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ORIGINAL ARTICLE

The fractional-order SIS epidemic model with variable population size

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Abstract In this work, we deal with the fractional-order SIS epidemic model with constant recruitment rate, mass action incidence and variable population size. The stability of equilibrium points is studied. Numerical solutions of this model are given. Numerical simulations have been used to verify the theoretical analysis.

MATHEMATICS SUBJECT CLASSIFICATION: 37N25; 34D20; 37M05

1. Introduction

The epidemic models incorporate constant recruitment, disease-induced death and mass action incidence rate.

Some infections do not confer any long lasting immunity. Such infections do not have a recovered state and individuals become susceptible again after infection. This type of disease can be modelled by SIS type. The total population N is divided into two compartments with $N = S + I$, where S is the number of individuals in the susceptible class, I is the number of individuals who are infectious [1,2].

The use of fractional-orders differential and integral operators in mathematical models has become increasingly widespread in recent years [3]. Several forms of fractional differential equations have been proposed in standard models.

Differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, economic, viscoelasticity, biology, physics and engineering. Recently, a large amount of literature has been developed concerning the application of fractional differential equations in nonlinear dynamics [3].

In this paper, we study the fractional-order SIS model. The stability of equilibrium points is studied. Numerical solutions of this model are given.

We like to argue that fractional-order equations are more suitable than integer order ones in modelling biological, economic and social systems (generally complex adaptive systems) where memory effects are important. In Section 2, the equilibrium points and their asymptotic stability of differential equations of fractional order are studied. In Sections 3 and 4, the model is presented and discussed. In Section 5 numerical solutions of the model are given.

Now we give the definition of fractional-order integration and fractional-order differentiation:

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Definition 1.1. The fractional integral of order $\beta \in R^+$ of the function $f(t)$, $t > 0$ is defined by

$$I^\beta f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, ds \tag{1}$$

and the fractional derivative of order $\alpha \in (n-1, n]$ of $f(t)$, $t > 0$ is defined by

$$D^\alpha f(t) = I^{n-\alpha} D^n f(t), \quad D = \frac{d}{dt}. \tag{2}$$

For the main properties of the fractional-orders derivatives and integrals [4–9].

2. Equilibrium points and their asymptotic stability

Let $\alpha \in (0, 1]$ and consider the system [10–15]

$$\begin{aligned} D^\alpha y_1(t) &= f_1(y_1, y_2) \\ D^\alpha y_2(t) &= f_2(y_1, y_2) \end{aligned} \tag{3}$$

with the initial values

$$y_1(0) = y_{o1} \quad \text{and} \quad y_2(0) = y_{o2}. \tag{4}$$

To evaluate the equilibrium points, let

$$D^\alpha y_i(t) = 0 \Rightarrow f_i(y_1^{eq}, y_2^{eq}) = 0, \quad i = 1, 2$$

from which we can get the equilibrium points y_1^{eq}, y_2^{eq}.

To evaluate the asymptotic stability, let

$$y_i(t) = y_i^{eq} + \varepsilon_i(t),$$

so the equilibrium point (y_1^{eq}, y_2^{eq}) is locally asymptotically stable if both the eigenvalues of the Jacobian matrix A

$$\begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix}$$

evaluated at the equilibrium point satisfies ($|\arg(\lambda_1)| > \alpha\pi/2$, $|\arg(\lambda_2)| > \alpha\pi/2$) [11,12,14–16]. The stability region of the fractional-order system with order α is illustrated in Fig. 1 (in which σ, ω refer to the real and imaginary parts of the eigenvalues, respectively, and $j = \sqrt{-1}$). From Fig. 1, it is easy to show that the stability region of the fractional-order case is greater than the stability region of the integer order case.

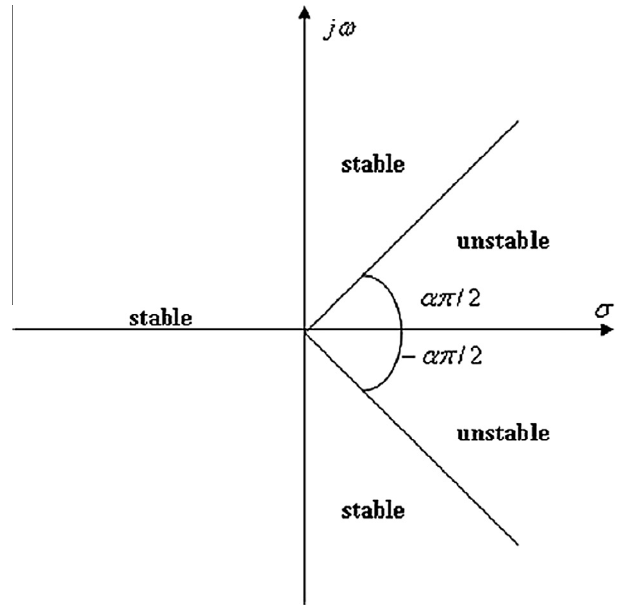

Figure 1 Stability region of the fractional-order system.

3. Fractional-order SIS model

Let $S(t)$ be the number of individuals in the susceptible class at time t, $I(t)$ be the number of individuals who are infectious at time t.

The fractional-order SIS model is given by

$$\begin{aligned} D^{\alpha_1} S(t) &= \Lambda - \beta SI - \mu S + \varphi I, \\ D^{\alpha_1} I(t) &= \beta SI - (\varphi + \mu + \alpha) I, \end{aligned} \tag{5}$$

where $0 < \alpha_1 \leqslant 1$ and the parameters are positive constants. The constant Λ is the recruitment rate of susceptible corresponding to births and immigration, μ is the per capita natural mortality rate. We assume that a disease may be fatal to some infectious, so deaths due to disease can be included in a model using the disease-related death rate from infectious class, α. Let φ be the rate at which individuals infectious and return to susceptible class. This together with $N = S + I$, implies

$$D^{\alpha_1} N = \Lambda - \mu N - \alpha I.$$

Thus, the total population size N may vary in time [2]. To evaluate the equilibrium points, let

$$D^{\alpha_1} S = 0,$$
$$D^{\alpha_1} I = 0,$$

then $(S_{eq}, I_{eq}) = \left(\frac{\Lambda}{\mu}, 0\right)$, (S_*, I_*), are the equilibrium points where,

$$S_* = \frac{1}{\beta}(\varphi + \mu + \alpha), \quad I_* = \frac{\Lambda}{(\mu + \alpha)} - \frac{\mu(\varphi + \mu + \alpha)}{\beta(\mu + \alpha)}.$$

For $(S_{eq}, I_{eq}) = \left(\frac{\Lambda}{\mu}, 0\right)$ we find that

$$A = \begin{bmatrix} -\mu & -\frac{\beta\Lambda}{\mu} + \varphi \\ 0 & \frac{\beta\Lambda}{\mu} - (\varphi + \mu + \alpha) \end{bmatrix},$$

and its eigenvalues are

$$\lambda_1 = -\mu < 0,$$

$$\lambda_2 = \frac{\beta\Lambda}{\mu} - (\varphi + \mu + \alpha) < 0 \quad \text{if} \quad \frac{\beta\Lambda}{\mu} < (\varphi + \mu + \alpha).$$

Hence the equilibrium point $(S_{eq}, I_{eq}) = \left(\frac{\Lambda}{\mu}, 0\right)$ is locally asymptotically stable if

$$\frac{\beta\Lambda}{\mu} < (\varphi + \mu + \alpha). \tag{6}$$

For $(S_{eq}, I_{eq}) = (S_*, I_*)$ we find that

$$A = \begin{bmatrix} -\frac{\beta\Lambda}{(\mu+\alpha)} + \frac{\mu(\varphi+\mu+\alpha)}{(\mu+\alpha)} - \mu & -(\mu + \alpha) \\ \frac{\beta\Lambda}{(\mu+\alpha)} - \frac{\mu(\varphi+\mu+\alpha)}{(\mu+\alpha)} & 0 \end{bmatrix},$$

and its eigenvalues are

$$\lambda_1 = \frac{1}{2(\mu+\alpha)}\left(-(\beta\Lambda - \mu\varphi) + \sqrt{(\beta\Lambda - \mu\varphi)^2 - 4(\mu+\alpha)^2[\beta\Lambda - \mu(\varphi + \mu + \alpha)]}\right),$$
$$\lambda_2 = \frac{1}{2(\mu+\alpha)}\left(-(\beta\Lambda - \mu\varphi) - \sqrt{(\beta\Lambda - \mu\varphi)^2 - 4(\mu+\alpha)^2[\beta\Lambda - \mu(\varphi + \mu + \alpha)]}\right).$$

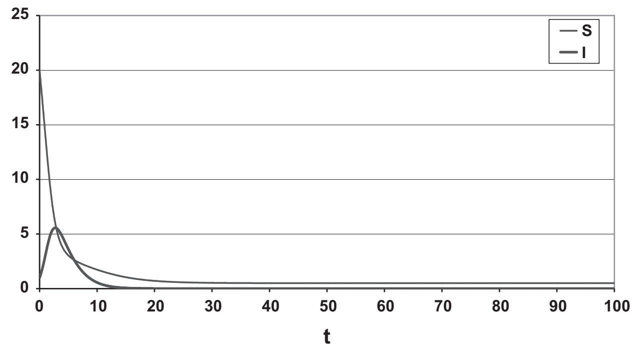

Figure 2 $\alpha_1 = 1.0$.

A sufficient condition for the local asymptotic stability of the equilibrium point $(S_{eq}, I_{eq}) = (S_*, I_*)$ is

$$|\arg(\lambda_1)| > \alpha_1\pi/2, |\arg(\lambda_2)| > \alpha_1\pi/2. \tag{7}$$

4. Existence of uniformly stable solution

Let

$$x_1(t) = S(t), \quad x_2(t) = I(t),$$

$$f_1(x_1(t), x_2(t)) = \Lambda - \beta x_1(t)x_2(t) - \mu x_1(t) + \varphi x_2(t),$$

and

$$f_2(x_1(t), x_2(t)) = \beta x_1(t)x_2(t) - (\varphi + \mu + \alpha)x_2(t).$$

Let

$$D = \{x_1, x_2 \in R : |x_i(t)| \leqslant a, \ t \in [0, T], i = 1, 2\},$$

then on D we have

$$\left|\frac{\partial}{\partial x_1} f_1(x_1, x_2)\right| \leqslant k_1, \quad \left|\frac{\partial}{\partial x_2} f_1(x_1, x_2)\right| \leqslant k_2,$$

$$\left|\frac{\partial}{\partial x_1} f_2(x_1, x_2)\right| \leqslant k_3 \quad \text{and} \left|\frac{\partial}{\partial x_2} f_2(x_1, x_2)\right| \leqslant k_4,$$

where k_1, k_2, k_3 and k_4 are positive constants.

This implies that each of the two functions f_1, f_2 satisfies the Lipschitz condition with respect to the two arguments x_1 and x_2, then each of the two functions f_1, f_2 is absolutely continuous with respect to the two arguments x_1 and x_2.

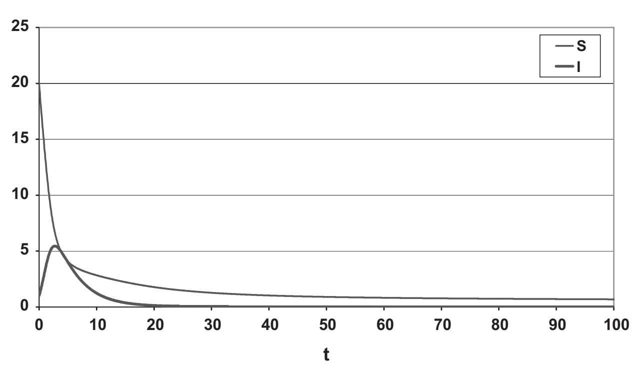

Figure 3 $\alpha_1 = 0.9$.

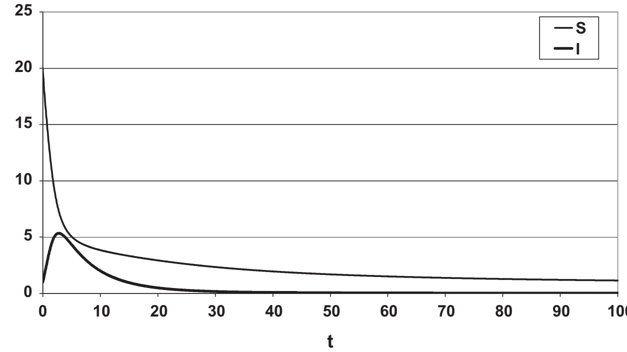

Figure 4 $\alpha_1 = 0.8$.

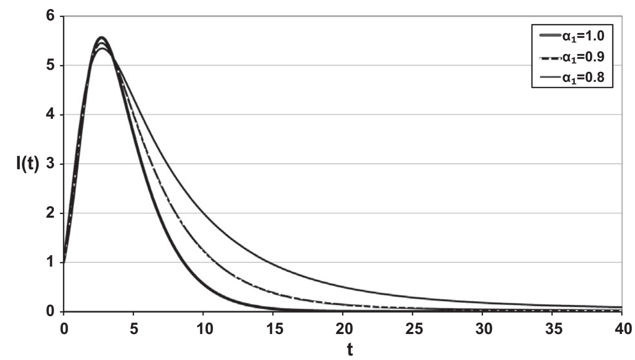

Figure 5

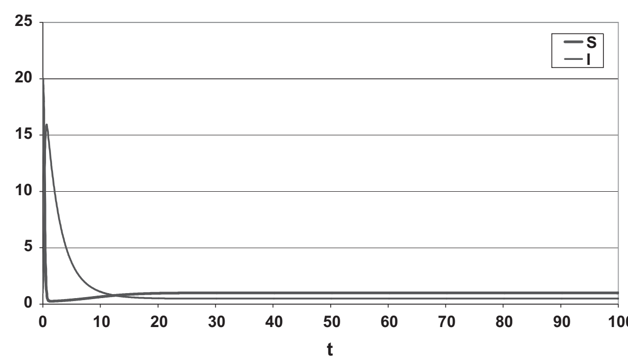

Figure 6 $\alpha_1 = 1.0$.

Consider the following initial value problem which represents the fractional-order SIS model (8) and (9)

$$D^{\alpha_1} x_1(t) = f_1(x_1(t), x_2(t)), \quad t > 0 \quad \text{and } x_1(0) = x_{o1}, \tag{8}$$

$$D^{\alpha_1} x_2(t) = f_2(x_1(t), x_2(t)), \quad t > 0 \quad \text{and } x_2(0) = x_{o2}. \tag{9}$$

Definition 4.1. By a solution of the fractional-order SIS model (8) and (9), we mean a column vector $(x_1(t)\ x_2(t))^{\tau}$, x_1 and $x_2 \in C[0, T]$, $T < \infty$ where $C[0, T]$ is the class of continuous functions defined on the interval $[0, T]$ and τ denote the transpose of the matrix.

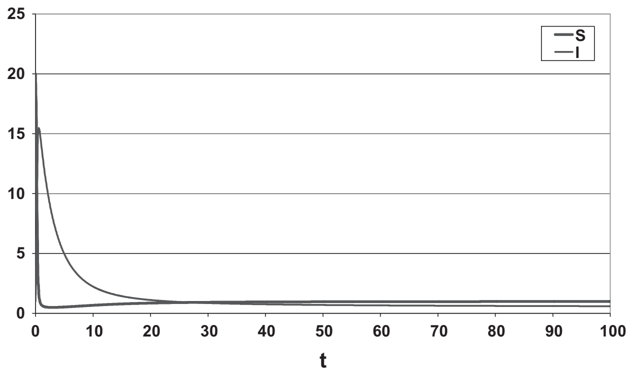

Figure 7 $\alpha_1 = 0.9$.

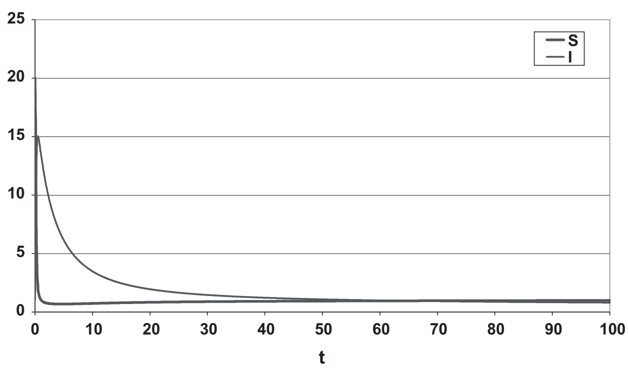

Figure 8 $\alpha_1 = 0.8$.

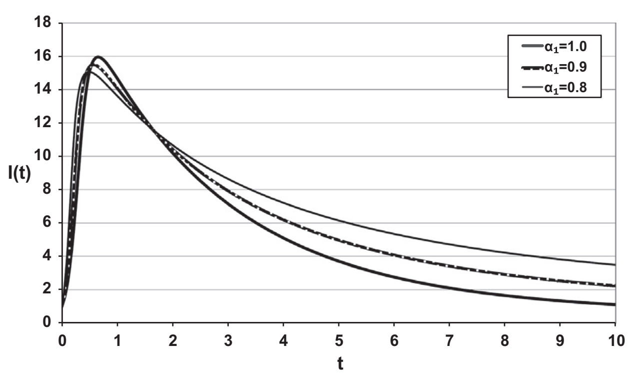

Figure 9

Now we have the following theorem

Theorem 4.1. *The fractional-order SIS model* (8) and (9) *has a unique uniformly Lyapunov stable solution.*

Proof. Write the model (8) and (9) in the matrix form

$$D^{\alpha_1}X(t) = F(X(t)), \quad t > 0 \quad \text{and } X(0) = X_o \tag{10}$$

where

$X(t) = (x_1(t)\ x_2(t))^{\tau}$,

and

$F(X(t)) = (f_1(x_1(t), x_2(t))\ f_2(x_1(t), x_2(t)))^{\tau}$.

Now applying Theorem 2.1 [17], we deduce that the fractional-order SIS model (8) and (9) has a unique solution. Also by Theorem 3.2 [17] this solution is uniformly Lyapunov stable. □

5. Numerical methods and results

An Adams-type predictor-corrector method has been introduced and investigated further in [18–20]. In this paper, we use an Adams-type predictor-corrector method for the numerical solution of fractional integral equation.

The key to the derivation of the method is to replace the original problem (5) by an equivalent fractional integral equations

$$\begin{aligned} S(t) &= S(0) + I^{\alpha_1}[\Lambda - \beta SI - \mu S + \varphi I], \\ I(t) &= I(0) + I^{\alpha_1}[\beta SI - (\varphi + \mu + \alpha)I], \end{aligned} \tag{11}$$

and then apply the **PECE** (Predict, Evaluate, Correct, Evaluate) method.

The approximate solutions are displayed in Figs. 2–9 for $S(0) = 20.0$, $I(0) = 1.0$ and different $0 < \alpha_1 \leqslant 1$. In Figs. 2–5 we take $\Lambda = 0.1, \beta = 0.1, \mu = 0.2, \varphi = 0.3, \alpha = 0.1$ and found that the equilibrium point $\left(\frac{\Lambda}{\mu}, 0\right) = (0.5, 0)$ is locally asymptotically stable where the condition (6) $\left(\frac{\beta\ \Lambda}{\mu} = 0.05 < (\varphi + \mu + \alpha) = 0.6\right)$ is satisfied. In Fig. 5 we found that in the fractional-order case, the peak of the infection is reduced. But the disease takes a longer time to be eradicated.

In Figs. 6–9 we take $\Lambda = 0.5$, $\beta = 0.5$, $\mu = 0.3$, $\varphi = 0.1$, $\alpha = 0.1$ and found that the equilibrium point $\left(\frac{\Lambda}{\mu}, 0\right) = (1.66667, 0)$ is unstable where the condition (6) is not satisfied $\left(\frac{\beta\ \Lambda}{\mu} = 0.833333 > (\varphi + \mu + \alpha) = 0.5\right)$ and the equilibrium point (S_*, I_*) is locally asymptotically stable where the condition (7) is satisfied where the equilibrium point and the eigenvalues are given as:

$$(S_*, I_*) = (1.0, 0.5),$$
$$\lambda_{1,2} = -0.275 \pm 0.156125i.$$

The equilibrium point $(S_*, I_*) = (1.0, 0.5)$, is locally asymptotically stable where$|\arg(\lambda_{2,3})| = 2.62524 > \alpha_1\pi/2$. In Fig. 9 we found that in the fractional-order case, the peak of the infection is reduced. But the disease takes a longer time to be eradicated.

6. Conclusions

In this paper we study the fractional-order SIS model. The stability of equilibrium points is studied. Numerical solutions of this model are given.

The reason for considering a fractional-order system instead of its integer order counterpart is that fractional-order differential equations are generalizations of integer order differential equations. Also using fractional-order differential equations can help us to reduce the errors arising from the neglected parameters in modelling real life phenomena.

We like to argue that fractional-order equations are more suitable than integer order ones in modelling biological, economic and social systems (generally complex adaptive systems) where memory effects are important.

The stability of equilibrium points is studied. Numerical solutions of these models are given. Numerical simulations have been used to verify the theoretical analysis.

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