An Asymptotic Formula for the Maximum Size of an $h$-Family in Products of Partially Ordered Sets

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An $h$-family of a partially ordered set $P$ is a subset of $P$ such that no $h + 1$ elements of the $h$-family lie on any single chain. Let $S_1, S_2, \ldots$ be a sequence of partially ordered sets which are not antichains and have cardinality less than a given finite value. Let $P_n$ be the direct product of $S_1, S_2, \ldots, S_n$. An asymptotic formula of the maximum size of an $h$-family in $P_n$ is given, where $h = o(\sqrt{n})$ and $n \to \infty$.

Let $P$ be a partially ordered set (poset). A subset $F_h \subseteq P$ is called an $h$-family if it does not contain a chain of $h + 1$ elements, i.e., there are not $x_0, \ldots, x_h \in F_h$ such that $x_0 < \cdots < x_h$. Let $d_h(P)$ be the maximum size of an $h$-family in $P$. If $P$ and $Q$ are posets, then the direct product $P \times Q$ is defined on the Cartesian product of the sets $P$ and $Q$ as follows:

$$(x_1, y_1) \leq_{P \times Q} (x_2, y_2) \text{ iff } x_1 \leq_P x_2 \text{ and } y_1 \leq_Q y_2.$$ 

In all that follows we consider a sequence $S_1, S_2, \ldots$ of nontrivial posets (i.e., they are not antichains) with bounded cardinalities. Let $k_i := |S_i| < C$ ($i = 1, 2, \ldots$). We put $P_n := S_1 \times \cdots \times S_n$ and $d_{n,h} := d_h(P_n)$. In this paper we will give an asymptotic formula for $d_{n,h}$ if $h = o(\sqrt{n})$ and $n \to \infty$. This generalizes a result of V. B. Alekseev [2] where the case $S_1 = S_2 = \cdots$ and $h = 1$ was settled.

In order to formulate our result we need the following definition. A representation of a poset $P$ is a mapping $z: P \to \mathbb{R}$ such that $z(x) - z(y) > 1$ if $x > y$. A representation is called optimal if $(1/|P|) \sum_{x \in P} (z(x) - \bar{z}(P))^2$ is an infimum (extending over all representations of $P$), where $\bar{z}(P) := (1/|P|) \sum_{x \in P} z(x)$. The infimum is denoted by $D(P)$.

Remark 1. In [2] and [3] it is proved that an optimal representation always exists.

In all that follows let $z_i$ be an optimal representation of $S_i$ such that $\bar{z}_i(S_i) = 0$. If $x \in S_i$, we can omit the index $i$ in $z_i(x)$ and write briefly $z(x)$ since the mapping is defined by $S_i$. Let $D_i := D(S_i)$ and $V_n := \sum_{i=1}^n D_i$. Our main result is the following

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**Theorem.** If \( h = o(\sqrt{n}) \), then
\[
d_{n,h} \sim \frac{k_1 \cdots k_n}{\sqrt{2\pi V_n}} \cdot h, \quad \text{where } n \to \infty.
\]

At first we will prove the

**Theorem A.** If \( h = o(\sqrt{n}) \), then
\[
d_{n,h} \sim \frac{k_1 \cdots k_n}{\sqrt{2\pi V_n}} \cdot h, \quad \text{where } n \to \infty.
\]

**Proof.** Let \( x = (x_1, \ldots, x_n) \in P_n \) and define \( z(x) := \sum_{i=1}^{n} z(x_i) \). Obviously \( \mathcal{F}_h := \{x : -h/2 < z(x) \leq h/2\} \) is an \( h \)-family. Hence, \( d_{n,h} \geq |\mathcal{F}_h| \). We will prove that
\[
|\mathcal{F}_h| \sim \frac{k_1 \cdots k_n}{\sqrt{2\pi V_n}} \cdot h. \tag{1}
\]

For that we define the following discrete random variables \( \eta_1, \eta_2, \ldots \) as follows:
\[
P(\eta_i = z_i') = \frac{1}{k_i},
\]
where \( z_i' := z(s_i') \) and \( S_i = \{s_1', \ldots, s_i'\} \). Let \( \eta_1, \eta_2, \ldots \) be independent and \( v_n := \eta_1 + \cdots + \eta_n \). Then the expected value and variance of \( v_n \) is equal to 0 and \( V_n \), respectively. We have \( |\mathcal{F}_h| = k_1 \cdots k_n \cdot P(-h/2 < v_n \leq h/2) \). Thus it is sufficient to prove that
\[
P \left( \frac{-h}{2} < v_n \leq \frac{h}{2} \right) = \frac{h}{\sqrt{2\pi V_n}} (1 + o(1)). \tag{2}
\]

In Lemmas 4 and 5 of [2] it is proved that \( \eta_1, \eta_2, \ldots \) have a lattice distribution and that the maximal spans of them are equal to \( 1/r_1, 1/r_2, \ldots \), where \( r_1, r_2, \ldots \) are integers. Obviously, there exists only a finite number of posets with cardinality less than \( C \). Thus the number of different distribution functions of \( \eta_1, \eta_2, \ldots \) is finite. Let \( 1/R_1, \ldots, 1/R_l \) be the corresponding maximal spans. If \( R \) is the least common multiple of \( R_1, \ldots, R_l \) and \( \zeta_i := R\eta_i + y_i \), then the maximal span of \( \zeta_i \) is equal to \( R/R_i \), hence an integer \( (i = 1, 2, \ldots) \). Thus \( y_i \) can be chosen such that \( \zeta_i \) is an integer-valued variable. If we put \( \mu_n := \sum_{i=1}^{l} \zeta_i \), then obviously \( W_n := R^2 V_n \) is the variance and \( M_n := y_1 + \cdots + y_n \) is the expected value of \( \mu_n \) (\( n = 1, 2, \ldots \)). Since the greatest common divisor of \( R/R_1, \ldots, R/R_l \) equals 1 we may use the limit theorem for
$k$-sequences of independent random variables (see [8, p. 189]) and conclude that

$$\sup_{n} \left| \sqrt{W_n} P(\mu_n = N) - \frac{1}{\sqrt{2\pi}} e^{-(N-M_n)^2/(2W_n)} \right| \to 0 \quad (3)$$

(the supremum extends over all integers $N$). Now we have

$$P\left(-\frac{h}{2} < v_n \leq \frac{h}{2}\right) = P\left(-\frac{hR}{2} + M_n < \mu_n \leq \frac{hR}{2} + M_n\right) = \sum_{N \in I} P(\mu_n = N), \quad (4)$$

where $I := \{N \in \mathbb{Z} : -hR/2 + M_n < N \leq hR/2 + M_n\}$.

Let $D$ and $\bar{D}$ be the smallest and largest value of $\{D_1, D_2, \ldots\}$, respectively (they exist since there is only a finite number of different distribution functions under $\eta_1, \eta_2, \ldots$). Because of $h = o(\sqrt{n})$ we conclude that for all $N \in I$

$$0 < \frac{(N - M_n)^2}{2W_n} \leq \frac{(hR)^2}{2R^2 V_n} \leq \frac{h^2}{2nD} < \varepsilon_1(n),$$

where $\varepsilon_1(n) \to 0$. (It is not the case that $D = 0$ since $S_1, S_2, \ldots$ are not antichains.) Thus (3) implies that for all $N \in I$

$$\frac{1}{\sqrt{2\pi}} (1 - \varepsilon_2(n)) \leq \sqrt{W_n} P(\mu_n = N) \leq \frac{1}{\sqrt{2\pi}} (1 + \varepsilon_3(n)), \quad (5)$$

where $\varepsilon_2(n) \to 0$ and $\varepsilon_3(n) \to 0$. Since $|I| = hR$ from (4) and (5) it follows

$$\frac{hR}{\sqrt{2\pi W_n}} (1 - \varepsilon_2(n)) \leq P\left(-\frac{h}{2} < v_n \leq \frac{h}{2}\right) \leq \frac{hR}{\sqrt{2\pi W_n}} (1 + \varepsilon_3(n)),$$

and because of $W_n = R^2 V_n$ we obtain

$$P\left(-\frac{h}{2} < v_n \leq \frac{h}{2}\right) = \frac{h}{\sqrt{2\pi V_n}} (1 + o(1)),$$

and (2) is proved. Q.E.D.

Now we will prove the more difficult

**Theorem B.** $d_{n,h} \leq (k_1 \cdots k_n/\sqrt{2\pi V_n}) \cdot h$, where $n \to \infty$.

**Proof.** It is sufficient to prove that

$$d_{n,1} \leq \frac{k_1 \cdots k_n}{\sqrt{2\pi V_n}}$$
since $d_{n,h} \leq h \cdot d_{n,1}$ (each $h$-family is the union of $h$ Sperner families, i.e., 1-families; see, for instance, [1, p. 271]). Let $N_v := \{x \in P_n : z(x) = v\}$ and consider the bipartite graph $G_v$ on the vertex-set $N_{v-1} \cup N_v$ in which $(x, y)$ is an edge iff $x < y$. Let $E_v$ be a maximum matching of $G_v$, i.e., a maximum set of pairwise non-adjacent edges of $G_v$. Now join adjacent edges of the matchings ..., $E_{v-1}, E_v, E_{v+1}$... so far as possible. In this way we obtain a partition of $P_n$ into chains (single points are regarded as chains too). Let $R_n$ be the set of such chains in the partition which have an element $x$ with $-\frac{1}{2} < z(x) \leq \frac{1}{2}$. Further let $R_1$ and $R_2$ be the set of such chains in the partition in which $z(x) \geq \frac{1}{2}$ and $z(x) \leq -\frac{1}{2}$ for all elements of the chain, respectively. Obviously, $d_{n,1} \leq |R_0| + |R_1| + |R_2|$, $|R_0| = |\{x : -\frac{1}{2} < z(x) \leq \frac{1}{2}\}|$. From (1) in the proof of Theorem A we obtain

$$|R_0| \sim \frac{1}{\sqrt{2\pi V_n}} k_1 \cdots k_n.$$ 

In all that follows we will prove that $|R_1| \leq k_1 \cdots k_n \cdot o(1/\sqrt{n})$. Then all is done since then

$$\frac{|R_1|}{|R_0|} \leq o \left( \frac{1}{\sqrt{n}} \right) \cdot \sqrt{2\pi V_n} \leq o \left( \frac{1}{\sqrt{n}} \right) \cdot \sqrt{2\pi nD} \to 0,$$

and the same follows for $|R_2|$ by duality.

Let $\delta_v$ be the number of elements of $N_v$ which are not covered by an edge of the maximum matching $E_v$. Associating to each chain of $R_1$ its smallest element we obtain

$$|R_1| = \sum_{v > 1/2} \delta_v. \quad (6)$$

For $X \subseteq N_v$ let $V(X) := \{y \in N_{v-1} : y < x \text{ for any } x \in X\}$. A set $X \subseteq N_v$ is called a critical set iff

$$|X| - |V(X)| = \max_{Y \subseteq N_v} (|Y| - |V(Y)|).$$

From well-known results on matchings (see [7, p. 138ff.]) it follows that there exists a unique minimal critical set $X_v$ which is contained in all other critical sets and for which

$$|X_v| - |V(X_v)| = \delta_v.$$

Now we will prove that special classes of elements, so-called statistics, are contained in $X_v$. At first we shall define these classes. Since $|S_i| < C$ for all $i$
we have in our sequence $S_1, S_2, \ldots$ only a finite number of different posets. Let $T_1, \ldots, T_l$ be these posets $(T_i = \{t^i_j; j = 1, \ldots, k_i\}, i = 1, \ldots, l)$. We can suppose that $T_1, \ldots, T_l$ are pairwise disjoint. Let $n_i$ be the number of factors $T_i$. Obviously,

$$P_n \cong T_1 \times \cdots \times T_1 \times \cdots \times T_l \times \cdots \times T_l.$$ 

Without loss of generality we may assume that $P_n$ is equal to this poset. Further, let $Q_i := (q^i_1, \ldots, q^i_{k_i})$ be a $k_i$-tuple of integers with $\sum_{i=1}^{k_i} q^i_j = n_i (i = 1, \ldots, l)$, and let $Q := (Q_1, \ldots, Q_l)$ be an $l$-tuple of such $k_i$-tuples. The statistic of $Q$ is defined to be the set of elements $x = (x_1, \ldots, x_n) \in P_n$ in which the element $t^i_j$ occurs exactly $q^i_j$ times $(j = 1, \ldots, k_i, i = 1, \ldots, l)$; it is denoted by $S(Q)$. Now we will prove that either no element of a statistic or the whole statistic, i.e., all elements of it, is contained in $X_v$. Let $\pi = (\pi_1, \ldots, \pi_l)$ be an element of the direct product of the symmetric groups on the sets $\{1, \ldots, n_i\}, i = 1, \ldots, l$. To such a $\pi$ we can associate an automorphism $\varphi_* of P_n$ by

$$\varphi_*(x_1^1, \ldots, x_n^1, \ldots, x_1^l, \ldots, x_n^l) := (x_{\pi_1(1)}, \ldots, x_{\pi_n(1)}, \ldots, x_{\pi_1(l)}, \ldots, x_{\pi_n(l)}).$$

Evidently, $|\varphi_*(X_v)| = |X_v|$ and $|V(\varphi_*(X_v))| = |V(X_v)|$, hence $|\varphi_*(X_v)| = |V(\varphi_*(X_v))| = \delta_v$. Since $X_v$ is a minimal critical set we have $X_v \subseteq \varphi_*(X_v)$, hence $X_v = \varphi_*(X_v)$. Consequently, if $x \in X_v$, then $\varphi_*(x) \in X_v$ for all such $\pi$, thus the whole statistic containing $x$ is contained in $X_v$.

Let $z(Q_i) := \sum_{j=1}^{k_i} q^i_j z(t^i_j)$ and $z(Q) := \sum_{i=1}^{l} z(Q_i)$. Obviously, $z(x) = z(Q)$ holds for all $x \in S(Q)$. Let $I_m$ be the set of all pairs $(i, j)$ for which $t^i_m > t^i_n$ and $z(t^i_m) - z(t^i_n) = 1 (m = 1, \ldots, l)$. In order to estimate $\delta_v$ (see (6)) we associate to each pair $(i, j)$ with $(i, j) \in I_m$ a number $\beta^i_{m(j)}$ (to be specified later) such that $\beta^i_m > 0 (m = 1, \ldots, l)$ and $\sum_{m=1}^{l} \sum_{(i, j) \in I_m} \beta^i_{m(j)} = 1$.

For $Q = (Q_1, \ldots, Q_l)$ and $(i, j) \in I_m$ we define $Q_{m}^{i}$ to be the tuple $(Q_1, \ldots, Q_{m-1}^{i}, Q_m', Q_{m+1}^{i}, \ldots, Q_l)$, where $Q_m' := (q^m_1, \ldots, q^m_{m-1}, q^m_{m+1}, \ldots, q^m_{k_m})$. Obviously the elements of $S(Q^m)$ can be obtained from the elements of $S(Q)$ by specifying some coordinate and changing $t^i_n$ to $t^i_m$ for each element in which $t^i_n$ occurs in that coordinate.

We set $r(Q) := |S(Q)| - \sum_{m=1}^{l} \sum_{(i, j) \in I_m} \beta^i_{m(j)} |S(Q^m)|$ if the right-hand side is not negative and $r(Q) := 0$, otherwise.

**Lemma 1.** $\delta_v \leq \sum_{Q: z(Q) = v} r(Q)$.

**Proof.** To each pair $(Q, Q')$ with $z(Q) = z(Q') + 1 = v$ and $Q' = Q^m$ for some $m \in \{1, \ldots, l\}$ and some $(i, j) \in I_m$ we associate the unique weight $\beta^i_{m(j)} |S(Q^m)|$. Counting the weights of pairs $(Q, Q')$ with $S(Q) \subset X_v$ in two different ways we obtain
\[
\sum_{Q: S(Q) = X_v} \sum_{m=1}^{l} \sum_{(i,j) \in I_m} \beta_{ij}^m |S(Q_{ij}^m)|
\]
\[
= \sum_{Q': S(Q') = V(X_v)} |S(Q')| \cdot \sum_{m=1}^{l} \sum_{(i,j) \in I_m} \beta_{ij}^m
\]
\[
\leq \sum_{Q': S(Q') = V(X_v)} |S(Q')| = |V(X_v)|.
\]

It follows

\[
\delta_v = |X_v| - |V(X_v)|
\]
\[
\leq \sum_{Q: S(Q) = X_v} |S(Q)| - \sum_{Q: S(Q) = X_v} \sum_{m=1}^{l} \sum_{(i,j) \in I_m} \beta_{ij}^m |S(Q_{ij}^m)|
\]
\[
= \sum_{Q: S(Q) = X_v} r(Q) \leq \sum_{Q: z(Q) = v} r(Q). \quad \text{Q.E.D.}
\]

Now we will prove that

\[
\sum_{Q: z(Q) > 1/2} r(Q) \leq k_1 \cdots k_n \cdot o \left( \frac{1}{\sqrt{n}} \right)
\]
which together with (6) and Lemma 1 will complete the proof. We will estimate the above sum in two steps. For that let, without loss of generality,

\[
n_1, \ldots, n_s > n^{1/3} \text{ and } n_{s+1}, \ldots, n_l \leq n^{1/3} \quad (7)
\]
(if \( n \) is large enough we have \( s \geq 1 \)). Let

\[
F_1 := \left\{ Q : \left| q_i^n - \frac{n_m}{k_m} \right| > 2 \sqrt{n_m} \ln n_m \text{ for some } m \in \{1, \ldots, s\} \text{ and some } i \in \{1, \ldots, \tilde{k}_m\} \right\},
\]
\[
F_2 := \left\{ Q : \left| q_i^n - \frac{n_m}{k_m} \right| \leq 2 \sqrt{n_m} \ln n_m \text{ for each } m \in \{1, \ldots, s\} \text{ and each } i \in \{1, \ldots, \tilde{k}_m\} \right\}.
\]

Then

\[
\sum_{Q: z(Q) > 1/2} r(Q) \leq \sum_{Q \in F_1} r(Q) + \sum_{Q \in F_2, z(Q) > 1/2} r(Q). \quad (8)
\]

**Lemma 2.** \( \sum_{Q \in F_1} r(Q) \leq k_1 \cdots k_n \cdot o(1/\sqrt{n}) \).
Proof. We have
\[ \sum_{Q \in F_1} r(Q) \leq \sum_{Q \in F_1} |S(Q)|. \] (9)

If \( G_i^m := \{ Q : |q_i^m - n_m/k_m| > 2 \sqrt{n_m \ln n_m} \} \) \((m = 1, \ldots, s, \ i = 1, \ldots, k_m)\), then obviously
\[ \sum_{Q \in F_1} |S(Q)| \sum_{m=1}^{s} \sum_{i=1}^{k_m} \sum_{Q \in G_i^m} |S(Q)| \leq \max_{m \in \{1, \ldots, s \}} \max_{i \in \{1, \ldots, k_m \}} \left( \sum_{Q \in G_i^m} |S(Q)| \right). \] (10)

If \( K_i^m(q) \) is the set of all \( x = (x_1, \ldots, x_n) \in P_n \) in which the element \( t_i^m \) occurs exactly \( q \) times, then
\[ \sum_{Q \in G_i^m} |S(Q)| = \sum_{q : |q - n_m/k_m| > 2 \sqrt{n_m \ln n_m}} |K_i^m(q)|. \]

In order to estimate these sums we consider the following identically distributed and independent random variables \( \lambda_1, \ldots, \lambda_{n_m} \), with
\[
P(\lambda_i = 0) = \frac{k_m - 1}{k_m},
\]
\[
P(\lambda_i = 1) = \frac{1}{k_m} \quad (i = 1, \ldots, n_m).
\]

Further let \( \zeta_{n_m} := \lambda_1 + \cdots + \lambda_{n_m} \). Obviously, \( |K_i^m(q)| = k_1 \cdots k_n \cdot P(\zeta_{n_m} = q) \), and thus
\[ \sum_{Q \in G_i^m} |S(Q)| = k_1 \cdots k_n \cdot P \left( \left| \zeta_{n_m} - \frac{n_m}{k_m} \right| > 2 \sqrt{n_m \ln n_m} \right). \] (11)

Since \( n_m/k_m \) is the expected value of \( \zeta_{n_m} \) it follows from Hoeffding’s exponential estimation for distributions of sums of independent random variables (see [8, p. 58, 8.]) that
\[ P \left( \left| \zeta_{n_m} - \frac{n_m}{k_m} \right| > 2 \sqrt{n_m \ln n_m} \right) \leq 2 \cdot e^{-n_m^2}. \] (12)

From (9)–(12) we now obtain
\[ \sum_{Q \in F_1} r(Q) \leq \max_{m \in \{1, \ldots, s \}} 2lC \cdot k_1 \cdots k_n \cdot e^{-n_m^2} \]
\[ \leq 2lC \cdot k_1 \cdots k_n \cdot e^{-(1/9)(\ln n)^2} = k_1 \cdots k_n \cdot o \left( \frac{1}{\sqrt{n}} \right). \] Q.E.D.
It remains to estimate the second sum in (8) $\sum_{Q \in P} z(Q) \geq 1/2 \ r(Q)$. For this we need the following lemma which can be obtained from the Theorem of Gale (see [4, p. 62] and [2, Lemma 12]) or from the Theorem of Kuhn and Tucker (see [3]).

**Lemma 3.** To each pair $(t^m_i, t^m_j)$ with $(i, j) \in I_m$ one can associate a number $f^m(i, j) \geq 0$ such that $f^m_n(i) = f^m_n(i) - z(t^m_n)$, where $f^m_n(i) := \sum_{j: (i, j) \in m} f^m_n(i, j)$ and $f^m_n(i) := \sum_{j: (i, j) \in m} f^m_n(j, i)$.

**Remark 2.** This is the only place where we use the fact that the poset representations are optimal.

Obviously there exist constants $\underline{F}$ and $\overline{F}$ such that

$$0 < \underline{F} \leq \sum_{(i, j) \in l_m} f^m_n(i, j) \leq \overline{F}$$

for all $m \in \{1, \ldots, l\}$ (13) 

($F$ can be chosen greater than 0 since $T_1, \ldots, T_l$ are not trivial posets). Now let

$$f := \frac{1}{n} \sum_{m=1}^{s} \sum_{(i, j) \in l_m} \frac{n_m}{k_m} f^m_n(i, j),$$

$$\beta^m_{ij} := \begin{cases} \frac{1}{n} \cdot \frac{f^m_n(i, j)}{f} \cdot \frac{n_m}{k_m}, & m = 1, \ldots, s, \\ 0, & m = s + 1, \ldots, l. \end{cases}$$

(15)

Obviously $\sum_{m=1}^{l} \sum_{(i, j) \in l_m} \beta^m_{ij} = 1$ and $\beta^m_{ij} \geq 0$. Further, from (13) and (14) it follows

$$\frac{1}{n} \cdot \frac{1}{C} \cdot (n_1 + \cdots + n_s) \cdot \underline{F} \leq f \leq \frac{1}{n} \cdot (n_1 + \cdots + n_s) \cdot \overline{F} \leq \overline{F}.$$

Since

$$\frac{n_1 + \cdots + n_s}{n} = 1 - \frac{n_{s+1} + \cdots + n_l}{n}$$

and

$$\frac{n_{s+1} + \cdots + n_l}{n} \leq \frac{l \cdot n^{1/j}}{n} \rightarrow 0,$$

there exists a constant $\overline{F}'$ such that

$$0 < \overline{F}' \leq f \leq \overline{F}.$$  (16)
Moreover we mention that there is a constant $Z$ such that $z(r^m) \leq Z$ for all $m \in \{1, \ldots, l\}$ and $i \in \{1, \ldots, \tilde{k}_m\}$. It follows

$$z(Q_m) \leq n_m \cdot Z \quad (m = 1, \ldots, l).$$

(17)

Now we are able to estimate the second sum in (8).

**Lemma 4.** If the numbers $\beta_{ij}^m$ are chosen as above,

$$\sum_{Q \in F_2 : z(Q) > 1/2} r(Q) \leq k_1 \cdots k_n \cdot o \left( \frac{1}{\sqrt{n}} \right).$$

**Proof.** We shall prove for a fixed $Q \in F_2$ with $z(Q) > \frac{1}{2}$, $r(Q) \leq |S(Q)| \cdot o(1/\sqrt{n})$, where the function $o(1/\sqrt{n})$ does not depend on $Q$. Then all is done since then

$$\sum_{Q \in F_2 : z(Q) > 1/2} r(Q) \leq \sum_Q |S(Q)| \cdot o \left( \frac{1}{\sqrt{n}} \right) = k_1 \cdots k_n \cdot o \left( \frac{1}{\sqrt{n}} \right).$$

If $r(Q) = 0$, we do not have to prove anything, thus let $r(Q) > 0$. Then

$$r(Q) = |S(Q)| - \sum_{m = 1}^l \sum_{(i, J) \in \ell_m} \beta_{ij}^m |S(Q_{ij}^m)|$$

$$= |S(Q)| \cdot \left( 1 - \sum_{m = 1}^l \sum_{(i, J) \in \ell_m} \beta_{ij}^m \frac{|S(Q_{ij}^m)|}{|S(Q)|} \right)$$

$$= |S(Q)| \cdot \left( 1 - \sum_{m = 1}^l \sum_{(i, J) \in \ell_m} \beta_{ij}^m \frac{q_j^m - \tilde{k}_m}{q_j^m + 1} \right)$$

$$= |S(Q)| \cdot \left( \sum_{m = 1}^l \sum_{(i, J) \in \ell_m} \beta_{ij}^m \frac{1}{q_j^m + 1} + \sum_{m = 1}^l \sum_{(i, J) \in \ell_m} \beta_{ij}^m \frac{(q_j^m - q_i^m)(n_m - \tilde{k}_m)}{n_m(q_j^m + 1)} + \sum_{m = 1}^l \sum_{(i, J) \in \ell_m} \beta_{ij}^m \frac{\tilde{k}_m}{n_m} \right).$$

Finally, we estimate the three sums in the parentheses using Lemma 3, (7), (15), (16), (17) and the facts that $k_i < C$, $Q \in F_2$ and $z(Q) \geq \frac{1}{2}$.

$$\sum_1 := \sum_{m = 1}^l \sum_{(i, J) \in \ell_m} \beta_{ij}^m \frac{1}{q_j^m + 1} = \frac{1}{2} \cdot \sum_{m = 1}^l \sum_{(i, J) \in \ell_m} \frac{f^m(i, J) n_m}{q_j^m + 1} \frac{n_m}{\tilde{k}_m}$$

$$\leq \frac{1}{2} \sum_{m = 1}^l \sum_{(i, J) \in \ell_m} \frac{f^m(i, J)}{1 - 2(\tilde{k}_m \ln n_m / \sqrt{n_m})}.$$
Because of (7) we have
\[ 1 - 2 \frac{\tilde{k}_m \ln n_m}{\sqrt{n_m}} > \frac{1}{2} \]

if \( n \) is large enough. Thus,
\[
\sum_1 \leq \frac{2}{nf} \cdot l \cdot \tilde{F} \leq \frac{2l\tilde{F}}{F'} \cdot \frac{1}{n} = o \left( \frac{1}{\sqrt{n}} \right) .
\]

\[
\sum_2 := \sum_{m=1}^l \sum_{(i,j) \in \Gamma_m} \beta_{m}^{ij} \frac{(q_j^m - q_i^m)(n_m - \tilde{k}_m q_j^m - \tilde{k}_m)}{n_m(q_j^m + 1)}
\]
\[
= \frac{1}{n\tilde{F}} \cdot \sum_{m=1}^s \sum_{(i,j) \in \Gamma_m} f_m(i,j)(q_j^m - q_i^m)(n_m - \tilde{k}_m q_j^m - \tilde{k}_m)
\]
\[
\leq \frac{1}{n\tilde{F}} \cdot \sum_{m=1}^s \sum_{(i,j) \in \Gamma_m} f_m(i,j) 4 \sqrt{n_m(\ln n_m)} \tilde{k}_m (2 \sqrt{n_m(\ln n_m)} + 1)
\]
\[
\leq \frac{1}{n\tilde{F}} \cdot \sum_{m=1}^s \sum_{(i,j) \in \Gamma_m} f_m(i,j) 4 \sqrt{n_m(\ln n_m)} (2 \sqrt{n_m(\ln n_m)} + 1)
\]

Since
\[
\frac{4 \sqrt{n(\ln n)(2 \sqrt{n(\ln n)} + 1)}}{n/C - 2 \sqrt{n \ln n}} \sim 8C \ln^2 n
\]
as \( n \to \infty \), for large enough \( n \) it holds
\[
\sum_2 \leq \frac{1}{n\tilde{F}} \cdot \sum_{m=1}^s \sum_{(i,j) \in \Gamma_m} f_m(i,j) \cdot 9C \ln^2 n_m
\]
\[
\leq \frac{l \cdot 9C \cdot \tilde{F}}{F'} \cdot \frac{\ln^2 n}{n} = o \left( \frac{1}{\sqrt{n}} \right) .
\]

Last but not least we estimate the third sum using Lemma 3.
\[
\sum_3 := \sum_{m=1}^l \sum_{(i,j) \in \Gamma_m} \beta_{m}^{ij} \frac{(q_j^m - q_i^m) \tilde{k}_m}{n_m}
\]
\[
= \frac{1}{n\tilde{F}} \cdot \sum_{m=1}^s \sum_{(i,j) \in \Gamma_m} f_m(i,j)(q_j^m - q_i^m)
\]
\[
= \frac{1}{n\tilde{F}} \cdot \sum_{m=1}^s \sum_{i=1} q_i^m (f_m^+(i) - f_m^-(i))
\]
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\[
= -\frac{1}{nf} \cdot \sum_{m=1}^{s} z(Q_m) = -\frac{1}{nf} z(Q) + \frac{1}{nf} \cdot \sum_{m=s+1}^{l} z(Q_m)
\]

\[
\leq 0 + \frac{Z}{f} \cdot \frac{n_{s+1} + \cdots + n_l}{n} \leq \frac{Z \cdot l \cdot n^{1/3}}{f} \cdot \frac{n}{n} = o\left(\frac{1}{\sqrt{n}}\right).
\]

Thus Lemma 4 and consequently Theorem B are proved. Q.E.D.

REFERENCES