $V \setminus V'$ with $y_1, y_2, y_3, \ldots, y_N$ where $N = \sum_{j=1}^{m-1} q_j + 2m$ such that $s(y_1) \le s(y_2) \le \ldots \le s(y_N)$ with $s(y) = \sum_{yz \in E(T)} \phi(yz)$, which can be considered as temporary weight of y_i in T. Define $\phi(y_1)$ recursively as follows.

$$\phi(y_1) = n_1 + 2 - s(y_1), \quad \text{which implies} \quad wt(y_1) = \phi(y_i) + s(y_i).$$

   For $2 \le i \le N$, then

$$wt(y_i) = max\{1, wt(y_{i-1}) + 1 - s(y_i)\} \text{ and } wt(y_i) = s(y_i) + \phi(y_i).$$

We conclude that:

1. ϕ is mapping from $V(T) \cup E(T)$ into $\{1, 2, \ldots, t_2\}$,
2. The weight of all pendant verties are $2, 3, 4, \ldots, n + 1$.

3. The weight of all remaining vertices form $n_1 + 2 = wt(y_1) < wt(y_2) < \ldots < wt(y_N)$ where $N = \sum_{j=1}^{m-1} q_j + 2m$.

Then, we have $tvs(T) \leq t_2$. □

2.3. Subdivision of amalgamation of stars

In this subsection, we determine the total vertex irregularity strength of a subdivision of amalgamation of stars. Let $\{G_1, G_2, \ldots, G_n\}$ be a finite collection of graphs and each G_i has a fixed vertex v_{0i}. The *vertex-amalgamation* of $G_1, G_2, \ldots, G_n$, denoted by $vertex-amal\{G_i; v_{0,i}; m\}$ is formed by taking all the G_i's and identifying their terminal vertices. Consider G_i is star graph K_{1,w_i} and $v_{0,i}$ be a pendant vertices in K_{1,w_i} for all i, denoted by $T \simeq vertex - amal\{K_{1,w_i}; v_{0,i}; m\}$ where m is the number of star graphs. All vertices degree one are called *pendant vertices*. All edges incident to pendant vertices are called *pendant edges*. All pendant edges incident to $v_{0,1}$ are called *fixed edges*.

Let $G = (V, E)$ be a connected graph and $e \in E(G)$. The subdivision of a graph G on the edge e in k times is a graph obtained from the graph G by replacing $e = uv$ with a path $u, x_1, x_2, \ldots, x_k, v$ on $k + 2$ vertices. The vertices x_i are called *subdivision vertices*. Now, denote by $T \simeq Sub(vertex - amal\{K_{1,w_i}; v_{0,i}; m\}, \{q_i\})$ where $1 \leq i \leq m$ and $q_i = (w_i - 1)$ the subdivision of amalgamation of star graphs on all of the fixed edges in $q_1, q_2, \ldots, q_m$ times, respectively. Denoted by x_{m,q_1+2} all the subdivision vertices. All vertices pendant vertices are labeled by $c_{1,1}, c_{1,2}, \ldots, c_{1,w_1}, c_{2,1}, c_{2,2}, \ldots, c_{2,w_2}, \ldots, c_{m,w_m}$. See Figure 3 as example.

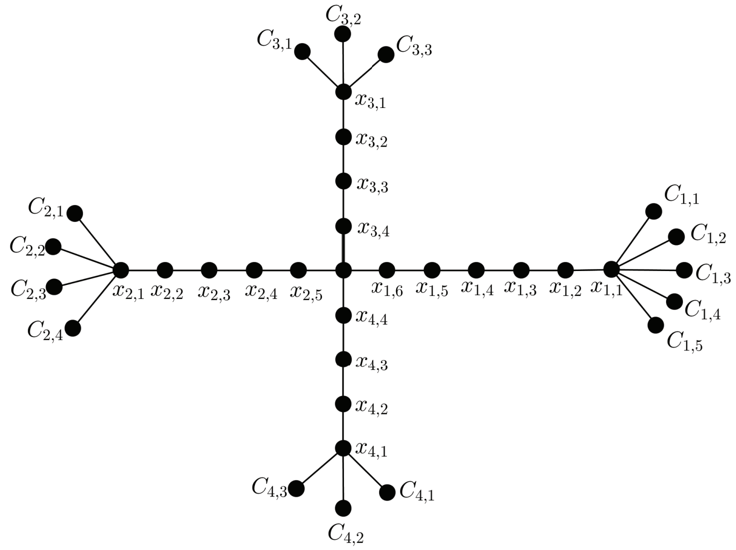

Fig. 3. $T \simeq Sub(Vertex - Amal\{K_{1,4}, K_{1,5}, K_{1,4}, K_{1,6}; v_{0,1}, v_{0,2}, v_{0,3}, v_{0,4}; 4\}, \{3, 4, 3, 5\})$

- Let $V(G) = \{x_{i,j} \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq q_i + 2\} \cup \{C_{i,l} \mid 1 \leq l \leq |w_i| - 1\}$ be a set of vertices in T.
- Let $E(T) = \{x_{i,j}x_{i,j+1} \mid 1 \leq i \leq n \text{ and } 1 \leq i \leq q_i + 1\} \cup \{x_{i,1}C_{i,l} \mid 1 \leq l \leq w_i - 1\}$ be a set of edges in T.

We can see that $x_{i,|q_i|+2}$ is one vertex for every $1 \leq i \leq m$.

Theorem 4. *Let $T \simeq Sub(vertex - amal\{K_{1,w_i}; v_{0,i}; m\}, \{q_i\})$ be a graph with $w_i, m \geq 3$ and $q_i = (w_i - 1)$ for each $1 \leq i \leq m$ is the number of subdivision in all of the backbone edges. The number of vertices degree one and two are $n_1 = \sum_{i=1}^{m} w_i - 1$ and $n_2 = \sum_{i=1}^{m} q_i$, respectively. For $n_2 \geq n_1$ and $n_{k+1} \leq \frac{1}{2}(n_k - 1)$ for $k \geq 2$. Then, $tvs(T) \leq t_2$.*

Proof. Define a labeling algorithm ψ as follows.

1. Label all pendant vertices and pendant edges by the following steps.

(a). Sort all w_i for $1 \geq i \geq m$ such that $w_1 \geq w_2 \geq \ldots \geq w_m$. This implies $q_1 \geq q_2 \geq \ldots \geq q_m$. Let $V_1 = \{c_{i,k} \mid 1 \leq i \leq m$ and $1 \leq k \leq w_i - 1\}$ be a set of pendant vertices in T.
(b). Label t_2 first pieces of pendant vertices in V_1 by 1.
(c). Label $(n_1 - t_2)$ remaining vertices in V_1 by $2, 3, \ldots, (n_1 - t_2 + 1)$.
(d). Let $E_1 = \{e_{i,k} \mid e_{i,k}$ edge incident to $c_{i,k}, \forall i, k\}$ where $1 \leq i \leq m$ and $1 \leq k \leq w_i - 1$ be a set of pendant edges. Label t_2 edges in E_1 by $1, 2, \ldots, t_2$ and label $(n_1 - t_2)$ remaining edges in E_1 by t_2.

2. Label all edges in $E(T) \setminus E_1$ by the following steps.

(a). For $1 \leq j \leq q_1$, then define $a_j = |\{q_i \mid q_i = j, 1 \leq i \leq m\}|$. Let $k_0 = n_1$ and $k_j = m - \sum_{s=1}^{j} a_s$ for $1 \leq j \leq q_1$.
For $1 \leq j \leq q_1 - 1$ and $1 \leq i \leq k_j$, then $\psi(x_{i,j+1}x_{i,j+2}) = \left\lceil \frac{1+i+\sum_{r=0}^{j-1} k_r}{3} \right\rceil$.
(b). Next, define $\psi(e) = t_2$ for all remaining edges in T.

3. Label all vertices in $V \setminus V_1$ by the following steps. Denoted all vertices in $V \setminus V_1$ by $y_1, y_2, y_3, \ldots, y_N$ where $N = \sum_{i\geq 1}^{m} q_i + m + 1$, such that $s(y_1) \leq s(y_2) \leq \ldots \leq s(y_N)$ with $s(y) = \sum_{yz \in E(T)} \psi(yz)$, which can be considered as the temporary weight of y_i in T. Define $\psi(y_1)$ recursively as follows.

$$\psi(y_1) = n_1 + 2 - s(y_1), \text{ which implies, } wt(y_1) = \psi(y_1) + s(y_1).$$

For $2 \leq i \leq N$

$$\psi(y_i) = max\{1, wt(y_{i-1}) + 1 - s(y_i)\}.$$

By this algorithm, we conclude that

1. ψ is mapping from $V(T) \cup E(T)$ into $\{1, 2, \ldots, t_2\}$,
2. The weight of pendant vertices are $2, 3, 4, \ldots, n + 1$, respectively.
3. The weight of all remaining vertices form $n_1 + 2 = wt(y_1) < wt(y_2) < \ldots < wt(y_N)$ where $N = \sum_{i\geq 1}^{m} q_i + m + 1$.

Then, $tvs(T) \leq t_2$. □

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