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An Asymptotic Formula for the Maximum Size of an h-Family in Products of Partially Ordered Sets

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An h-family of a partially ordered set P is a subset of P such that no $h+1$ elements of the h-family lie on any single chain. Let $S_1, S_2,...$ be a sequence of partially ordered sets which are not antichains and have cardinality less than a given finite value. Let P_n be the direct product of $S_1,..., S_n$. An asymptotic formula of the maximum size of an h-family in P_n is given, where $h = o(\sqrt{n})$ and $n \to \infty$.

Let P be a partially ordered set (poset). A subset $\mathfrak{F}_h \subseteq P$ is called an h-family if it does not contain a chain of $h+1$ elements, i.e., there are not $x_0,..., x_h \in \mathfrak{F}_h$ such that $x_0 < \cdots < x_h$. Let $d_h(P)$ be the maximum size of an h-family in P. If P and Q are posets, then the *direct product* $P \times Q$ is defined on the Cartesian product of the sets P and Q as follows: $(x_1, y_1) \leqslant_{P \times Q} (x_2, y_2)$ iff $x_1 \leqslant_P x_2$ and $y_1 \leqslant_Q y_2$.

In all that follows we consider a sequence $S_1, S_2,...$ of nontrivial posets (i.e., they are not antichains) with bounded cardinalities. Let $k_i := |S_i| < C$ $(i = 1, 2,...)$. We put $P_n := S_1 \times \cdots \times S_n$ and $d_{n,h} := d_h(P_n)$. In this paper we will give an asymptotic formula for $d_{n,h}$ if $h = o(\sqrt{n})$ and $n \to \infty$. This generalizes a result of V. B. Alekseev [2] where the case $S_1 = S_2 = \cdots$ and $h = 1$ was settled.

In order to formulate our result we need the following definition. A *representation* of a poset P is a mapping $z: P \to \mathbb{R}$ such that $z(x) - z(y) \geqslant 1$ if $x > y$. A representation is called *optimal* if $(1/|P| \sum_{x \in P} (z(x) - \bar{z}(P))^2$ is an infimum (extending over all representations of P), where $\bar{z}(P) := (1/|P| \sum_{x \in P} z(x)$. The infimum is denoted by $D(P)$.

Remark 1. In [2] and [3] it is proved that an optimal representation always exists.

In all that follows let z_i be an optimal representation of S_i such that $\bar{z}_i(S_i) = 0$. If $x \in S_i$, we can omit the index i in $z_i(x)$ and write biefly $z(x)$ since the mapping is defined by S_i. Let $D_i := D(S_i)$ and $V_n := \sum_{i=1}^n D_i$. Our main result is the following

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THEOREM. *If $h = o(\sqrt{n})$, then*

$$d_{n,h} \sim \frac{k_1 \cdots k_n}{\sqrt{2\pi V_n}} \cdot h, \qquad \text{where } n \to \infty.$$

At first we will prove the

THEOREM A. *If $h = o(\sqrt{n})$, then*

$$d_{n,h} \gtrsim \frac{k_1 \cdots k_n}{\sqrt{2\pi V_n}} \cdot h, \qquad \textit{where } n \to \infty.$$

Proof. Let $\mathbf{x} = (x_1, ..., x_n) \in P_n$ and define $z(\mathbf{x}) := \sum_{i=1}^n z(x_i)$. Obviously $\mathfrak{F}_h := \{\mathbf{x}: -h/2 < z(\mathbf{x}) \leqslant h/2\}$ is an h-family. Hence, $d_{n,h} \geqslant |\mathfrak{F}_h|$. We will prove that

$$|\mathfrak{F}_h| \sim \frac{k_1 \cdots k_n}{\sqrt{2\pi V_n}} \cdot h. \tag{1}$$

For that we define the following discrete random variables $\eta_1, \eta_2, ...$ as follows:

$$P(\eta_i = z_j^i) = \frac{1}{k_i},$$

where $z_j^i := z(s_j^i)$ and $S_i = \{s_1^i, ..., s_{k_i}^i\}$. Let $\eta_1, \eta_2, ...$ be independent and $\nu_n := \eta_1 + \cdots + \eta_n$. Then the expected value and variance of ν_n is equal to 0 and V_n, respectively. We have $|\mathfrak{F}_h| = k_1 \cdots k_n \cdot P(-h/2 < \nu_n \leqslant h/2)$. Thus it is sufficient to prove that

$$P\left(-\frac{h}{2} < \nu_n \leqslant \frac{h}{2}\right) = \frac{h}{\sqrt{2\pi V_n}}(1 + o(1)). \tag{2}$$

In Lemmas 4 and 5 of [2] it is proved that $\eta_1, \eta_2, ...$ have a lattice distribution and that the maximal spans of them are equal to $1/r_1$, $1/r_2, ...$, where $r_1, r_2, ...$ are integers. Obviously, there exists only a finite number of posets with cardinality less than C. Thus the number of different distribution functions of $\eta_1, \eta_2, ...$ is finite. Let $1/R_1, ..., 1/R_l$ be the corresponding maximal spans. If R is the least common multiple of $R_1, ..., R_l$ and $\xi_i := R\eta_i + y_i$, then the maximal span of ξ_i is equal to R/R_i, hence an integer ($i = 1, 2, ...$). Thus y_i can be chosen such that ξ_i is an integer-valued variable. If we put $\mu_n := \sum_{i=1}^n \xi_i$, then obviously $W_n := R^2 V_n$ is the variance and $M_n := y_1 + \cdots + y_n$ is the expected value of μ_n ($n = 1, 2, ...$). Since the greatest common divisor of $R/R_1, ..., R/R_l$ equals 1 we may use the limit theorem for

k-sequences of independent random variables (see [8, p. 189]) and conclude that

$$\sup_N \left| \sqrt{W_n}\, P(\mu_n = N) - \frac{1}{\sqrt{2\pi}} e^{-(N-M_n)^2/2W_n} \right| \to 0 \tag{3}$$

(the supremum extends over all integers N). Now we have

$$P\left(-\frac{h}{2} < \nu_n \leqslant \frac{h}{2}\right) = P\left(-\frac{hR}{2} + M_n < \mu_n \leqslant \frac{hR}{2} + M_n\right)$$
$$= \sum_{N \in I} P(\mu_n = N), \tag{4}$$

where $I := \{N \in \mathbb{Z} : -hR/2 + M_n < N \leqslant hR/2 + M_n\}$.

Let $\underline{D}$ and $\bar{D}$ be the smallest and largest value of $\{D_1, D_2, \ldots\}$, respectively (they exist since there is only a finite number of different distribution functions under $\eta_1, \eta_2, \ldots$). Because of $h = o(\sqrt{n})$ we conclude that for all $N \in I$

$$0 < \frac{(N-M_n)^2}{2W_n} \leqslant \frac{(hR)^2}{2R^2 V_n} \leqslant \frac{h^2}{2n\underline{D}} < \varepsilon_1(n),$$

where $\varepsilon_1(n) \to 0$. (It is not the case that $\underline{D} = 0$ since $S_1, S_2, \ldots$ are not antichains.) Thus (3) implies that for all $N \in I$

$$\frac{1}{\sqrt{2\pi}}(1 - \varepsilon_2(n)) \leqslant \sqrt{W_n}\, P(\mu_n = N) \leqslant \frac{1}{\sqrt{2\pi}}(1 + \varepsilon_3(n)), \tag{5}$$

where $\varepsilon_2(n) \to 0$ and $\varepsilon_3(n) \to 0$. Since $|I| = hR$ from (4) and (5) it follows

$$\frac{hR}{\sqrt{2\pi W_n}}(1 - \varepsilon_2(n)) \leqslant P\left(-\frac{h}{2} < \nu_n \leqslant \frac{h}{2}\right) \leqslant \frac{hR}{\sqrt{2\pi W_n}}(1 + \varepsilon_3(n)),$$

and because of $W_n = R^2 V_n$ we obtain

$$P\left(-\frac{h}{2} < \nu_n \leqslant \frac{h}{2}\right) = \frac{h}{\sqrt{2\pi V_n}}(1 + o(1)),$$

and (2) is proved. Q.E.D.

Now we will prove the more difficult

THEOREM B. $d_{n,h} \lesssim (k_1 \cdots k_n/\sqrt{2\pi V_n}) \cdot h$, *where* $n \to \infty$.

Proof. It is sufficient to prove that

$$d_{n,1} \lesssim \frac{k_1 \cdots k_n}{\sqrt{2\pi V_n}}$$

since $d_{n,h} \leqslant h \cdot d_{n,1}$ (each h-family is the union of h Sperner families, i.e., 1-families; see, for instance, [1, p. 271]). Let $N_v := \{\mathbf{x} \in P_n : z(\mathbf{x}) = v\}$ and consider the bipartite graph G_v on the vertex-set $N_{v-1} \cup N_v$ in which $(\mathbf{x}, \mathbf{y})$ is an edge iff $\mathbf{x} < \mathbf{y}$. Let E_v be a maximum matching of G_v, i.e., a maximum set of pairwise non-adjacent edges of G_v. Now join adjacent edges of the matchings ..., E_{v-1}, E_v, E_{v+1},... so far as possible. In this way we obtain a partition of P_n into chains (single points are regarded as chains too). Let R_0 be the set of such chains in the partition which have an element $\mathbf{x}$ with $-\frac{1}{2} < z(\mathbf{x}) \leqslant \frac{1}{2}$. Further let R_1 and R_2 be the set of such chains in the partition in which $z(\mathbf{x}) \geqslant \frac{1}{2}$ and $z(\mathbf{x}) \leqslant -\frac{1}{2}$ for all elements of the chain, respectively. Obviously, $d_{n,1} \leqslant |R_0| + |R_1| + |R_2|$, $|R_0| = |\{\mathbf{x}: -\frac{1}{2} < z(\mathbf{x}) \leqslant \frac{1}{2}\}|$. From (1) in the proof of Theorem A we obtain

$$|R_0| \sim \frac{1}{\sqrt{2\pi V_n}} k_1 \cdots k_n.$$

In all that follows we will prove that $|R_1| \leqslant k_1 \cdots k_n \cdot o(1/\sqrt{n})$. Then all is done since then

$$\frac{|R_1|}{|R_0|} \leqslant o\left(\frac{1}{\sqrt{n}}\right) \cdot \sqrt{2\pi V_n} \leqslant o\left(\frac{1}{\sqrt{n}}\right) \cdot \sqrt{2\pi n \bar{D}} \to 0,$$

and the same follows for $|R_2|$ by duality.

Let δ_v be the number of elements of N_v which are not covered by an edge of the maximum matching E_v. Associating to each chain of R_1 its smallest element we obtain

$$|R_1| = \sum_{v \geqslant 1/2} \delta_v. \tag{6}$$

For $X \subseteq N_v$ let $V(X) := \{\mathbf{y} \in N_{v-1} : \mathbf{y} < \mathbf{x} \text{ for any } \mathbf{x} \in X\}$. A set $X \subseteq N_v$ is called a *critical set* iff

$$|X| - |V(X)| = \max_{Y \subseteq N_v} (|Y| - |V(Y)|).$$

From well-known results on matchings (see [7, p. 138ff.]) it follows that there exists a ***unique minimal critical set*** X_v which is contained in all other critical sets and for which

$$|X_v| - |V(X_v)| = \delta_v.$$

Now we will prove that special classes of elements, so-called statistics, are contained in X_v. At first we shall define these classes. Since $|S_i| < C$ for all i

we have in our sequence $S_1, S_2, \ldots$ only a finite number of different posets. Let $T_1, \ldots, T_l$ be these posets ($T_i = \{t_j^i; j = 1, \ldots, \tilde{k}_i\}$, $i = 1, \ldots, l$). We can suppose that $T_1, \ldots, T_l$ are pairwise disjoint. Let n_i be the number of factors T_i. Obviously,

$$P_n \cong \underbrace{T_1 \times \cdots \times T_1}_{n_1} \times \cdots \times \underbrace{T_l \times \cdots \times T_l}_{n_l}.$$

Without loss of generality we may assume that P_n is equal to this poset. Further, let $Q_i := (q_1^i, \ldots, q_{\tilde{k}_i}^i)$ be a $\tilde{k}_i$-tuple of integers with $\sum_{j=1}^{\tilde{k}_i} q_j^i = n_i$ ($i = 1, \ldots, l$), and let $\mathbf{Q} := (Q_1, \ldots, Q_l)$ be an l-tuple of such $\tilde{k}_i$-tuples. The *statistic* of Q is defined to be the set of elements $\mathbf{x} = (x_1, \ldots, x_n) \in P_n$ in which the element t_j^i occurs exactly q_j^i times ($j = 1, \ldots, \tilde{k}_i$, $i = 1, \ldots, l$); it is denoted by $S(\mathbf{Q})$. Now we will prove that either no element of a statistic or the whole statistic, i.e., all elements of it, is contained in X_v. Let $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_l)$ be an element of the direct product of the symmetric groups on the sets $\{1, \ldots, n_i\}$, $i = 1, \ldots, l$. To such a $\boldsymbol{\pi}$ we can associate an automorphism $\varphi_{\boldsymbol{\pi}}$ of P_n by

$$\varphi_{\boldsymbol{\pi}}(x_1^1, \ldots, x_{n_1}^1, \ldots, x_1^l, \ldots, x_{n_l}^l) := (x_{\pi_1(1)}^1, \ldots, x_{\pi_1(n_1)}^1, \ldots, x_{\pi_l(1)}^l, \ldots, x_{\pi_l(n_l)}^l).$$

Evidently, $|\varphi_{\boldsymbol{\pi}}(X_v)| = |X_v|$ and $|V(\varphi_{\boldsymbol{\pi}}(X_v))| = |V(X_v)|$, hence $|\varphi_{\boldsymbol{\pi}}(X_v)| - |V(\varphi_{\boldsymbol{\pi}}(X_v))| = \delta_v$. Since X_v is a minimal critical set we have $X_v \subseteq \varphi_{\boldsymbol{\pi}}(X_v)$, hence $X_v = \varphi_{\boldsymbol{\pi}}(X_v)$. Consequently, if $\mathbf{x} \in X_v$, then $\varphi_{\boldsymbol{\pi}}(\mathbf{x}) \in X_v$ for all such $\boldsymbol{\pi}$, thus the whole statistic containing $\mathbf{x}$ is contained in X_v.

Let $z(Q_i) := \sum_{j=1}^{\tilde{k}_i} q_j^i z(t_j^i)$ and $z(\mathbf{Q}) := \sum_{i=1}^{l} z(Q_i)$. Obviously, $z(\mathbf{x}) = z(\mathbf{Q})$ holds for all $\mathbf{x} \in S(\mathbf{Q})$. Let I_m be the set of all pairs (i,j) for which $t_i^m > t_j^m$ and $z(t_i^m) - z(t_j^m) = 1$ ($m = 1, \ldots, l$). In order to estimate δ_v (see (6)) we associate to each pair (t_i^m, t_j^m) with $(i,j) \in I_m$ a number β_{ij}^m (to be specified later) such that $\beta_{ij}^m \geqslant 0$ ($m = 1, \ldots, l$) and $\sum_{m=1}^{l} \sum_{(i,j) \in I_m} \beta_{ij}^m = 1$.

For $\mathbf{Q} = (Q_1, \ldots, Q_l)$ and $(i,j) \in I_m$ we define $\mathbf{Q}_{ij}^m$ to be the tuple $(Q_1, \ldots, Q_{m-1}, Q_m', Q_{m+1}, \ldots, Q_l)$, where $Q_m' := (q_1^m, \ldots, q_i^m - 1, \ldots, q_j^m + 1, \ldots, q_{\tilde{k}_m}^m)$. Obviously the elements of $S(\mathbf{Q}_{ij}^m)$ can be obtained from the elements of $S(\mathbf{Q})$ by specifying some coordinate and changing t_i^m to t_j^m for each element in which t_i^m occurs in that coordinate.

We set $r(\mathbf{Q}) := |S(\mathbf{Q})| - \sum_{m=1}^{l} \sum_{(i,j) \in I_m} \beta_{ij}^m |S(\mathbf{Q}_{ij}^m)|$ if the right-hand side is not negative and $r(\mathbf{Q}) := 0$, otherwise.

LEMMA 1. $\delta_v \leqslant \sum_{\mathbf{Q}: z(\mathbf{Q}) = v} r(\mathbf{Q})$.

Proof. To each pair $(\mathbf{Q}, \mathbf{Q}')$ with $z(\mathbf{Q}) = z(\mathbf{Q}') + 1 = v$ and $\mathbf{Q}' = \mathbf{Q}_{ij}^m$ for some $m \in \{1, \ldots, l\}$ and some $(i,j) \in I_m$ we associate the unique weight $\beta_{ij}^m |S(\mathbf{Q}_{ij}^m)|$. Counting the weights of pairs $(\mathbf{Q}, \mathbf{Q}')$ with $S(\mathbf{Q}) \subset X_v$ in two different ways we obtain

$$\sum_{\mathbf{Q}:S(\mathbf{Q})\subset X_v} \sum_{m=1}^{l} \sum_{(i,j)\in I_m} \beta_{ij}^m |S(\mathbf{Q}_{ij}^m)|$$

$$= \sum_{\mathbf{Q}':S(\mathbf{Q}')\subset V(X_v)} |S(\mathbf{Q}')| \cdot \sum_{m=1}^{l} \sum_{(i,j)\in I_m:\,\exists \mathbf{Q} \text{ with } S(\mathbf{Q})\subset X_v \text{ and } \mathbf{Q}_{ij}^m=\mathbf{Q}'} \beta_{ij}^m$$

$$\leqslant \sum_{\mathbf{Q}':S(\mathbf{Q}')\subset V(X_v)} |S(\mathbf{Q}')| = |V(X_v)|.$$

It follows

$$\delta_v = |X_v| - |V(X_v)|$$

$$\leqslant \sum_{\mathbf{Q}:S(\mathbf{Q})\subset X_v} |S(\mathbf{Q})| - \sum_{\mathbf{Q}:S(\mathbf{Q})\subset X_v} \sum_{m=1}^{l} \sum_{(i,j)\in I_m} \beta_{ij}^m |S(\mathbf{Q}_{ij}^m)|$$

$$= \sum_{\mathbf{Q}:S(\mathbf{Q})\subset X_v} r(\mathbf{Q}) \leqslant \sum_{\mathbf{Q}:z(\mathbf{Q})=v} r(\mathbf{Q}). \qquad \text{Q.E.D.}$$

Now we will prove that

$$\sum_{\mathbf{Q}:z(\mathbf{Q})>1/2} r(\mathbf{Q}) \leqslant k_1 \cdots k_n \cdot o\left(\frac{1}{\sqrt{n}}\right)$$

which together with (6) and Lemma 1 will complete the proof. We will estimate the above sum in two steps. For that let, without loss of generality,

$$n_1, ..., n_s > n^{1/3} \qquad \text{and} \qquad n_{s+1}, ..., n_l \leqslant n^{1/3} \tag{7}$$

(if n is large enough we have $s \geqslant 1$). Let

$$F_1 := \left\{\mathbf{Q}: \left| q_i^m - \frac{n_m}{\tilde{k}_m} \right| > 2\sqrt{n_m} \ln n_m \text{ for some } m \in \{1, ..., s\} \text{ and some } i \in \{1, ..., \tilde{k}_m\}\right\},$$

$$F_2 := \left\{\mathbf{Q}: \left| q_i^m - \frac{n_m}{\tilde{k}_m} \right| \leqslant 2\sqrt{n_m} \ln n_m \text{ for each } m \in \{1, ..., s\} \text{ and each } i \in \{1, ..., \tilde{k}_m\}\right\}.$$

Then

$$\sum_{\mathbf{Q}:z(\mathbf{Q})>1/2} r(\mathbf{Q}) \leqslant \sum_{\mathbf{Q}\in F_1} r(\mathbf{Q}) + \sum_{\mathbf{Q}\in F_2:z(\mathbf{Q})>1/2} r(\mathbf{Q}). \tag{8}$$

LEMMA 2. $\sum_{\mathbf{Q}\in F_1} r(\mathbf{Q}) \leqslant k_1 \cdots k_n \cdot o(1/\sqrt{n})$.

Proof. We have

$$\sum_{\mathbf{Q}\in F_1} r(\mathbf{Q}) \leqslant \sum_{\mathbf{Q}\in F_1} |S(\mathbf{Q})|. \tag{9}$$

If $G_i^m := \{\mathbf{Q}: |q_i^m - n_m/\tilde{k}_m| > 2\sqrt{n_m}\ln n_m\}(m = 1,..., s, \quad i = 1,..., \tilde{k}_m)$, then obviously

$$\sum_{\mathbf{Q}\in F_1} |S(\mathbf{Q})| \sum_{m=1}^{s} \sum_{i=1}^{\tilde{k}_m} \sum_{\mathbf{Q}\in G_i^m} |S(\mathbf{Q})| \leqslant lC \cdot \max_{\substack{m\in\{1,\dots,s\}\\ i\in\{1,\dots,\tilde{k}_m\}}} \left(\sum_{\mathbf{Q}\in G_i^m} |S(\mathbf{Q})| \right). \tag{10}$$

If $K_i^m(q)$ is the set of all $\mathbf{x} = (x_1,..., x_n) \in P_n$ in which the element t_i^m occurs exactly q times, then

$$\sum_{\mathbf{Q}\in G_i^m} |S(\mathbf{Q})| = \sum_{q:|q-n_m/\tilde{k}_m| > 2\sqrt{n_m}\ln n_m} |K_i^m(q)|.$$

In order to estimate these sums we consider the following identically distributed and independent random variables $\lambda_1,..., \lambda_{n_m}$, with

$$P(\lambda_i = 0) = \frac{\tilde{k}_m - 1}{\tilde{k}_m},$$

$$P(\lambda_i = 1) = \frac{1}{\tilde{k}_m} \qquad (i = 1,..., n_m).$$

Further let $\zeta_{n_m} := \lambda_1 + \cdots + \lambda_{n_m}$. Obviously, $|K_i^m(q)| = k_1 \cdots k_n \cdot P(\zeta_{n_m} = q)$, and thus

$$\sum_{\mathbf{Q}\in G_i^m} |S(\mathbf{Q})| = k_1 \cdots k_n \cdot P\left(\left| \zeta_{n_m} - \frac{n_m}{\tilde{k}_m} \right| > 2\sqrt{n_m}\ln n_m \right). \tag{11}$$

Since $n_m/\tilde{k}_m$ is the expected value of ζ_{n_m} it follows from Hoeffding's exponential estimation for distributions of sums of independent random variables (see [8, p. 58, 8.]) that

$$P\left(\left| \zeta_{n_m} - \frac{n_m}{\tilde{k}_m} \right| > 2\sqrt{n_m}\ln n_m \right) \leqslant 2 \cdot e^{-\ln^2 n_m}. \tag{12}$$

From (9)–(12) we now obtain

$$\sum_{\mathbf{Q}\in F_1} r(\mathbf{Q}) \leqslant \max_{m\in\{1,\dots,s\}} 2lC \cdot k_1 \cdots k_n \cdot e^{-\ln^2 n_m}$$

$$\leqslant 2lC \cdot k_1 \cdots k_n \cdot e^{-(1/9)\ln^2 n} = k_1 \cdots k_n \cdot o\left(\frac{1}{\sqrt{n}}\right). \qquad \text{Q.E.D.}$$

It remains to estimate the second sum in (8) $\sum_{Q \in F_2 : z(Q) \geqslant 1/2} r(Q)$. For this we need the following lemma which can be obtained from the Theorem of Gale (see [4, p. 62] and [2, Lemma 12]) or from the Theorem of Kuhn and Tucker (see [3]).

LEMMA 3. *To each pair* (t_i^m, t_j^m) *with* $(i,j) \in I_m$ *one can associate a number* $f^m(i,j) \geqslant 0$ *such that* $f_-^m(i) - f_+^m(i) = z(t_i^m)$, *where* $f_-^m(i) := \sum_{j:(i,j) \in I_m} f^m(i,j)$ *and* $f_+^m(i) := \sum_{j:(j,i) \in I_m} f^m(j,i)$.

Remark 2. This is the only place where we use the fact that the poset representations are optimal.

Obviously there exist constants $\underline{F}$ and $\bar{F}$ such that

$$0 < \underline{F} \leqslant \sum_{(i,j) \in I_m} f^m(i,j) \leqslant \bar{F} \qquad \text{for all } m \in \{1,..., l\} \tag{13}$$

($\underline{F}$ can be chosen greater than 0 since $T_1,..., T_l$ are not trivial posets). Now let

$$f := \frac{1}{n} \sum_{m=1}^{s} \sum_{(i,j) \in I_m} \frac{n_m}{\tilde{k}_m} f^m(i,j), \tag{14}$$

$$\beta_{ij}^m := \begin{cases} \dfrac{1}{n} \cdot \dfrac{f^m(i,j)}{f} \cdot \dfrac{n_m}{\tilde{k}_m}, & m = 1,..., s, \\ 0, & m = s+1,..., l. \end{cases} \tag{15}$$

Obviously $\sum_{m=1}^{l} \sum_{(i,j) \in I_m} \beta_{ij}^m = 1$ and $\beta_{ij}^m \geqslant 0$. Further, from (13) and (14) it follows

$$\frac{1}{n} \cdot \frac{1}{C} \cdot (n_1 + \cdots + n_s) \cdot \underline{F} \leqslant f \leqslant \frac{1}{n} \cdot (n_1 + \cdots + n_s) \cdot \bar{F} \leqslant \bar{F}.$$

Since

$$\frac{n_1 + \cdots + n_s}{n} = 1 - \frac{n_{s+1} + \cdots + n_l}{n}$$

and

$$\frac{n_{s+1} + \cdots + n_l}{n} \leqslant \frac{l \cdot n^{1/3}}{n} \to 0,$$

there exists a constant $\underline{F}'$ such that

$$0 < \underline{F}' \leqslant f \leqslant \bar{F}. \tag{16}$$

Moreover we mention that there is a constant Z such that $z(t_i^m) \leqslant Z$ for all $m \in \{1,..., l\}$ and $i \in \{1,..., \tilde{k}_m\}$. It follows

$$z(\mathbf{Q}_m) \leqslant n_m \cdot Z \qquad (m = 1,..., l). \tag{17}$$

Now we are able to estimate the second sum in (8).

LEMMA 4. *If the numbers β_{ij}^m are chosen as above,*

$$\sum_{\mathbf{Q} \in F_2 : z(\mathbf{Q}) \geqslant 1/2} r(\mathbf{Q}) \leqslant k_1 \cdots k_n \cdot o\left(\frac{1}{\sqrt{n}}\right).$$

Proof. We shall prove for a fixed $\mathbf{Q} \in F_2$ with $z(\mathbf{Q}) \geqslant \frac{1}{2}$, $r(\mathbf{Q}) \leqslant |S(\mathbf{Q})| \cdot o(1/\sqrt{n})$, where the function $o(1/\sqrt{n})$ does not depend on $\mathbf{Q}$. Then all is done since then

$$\sum_{\mathbf{Q} \in F_2 : z(\mathbf{Q}) \geqslant 1/2} r(\mathbf{Q}) \leqslant \sum_{\mathbf{Q}} |S(\mathbf{Q})| \cdot o\left(\frac{1}{\sqrt{n}}\right) = k_1 \cdots k_n \cdot o\left(\frac{1}{\sqrt{n}}\right).$$

If $r(\mathbf{Q}) = 0$, we do not have to prove anything, thus let $r(\mathbf{Q}) > 0$. Then

$$\begin{aligned}
r(\mathbf{Q}) &= |S(\mathbf{Q})| - \sum_{m=1}^{l} \sum_{(i,j) \in I_m} \beta_{ij}^m |S(\mathbf{Q}_{ij}^m)| \\
&= |S(\mathbf{Q})| \cdot \left(1 - \sum_{m=1}^{l} \sum_{(i,j) \in I_m} \beta_{ij}^m \frac{|S(\mathbf{Q}_{ij}^m)|}{|S(\mathbf{Q})|}\right) \\
&= |S(\mathbf{Q})| \cdot \left(1 - \sum_{m=1}^{l} \sum_{(i,j) \in I_m} \beta_{ij}^m \frac{q_i^m}{q_j^m + 1}\right) \\
&= |S(\mathbf{Q})| \cdot \left(\sum_{m=1}^{l} \sum_{(i,j) \in I_m} \beta_{ij}^m \frac{1}{q_j^m + 1}\right. \\
&\quad + \sum_{m=1}^{l} \sum_{(i,j) \in I_m} \beta_{ij}^m \frac{(q_j^m - q_i^m)(n_m - \tilde{k}_m q_j^m - \tilde{k}_m)}{n_m (q_j^m + 1)} \\
&\quad \left. + \sum_{m=1}^{l} \sum_{(i,j) \in I_m} \beta_{ij}^m \frac{(q_j^m - q_i^m)\, \tilde{k}_m}{n_m}\right).
\end{aligned}$$

Finally, we estimate the three sums in the parentheses using Lemma 3, (7), (15), (16), (17) and the facts that $k_i < C$, $\mathbf{Q} \in F_2$ and $z(\mathbf{Q}) \geqslant \frac{1}{2}$.

$$\begin{aligned}
\sum\nolimits_1 &:= \sum_{m=1}^{l} \sum_{(i,j) \in I_m} \beta_{ij}^m \frac{1}{q_j^m + 1} = \frac{1}{nf} \cdot \sum_{m=1}^{s} \sum_{(i,j) \in I_m} \frac{f^m(i,j)\, n_m}{(q_j^m + 1)\, \tilde{k}_m} \\
&\leqslant \frac{1}{nf} \sum_{m=1}^{s} \sum_{(i,j) \in I_m} \frac{f^m(i,j)}{1 - 2(\tilde{k}_m \ln n_m / \sqrt{n_m})}.
\end{aligned}$$

Because of (7) we have

$$1 - 2\frac{\tilde{k}_m \ln n_m}{\sqrt{n_m}} > \frac{1}{2}$$

if n is large enough. Thus,

$$\sum\nolimits_1 \leqslant \frac{2}{nf} \cdot l \cdot \bar{F} \leqslant \frac{2l\bar{F}}{\underline{F}'} \cdot \frac{1}{n} = o\left(\frac{1}{\sqrt{n}}\right).$$

$$\begin{aligned}\sum\nolimits_2 &:= \sum_{m=1}^{l} \sum_{(i,j)\in I_m} \beta_{ij}^m \frac{(q_j^m - q_i^m)(n_m - \tilde{k}_m q_j^m - \tilde{k}_m)}{n_m(q_j^m + 1)}\\
&= \frac{1}{nf} \cdot \sum_{m=1}^{s} \sum_{(i,j)\in I_m} \frac{f^m(i,j)(q_j^m - q_i^m)(n_m - \tilde{k}_m q_j^m - \tilde{k}_m)}{\tilde{k}_m(q_j^m + 1)}\\
&\leqslant \frac{1}{nf} \cdot \sum_{m=1}^{s} \sum_{(i,j)\in I_m} \frac{f^m(i,j)\, 4\sqrt{n_m}(\ln n_m)\, \tilde{k}_m (2\sqrt{n_m}(\ln n_m) + 1)}{\tilde{k}_m(n_m/\tilde{k}_m - 2\sqrt{n_m}\ln n_m)}\\
&\leqslant \frac{1}{nf} \cdot \sum_{m=1}^{s} \sum_{(i,j)\in I_m} \frac{f^m(i,j)\, 4\sqrt{n_m}(\ln n_m)(2\sqrt{n_m}(\ln n_m) + 1)}{n_m/C - 2\sqrt{n_m}\ln n_m}.\end{aligned}$$

Since

$$\frac{4\sqrt{n}(\ln n)(2\sqrt{n}(\ln n) + 1)}{n/C - 2\sqrt{n}\ln n} \sim 8C \ln^2 n$$

as $n \to \infty$, for large enough n it holds

$$\begin{aligned}\sum\nolimits_2 &\leqslant \frac{1}{nf} \cdot \sum_{m=1}^{s} \sum_{(i,j)\in I_m} f^m(i,j) \cdot 9C \ln^2 n_m\\
&\leqslant \frac{l \cdot 9C \cdot \bar{F}}{\underline{F}'} \cdot \frac{\ln^2 n}{n} = o\left(\frac{1}{\sqrt{n}}\right).\end{aligned}$$

Last but not least we estimate the third sum using Lemma 3.

$$\begin{aligned}\sum\nolimits_3 &:= \sum_{m=1}^{l} \sum_{(i,j)\in I_m} \beta_{ij}^m \frac{(q_j^m - q_i^m)\,\tilde{k}_m}{n_m}\\
&= \frac{1}{nf} \cdot \sum_{m=1}^{s} \sum_{(i,j)\in I_m} f^m(i,j)(q_j^m - q_i^m)\\
&= \frac{1}{nf} \cdot \sum_{m=1}^{s} \sum_{i=1}^{\tilde{k}_m} q_i^m (f_+^m(i) - f_-^m(i))\end{aligned}$$

$$= -\frac{1}{nf} \cdot \sum_{m=1}^{s} z(Q_m) = -\frac{1}{nf} z(\mathbf{Q}) + \frac{1}{nf} \cdot \sum_{m=s+1}^{l} z(Q_m)$$

$$\leqslant 0 + \frac{Z}{f} \cdot \frac{n_{s+1} + \cdots + n_l}{n} \leqslant \frac{Z \cdot l}{f} \cdot \frac{n^{1/3}}{n} = o\left(\frac{1}{\sqrt{n}}\right).$$

Thus Lemma 4 and consequently Theorem B are proved. Q.E.D.

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