Abstract

Motivated by the notion of the irregularity strength of a graph introduced by Chartrand et al.\cite{3} in 1988 and various kind of other total labelings, Baca et al.\cite{1} introduced the total vertex irregularity strength of a graph.

In 2010, Nurdin, Baskoro, Salman and Gaos\cite{5} determined the total vertex irregularity strength for various types of trees, namely complete \( k \)-ary trees, a subdivision of stars, and subdivision of particular types of caterpillars. In other paper\cite{6}, they conjectured that the total vertex irregularity strength of any tree \( T \) is only determined by the number of vertices of degree 1, 2, and 3 in \( T \). In this paper, we attempt to verify this conjecture by considering a subdivision of several types of trees, namely caterpillars, firecrackers, and amalgamation of stars.

Key words: Total vertex irregularity strength, tree, subdivision.
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1. Introduction

The following problem was proposed by Chartrand, Jacobson, Lehel, Oellermann, Ruiz and Sab\textsuperscript{a}\cite{3}. Assign positive integer labels to the edges of a simple connected graph of order at least 3 in such a way that the graph becomes irregular, i.e., the weights (label sums) at each vertex are distinct. The minimum value of the largest label over all such irregular assignments is known as the irregularity strength of such a graph. Finding the irregularity strength of a graph seems to be rather hard\cite{2} even for simple graphs. Later, Baca, Jendrol, Miller and Ryan\cite{1} introduced the total vertex irregularity strength of a graph as follows. Let \( G(V,E) \) be a simple graph. For a labeling \( \lambda : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, k\} \) the weight of a vertex \( x \) is defined as \( wt(x) = \lambda(x) + \sum_{xz \in E} \lambda(xz) \). The mapping \( \lambda \) is called a vertex irregular total \( k \)-labeling if for every pair of distinct two vertices \( x \) and \( y \) we have \( wt(x) \neq wt(y) \). The minimum \( k \) for which the graph \( G \) has a vertex irregular total \( k \)-labeling is called the total vertex irregularity strength of \( G \) and is denoted by \( tvs(G) \). Baca et al.\cite{1} proved that \( tvs(C_n) = \left\lceil \frac{n+2}{3} \right\rceil, n \geq 2, tvs(K_n) = 2 \) for any \( n \geq 3 \),
Proof. Define a labeling algorithm $\lambda$ as follows.

Theorem 2. Let $T$ be a tree with maximum degree $\Delta$. Let $n_i$ be the number of vertices of degree $i$, then

$$tvs(T) \geq \max\{t_1, t_2, \ldots, t_3\},$$

where $t_i = [(i + \sum_{j=1}^{i} n_j)/(i + 1)]$ and $n_i$ be the number of vertices of degree $i \in [1, \Delta]$.

2. Subdivision of caterpillars

In this subsection, we determine the total vertex irregularity strength of a subdivision graph of a non-homogeneous caterpillar. For integers $m, k_1, k_2, \ldots, k_m \geq 2$, define a caterpillar $C_{m}^{k_1, k_2, \ldots, k_m}$ as a graph obtained by attaching $k_i$ vertices to each vertex of $c_i$ of the path $P_m$, for $i \in [1, m]$. The path $P_m$ in $C_{m}^{k_1, k_2, \ldots, k_m}$ is called the backbone of the caterpillar. All vertices degree one are called pendant vertices. All pendant edges adjacent to $c_i$ are labeled by $c_{i1}, c_{i2}, \ldots, c_{ij}$ where $1 \leq j \leq |k_i|$. Edges $c_{ij}$ and $c_{i+1,j}$ are called pendant edges and backbone edges, respectively. If $k_1 = k_2 = \ldots = k_m = r$, then the caterpillar is called to be homogeneous, and it is denoted by $C_{m}^{r}$. Otherwise, the caterpillar is called to be non-homogeneous. Let $G = (V, E)$ be a connected graph and $e \in E(G)$. The subdivision of a graph $G$ on the edge $e$ in $k$ times is a graph obtained from the graph $G$ by replacing edge $e = uv$ with a path $u, x_1, x_2, \ldots, x_k, v$ on $k + 2$ vertices. The vertices $x_i$ are called subdivision vertices. Now, denote by $Sub(C_m^{k_1, k_2, \ldots, k_m}, \{w_1, w_2, w_3, \ldots, w_{m-1}\})$ the subdivision of a non-homogeneous caterpillar on all the backbone edges in $w_1, w_2, \ldots, w_{m-1}$ times, respectively. Denote by $x_i$ all the subdivision vertices. See Figure 1 as examples.

Theorem 2. Let $T \approx Sub(C_m^{k_1, k_2, \ldots, k_m}, \{w_1, w_2, w_3, \ldots, w_{m-1}\})$ where $m \geq 2$ and $w_i, k_i \geq 1$ for $1 \leq k \leq m - 1$ and $1 \leq i \leq m$. The number of vertices degree one and two are $n_1 = \sum_{i=1}^{m} k_i$ and $n_2 = \sum_{k=1}^{m-1} w_k$, respectively. For $n_2 \geq n_1$ and $n_{j+1} \leq \frac{1}{2}(n_j - 1)$ for $j \geq 2$, then $tvs(T) \leq t_2$. 


We conclude that:

of vertices degree one in fire crackers. For integers $2$. Subdivision of fire crackers

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Let $E_1 = \{e_{ij} | e_{ij} \in E \}$, where $e_{ij} \in E_1$ for each $2$

Let $E_2 = \{x_i y_{n+1} | n \in [1, w_i - 1], i \in [1, m - 1]\}$. Define $k_1 = n_1$ and $k_i = k_{i-1} + \sum_{r=1}^{i-1} w_r - 1$ for each $2 \leq i \leq m - 1$. Now, for $1 \leq i \leq m - 1$ and $1 \leq n \leq w_i - 1$, define $\lambda(x_n x_{n+1}) = \lceil \frac{1 + n + k_i}{3} \rceil$. Next, define $\lambda(e) = t_2$, for $e \in E \setminus (E_1 + E_2)$.

Label all vertices in $V \setminus V_1$ by the following steps.

Denote all vertices in $V \setminus V_1$ by $y_1, y_2, \ldots, y_N$, where $N = \sum_{i=1}^{m-1} w_i + m$, such that $s(y_1) \leq s(y_2) \leq \ldots \leq s(y_N)$ with $s(y) = \sum_{y \in E(T)} \lambda(yz)$, which can be considered as the temporary weight of $y_i$ in $T$.

Define $\lambda(y_1)$ recursively as follows.

\[ \lambda(y_1) = n_1 + 2 - s(y_1), \]

which implies $wt(y_1) = \lambda(y_1) + s(y_1)$. For $2 \leq i \leq N$

\[ \lambda(y_i) = \max\{1, wt(y_{i-1}) + 1 - s(y_i)\}. \]

We conclude that:

1. $\lambda$ is a mapping from $V(T) \cup E(T)$ into $\{1, 2, \ldots, t_2\}$,
2. The weights of pendant vertices are $2, 3, 4, \ldots, n_1 + 1$, respectively.
3. The weight of all remaining vertices forms $n_1 + 2 = wt(y_1) < wt(y_2) < \ldots < wt(y_N)$ where $N = \sum_{i=1}^{m-1} w_i + m$.

Then, $tvs(T) \leq t_2$. □

2.2. Subdivision of fire crackers

In this subsection, we determined the total vertex irregularity strength of subdivision graph of a non-homogeneous fire crackers. For integers $m, w_1, w_2, \ldots, w_m \geq 3$. Let $G_1, G_2, \ldots, G_m$ be a family of disjoint stars and $w_i$ be the number of vertices degree one in $G_i$. Let $a_i$ be a pendant vertex of $G_i$, $1 \leq i \leq m$, define a fire crackers $F_m^{[w_1,w_2,\ldots,w_m]}$ as a tree graph which contains all the $m$ stars and a path joining $a_1, a_2, \ldots, a_m$. Fig. 1. Graf $T \approx Sub(C_3^{(1,2)} \cup \{1,2\})$ dan $T \approx Sub(C_3^{(1,3,3)} \cup \{1,2\})$
The vertices in path $P_m$ in $F_m^{[w_1,w_2,...,w_n]}$ is called the backbone vertices of the fire cracker and labeled by $x_1, x_2, \ldots, x_m$. All vertices degree one are called pendant vertices. All vertices incident to pendant vertices are called central vertices and labeled by $c_1, c_2, \ldots, c_m$. All pendant vertices attached to $c_i$ where $1 \leq i \leq m$ are labeled by $a_{i1}, a_{i2}, \ldots, a_{iw_i}$. Edges $c_i a_{iw_i}$ and $x_k x_{k+1}$ where $1 \leq k + m$ are called pendant edges and backbone edges, respectively. If $w_1 = w_2 = \ldots = w_m = w$, then the fire cracker is called to be homogeneous, and it is denoted by $F_m^w$. Otherwise, the fire cracker is called to be non-homogeneous. Let $G = (V, E)$ be a connected graph and $e \in E(G)$. The subdivision of a graph $G$ on the edge $e$ in $k$ times is a graph obtained from the graph $G$ by replacing edge $e = uv$ with a path $(u, x_1, x_2, \ldots, x_k, v)$ on $k + 2$ vertices. Now, denoted by $T = Sub(F_m^{[w_1,w_2,...,w_n]}, \{q_1, q_2, \ldots, q_{m-1}\})$ the subdivision of a non-homogeneous fire cracker on all the backbone edges in $q_1, q_2, \ldots,$ or $q_{m-1}$ times, respectively. See Figure 2 for illustration.

**Theorem 3.** Let $T = Sub(F_m^{[w_1,w_2,...,w_n]}, \{q_1, q_2, \ldots, q_{m-1}\})$ with $m, w_i \geq 3$ for $1 \leq i \leq m$ and $q_j \geq 2$ for $1 \leq j \leq m - 1$. The number of vertices degree one and two are $n_k = \sum_{i=1}^m w_i$ and $n_2 = \sum_{j=1}^{m-1} q_j + 2$, respectively. For $n_2 \geq n_1$ and $n_{k+1} \leq \frac{1}{2}(n_k - 1)$ for $k \geq 2$, then $tvs(T) \leq t_2$.

**Proof.** Define a labeling algorithm $\phi$ as follows.

1. Label all pendant vertices and pendant edges by the following steps.
   
   (a). Let $V_1 = \{c_1, c_2, \ldots, c_l\}$ be a set of central vertices where $d(c_i) \geq d(c_{i+1})$. Let $V_1^* = \{a_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq w_i, \forall i\}$ be a set of pendant vertices in $T$.
   
   (b). Label $t_2$ first pendant vertices in $V_1^*$ by 1.
   
   (c). Label $(n_1 - t_2)$ remaining vertices in $V_1$ by 2, 3, \ldots, $n_1 - t_2 + 1$.
   
   (d). Let $E_1 = \{e_{ij, k} \mid e_{ij} \text{ edges incident to } a_{i,j}, 1 \leq i \leq m \text{ and } 1 \leq j \leq w_i\}$ be a set of pendant edges. Label $t_2$ edges in $E_1$ by 1, 2, \ldots, $t_2$ and label $(n_1 - t_2)$ edges in $E_1$ by $t_2$.

2. Let $E_2 = \{x_i x_{i+1} \mid 1 \leq i \leq \sum_{j=1}^{m-1} q_j + m - 1\}$ be a set of backbone edges in $P_m$. Define $\phi(x_i x_{i+1}) = \lceil \frac{i+1+n_1}{3} \rceil$.

3. Next, define $\phi = t_2$ for $e \in E(E_1 + E_2)$.

4. Label all vertices in $V \setminus V_1$ by the following steps. Denote vertices $V \setminus V'$ with $y_1, y_2, y_3, \ldots, y_N$ where $N = \sum_{j=1}^{m-1} q_j + 2m$ such that $s(y_1) \leq s(y_2) \leq \ldots \leq s(y_N)$ with $s(y) = \sum_{yz \in E(T)} \phi(yz)$, which can be considered as temporary weight of $y$ in $T$. Define $\phi(y_1)$ recursively as follows.

   $$
   \phi(y_1) = n_1 + 2 - s(y_1), \quad \text{which implies} \quad wt(y_1) = \phi(y_1) + s(y_1).
   $$

   For $2 \leq i \leq N$, then

   $$
   wt(y_i) = \max[1, wt(y_{i-1}) + 1 - s(y_i)] \quad \text{and} \quad wt(y_i) = s(y_i) + \phi(y_i).
   $$

   We conclude that:

   1. $\phi$ is mapping from $V(T) \cup E(T)$ into $\{1, 2, \ldots, t_2\}$.
   2. The weight of all pendant vertices are $2, 3, 4, \ldots, n + 1$. 

Fig. 2. $T = Sub(F_3^{[3,3,4]}, \{2, 5\})$
3. The weight of all remaining vertices form \( n_1 + 2 = wt(y_1) < wt(y_2) < \ldots < wt(y_N) \) where \( N = \sum_{j=1}^{m} q_j + 2m \).

Then, we have \( tvs(T) \leq t_2 \).

2.3. Subdivision of amalgamation of stars

In this subsection, we determine the total vertex irregularity strength of a subdivision of amalgamation of stars. Let \( \{G_1, G_2, \ldots, G_n\} \) be a finite collection of graphs and each \( G_i \) has a fixed vertex \( v_{0i} \). The vertex-amalgamation of \( G_1, G_2, \ldots, G_n \), denoted by vertex- \( amal \{G_i; v_{0i}; m\} \) is formed by taking all the \( G_i \)’s and identifying their terminal vertices. Consider \( G_i \) is star graph \( K_{1,w_i} \) and \( v_{0,i} \) be a pendant vertices in \( K_{1,w_i} \) for all \( i \), denoted by \( T \simeq vertex – amal[K_{1,w_i}; v_{0,i}; m] \) where \( m \) is the number of star graphs. All vertices degree one are called pendant vertices. All pendant edges incident to \( v_{0,i} \) are called fixed edges.

Let \( G = (V, E) \) be a connected graph and \( e \in E(G) \). The subdivision of a graph \( G \) on the edge \( e \) in \( k \) times is a graph obtained from the graph \( G \) by replacing \( e = uv \) with a path \( u, x_1, x_2, \ldots, x_k, v \) on \( k + 2 \) vertices. The vertices \( x_i \) are called subdivision vertices. Now, denote by \( T \simeq Sub(\text{vertex} – \text{amal}[K_{1,w_i}; v_{0,i}; m], \{q_i\}) \) where \( 1 \leq i \leq m \) and \( q_i = (w_i - 1) \) the subdivision of amalgamation of star graphs on all of the fixed edges in \( q_1, q_2, \ldots, q_m \) times, respectively. Denoted by \( x_{m,q_i+2} \) all the subdivision vertices. All vertices pendant vertices are labeled by \( c_{1,1}, c_{1,2}, \ldots, c_{1,w_i}, c_{2,1}, c_{2,2}, \ldots, c_{2,w_i}, \ldots, c_{m,w_i} \).

![Fig. 3. T \simeq Sub(\text{Vertex} – \text{Amal}[K_{1,4}, K_{1,5}, K_{1,6}, v_{0,1}, v_{0,2}, v_{0,3}, v_{0,4}, 4], \{3, 4, 5\})](image)

- Let \( V(G) = \{x_{i,j} \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq q_i + 2\} \cup \{C_{i,l} \mid 1 \leq l \leq |w_i| - 1\} \) be a set of vertices in \( T \).
- Let \( E(T) = \{x_{i,j}x_{i,j+1} \mid 1 \leq i \leq n \text{ and } 1 \leq i \leq q_i + 1\} \cup \{x_{i,l}C_{i,l} \mid 1 \leq l \leq w_i - 1\} \) be a set of edges in \( T \).

We can see that \( x_{i,q_i+2} \) is one vertex for every \( 1 \leq i \leq m \).

**Theorem 4.** Let \( T \simeq Sub(\text{vertex} – \text{amal}[K_{1,w_i}; v_{0,i}; m], \{q_i\}) \) be a graph with \( w_i, m \geq 3 \) and \( q_i = (w_i - 1) \) for each \( 1 \leq i \leq m \) is the number of subdivision in all of the backbone edges. The number of vertices degree one and two are \( n_1 = \sum_{i=1}^{m} w_i - 1 \) and \( n_2 = \sum_{i=1}^{m} q_i \), respectively. For \( n_2 \geq n_1 \) and \( n_{k+1} \leq \frac{1}{2}(m_k - 1) \) for \( k \geq 2 \). Then, \( tvs(T) \leq t_2 \).

**Proof.** Define a labeling algorithm \( \psi \) as follows.

1. Label all pendant vertices and pendant edges by the following steps.
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References