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Exact solutions of a remarkable fin equation

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Abstract

A model 'remarkable' fin equation is singled out from a class of nonlinear $(1+1)$-dimensional fin equations. For this equation a number of exact solutions are constructed by means of using both the classical Lie algorithm and different modern techniques (functional separation of variables, generalized conditional symmetries, hidden symmetries etc.).

Keywords: Fin equations; Lie reductions; Non-Lie reductions; Nonclassical symmetries; Exact solutions

1. Introduction

Mathematical models of conductivity and diffusion processes are traditional objects for investigation with symmetry methods and approaches related to them [10]. Since these models are often formulated in terms of nonlinear differential equations which are, as a rule, nonintegrable and cannot be linearized, symmetry methods are important for construction of their exact solutions. It is indicative that modern development of group analysis of differential equations was begun from the study of a class of $(1+1)$-dimensional nonlinear diffusion equations [17]. Afterward a wide range of diffusion equations were investigated within the symmetry framework (see e.g. [2,4,9,10,15,23]). At the same time, even 'simplified' $(1+1)$-dimensional nonlinear diffusion models are fraught with a great many 'symmetry mysteries' which remain to be solved.

Recently the class of nonlinear fin equations of the general form

$$u_t = (D(u)u_x)_x + h(x)u, \tag{1}$$

was investigated from the symmetry point of view in a number of papers [3,19,20,24]. Here u is treated as the dimensionless temperature, t and x the dimensionless time and space variables, D the thermal conductivity, $h = -N^2 f(x)$, N the fin parameter and f the heat transfer coefficient. (See e.g. [3] for references on the physical meaning and applications of equations from class (1).)

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Note that certain equations from the class (1) were studied formerly. For example, the condition $D_u = 0$ corresponds to the linear case of (1) which was completely investigated from the Lie symmetry point of view a long time ago [14,18]. The problem of group classification for the class of nonlinear one-dimensional diffusion equations (the degenerate case $h = 0$) was first solved by Ovsiannikov [17,18]. The class of diffusion–reaction equations classified by Dorodnitsyn [4,10] contains the equations of form (1) with h being a constant. Group classification of the subclass where the thermal conductivity is a power function of the temperature was carried out in [23]. Large sets of exact solutions were constructed for the above equations and collected e.g. in [10,21].

In contrast to the study in [3,19,20], that in [24] was concentrated on rigorous and exhaustive group classification of the whole class (1) and on construction of exact solutions for truly nonlinear and 'variable-coefficient' equations from this class. To find exact solutions, both classical Lie reduction and different modern approaches were applied. Although some interesting exact solutions were constructed, almost all of them are either stationary or scale invariant. Therefore, the problem of finding more complicated exact solutions of equations from class (1) with $D_u h_x \neq 0$ remains open, even for partial values of the parameter functions.

In this letter we single out a model 'remarkable' fin equation (1) with fixed values of the parameter functions $D = u^{-3/2}$ and $h = x^{-1}$, i.e.

$$u_t = (|u|^{-3/2} u_x)_x + x^{-1} u, \tag{2}$$

and investigate it in detail. A number of its exact solutions are constructed with a variety of symmetry techniques (Lie reduction, nonlinear separation of variables, generalized conditional symmetries, hidden symmetries etc.) in a closed form.

2. Lie invariance

Eq. (2) is not an exceptional member of class (1), from the Lie symmetry point of view. The maximal Lie invariance group of (2) is $A_1 = \langle \partial_t,\ D = t\partial_t + x\partial_x - \frac{2}{3}u\partial_u \rangle$, i.e. its Lie symmetry group G_1 consists of the transformations

$$\tilde{t} = \mathrm{e}^{\delta_1} t + \delta_0, \qquad \tilde{x} = \mathrm{e}^{\delta_1} x, \qquad \tilde{u} = \mathrm{e}^{-\frac{2}{3}\delta_1} u,$$

where δ_0 and δ_1 are arbitrary constants. At the same time, Eq. (2) has remarkable properties connected with different kinds of non-Lie symmetries that allow us to construct a number of its exact solutions. Sources of this singularity should be investigated additionally.

Due to the possibility of changing the sign of u and due to the physical sense of the equation, we can assume u to be positive and omit the modulus from the expression $|u|^{-3/2}$.

Instead of Eq. (2), we can investigate the equivalent equation

$$v_t = v v_{xx} - \frac{2}{3}(v_x)^2 - \frac{3}{2}\frac{v}{x}, \tag{3}$$

where $v = u^{-3/2}$ and, therefore, v is positive. In the variables (t, x, v) the operator ∂_t has the same form and $D = t\partial_t + x\partial_x + v\partial_v$. The inverse transformation from (3) to (2) is $u = v^{-2/3}$.

3. List of exact solutions

For usability we collect all constructed solutions of (2) together and then discuss methods of finding them, including both Lie and non-Lie techniques. The adduced solutions are inequivalent with respect to the group G_1 and can be extended to parametric sets of solutions of Eq. (2) with transformations from this group:

(1) $u = \left(-\varepsilon^2 x^3 + 3\varepsilon x^2 - \frac{9}{4}x\right)^{-2/3}$, $\varepsilon \in \{-1, 0, 1\} \bmod G_1$,

(2) $u = \left(\frac{3}{2}\frac{x^2}{t} - \frac{9}{4}x\right)^{-2/3}$,

(3) $u = \left(x^3 - 3x^2 \tan 2t - \frac{9}{4}x\right)^{-2/3}$,

(4) $u = \left(-x^3 + 3x^2 \tanh 2t - \frac{9}{4}x\right)^{-2/3}$,

(5) $u=\left(-x^3+3x^2\coth 2t-\frac{9}{4}x\right)^{-2/3}$,
(6) $\sqrt{\psi-\psi^2}-\frac{1}{2}\arcsin(2\psi-1)=\pm\frac{1}{x}+C_0,\ \psi:=-\frac{u^{-1/2}}{x},\ 0<\psi<1$,
(7) $\sqrt{\psi+\psi^2}-\frac{1}{2}\ln(2\psi+1+\sqrt{\psi+\psi^2})=\pm\frac{1}{x}+C_0,\ \psi:=-\frac{u^{-1/2}}{x},\ \psi<-1$ or $\psi>0$.

Solutions (1)–(5) should be considered only for the values of (t,x) where the corresponding basis of power $-2/3$ is positive.

Note that solutions of (2) can be transformed to solutions of equations which are pointwise equivalent to Eq. (2). For example, the transformation $\tilde t=t$, $\tilde x=x^{-1}$, $\tilde u=x^2u$ links (2) with the equation

$$\tilde x^{-1}\tilde u_{\tilde t}=(\tilde u^{-3/2}\tilde u_{\tilde x})_{\tilde x}+\tilde u.$$

More generally, the above equation and (2) belong to the class of variable-coefficient diffusion–reaction equations which were investigated in [23]. Therefore, Eq. (2) can be extended by transformations from the corresponding equivalence group to a subclass of variable-coefficient diffusion–reaction equations with simultaneous extension of the above solutions.

4. Lie reductions

In this section we discuss ways of finding the above solutions starting with the classical Lie method. A_1 is a non-Abelian two-dimensional Lie algebra. A complete list of inequivalent subalgebras of A_1 is exhausted by the one-dimensional subalgebras $\langle\partial_t\rangle$ and $\langle D\rangle$ and the algebra A_1 itself.

With A_1 we construct the ansatz $u=\varphi x^{-2/3}$. In view of the positivity of u, φ should also be positive. Since A_1 has a single functionally independent invariant, there are no invariant independent variables in the ansatz and φ is a constant function. Therefore, the ansatz reduces Eq. (2) to the algebraic equation $4\varphi^{-3/2}+9\,\mathrm{sign}\,x=0$ with respect to φ, which has a solution only on the negative semiaxis $x<0$. As a result, we obtain solution (1) with $\varepsilon=0$. It is interesting in the sense that all other solutions (1)–(5) are modifications of it with additional terms.

The solutions, being invariant with respect to the subalgebra $\langle\partial_t\rangle$, are nothing but stationary solutions. The corresponding ansatz $u=\varphi(\omega)$, where $\omega=x$, gives the reduced ODE

$$(\varphi^{-3/2}\varphi_\omega)_\omega+\omega^{-1}\varphi=0 \tag{4}$$

which is integrable in quadratures. Eq. (4) is connected with equations 6.101 and 6.205 of [11]. Integrability of (4) can be explained in the framework of the symmetry approach. The Lie invariance algebra of (4) is generated by the operators $\hat D=3\omega\partial_\omega-2\varphi\partial_\varphi$ and $\hat\Pi=\omega^2\partial_\omega-2\omega\varphi\partial_\varphi$, i.e. it is two dimensional. It is enough for Eq. (4) to be integrable within the framework of the Lie method.

The invariance algebra A_1 induces only the subalgebra spanned by the operator $\hat D$. Therefore, $\hat\Pi$ is a pure *hidden* symmetry operator of the initial equation (2). Let us note that the first nontrivial example of hidden symmetries connected with reduction of PDEs was found by Kapitanskiy [12,13] for the Navier–Stokes equations. Wide classes of hidden symmetries of the Navier–Stokes equations were constructed in [6]. See also [1] for different notions of hidden symmetries of ODEs, and other references therein.

To reduce Eq. (4) to an integrable form, we change the unknown function via

$$\psi=-\frac{\varphi^{-1/2}}{\omega},\quad \text{i.e.}\quad \varphi=\frac{1}{\omega^2\psi^2}.$$

In view of (4), $\psi_\omega\neq0$ and ψ satisfies the equation $(\omega^4\psi_\omega^2)_\omega=(\psi^{-1})_\omega$ which is simply integrated once. Further integration of the first-order ODE obtained, $\omega^4\psi_\omega^2=\psi^{-1}+C_1$, where C_1 is an arbitrary constant, with separation of variables, results in the implicit solution

$$\int\frac{\mathrm d\psi}{\sqrt{\psi^{-1}+C_1}}=\pm\frac{1}{\omega}+C_0.$$

The integration constant C_1 is normalized to $\{-1,0,1\}$ using induced scale transformations. Let us note also that all values of C_0 are equivalent with respect to the hidden symmetry group generated by the operator $\hat\Pi$. Calculation of the integral depends on the value of C_1.

If $C_1 = 0$ then $\psi > 0$, i.e. in view of the definition of ψ, solutions may exist only for negative values of ω. Integration under the condition $C_1 = 0$ results in the solution $\psi = (C_0 - \frac{3}{2}x^{-1})^{\frac{2}{3}}$ corresponding to solution (1) of Eq. (2). The integration constant C_0 is normalized to $\varepsilon \in \{-1, 0, 1\}$ using scale transformations associated with the operator $\hat{D}$.

The condition $C_1 = -1$ implies $0 < \psi < 1$, i.e. solutions again exist only for negative values of ω and have an implicit form giving solution (6) of Eq. (2). In the case $C_1 = 1$ we have the constraints $\psi < -1$ or $\psi > 0$ and derive solution (7).

The other kinds of Lie invariant solutions are the similarity solutions which are invariant with respect to scale transformations. It is more convenient here to work in terms of the variables (t, x, v). The ansatz constructed with the subalgebra $\langle D\rangle$ has the form $v = t\varphi(\omega)$, where $\omega = x/t$. After substituting it into (3), we obtain the reduced ODE

$$\varphi\varphi_{\omega\omega} - \frac{2}{3}(\varphi_\omega)^2 + \omega\varphi_\omega - \frac{3}{2}\frac{\varphi}{\omega} - \varphi = 0.$$

It has two polynomial solutions $\varphi = -\frac{9}{4}\omega$ and $\varphi = \frac{3}{2}\omega^2 - \frac{9}{4}\omega$ which correspond to solutions $(1)_{\varepsilon=0}$ and (2) of Eq. (2).

5. Non-Lie ansatz

The form of Lie invariant solutions (1) and (2) leads us to look for more general polynomial solutions of Eq. (3). As a result, we find the ansatz

$$v = \varphi^1(t)x^3 + \varphi^2(t)x^2 - \frac{9}{4}x,$$

which reduces Eq. (3) to the system of two ODEs

$$\varphi^1_t = 0, \qquad \varphi^2_t = -6\varphi^1 - \frac{2}{3}(\varphi^2)^2.$$

Up to translations with respect to t and scale transformations induced by the Lie symmetry group of the initial equation, the above system has the following nonequivalent solutions:

$$(\varepsilon^2, \varepsilon), \qquad \left(0, -\frac{3}{2}t\right), \qquad (1, -3\tan 2t), \qquad (-1, 3\tanh 2t), \qquad (-1, 3\coth 2t)$$

which correspond to solutions (1)–(5) of Eq. (2). Solutions (3)–(5) are non-Lie ones.

The above ansatz is rewritten in terms of the function u as

$$u = \left(\varphi^1(t)x^3 + \varphi^2(t)x^2 - \frac{9}{4}x\right)^{-2/3}.$$

This ansatz can be interpreted in the frameworks of a number of different approaches for finding exact solutions of nonlinear PDEs, such as nonlinear variable separation [9], the method of differential constraints [22], anti-reduction [8] or generalized conditional symmetries [5,27]. (See also [16] for connections between these approaches.) Thus, the differential constraint $2x^3(x^{-2}u^{-3/2})_{xx} = -9$ corresponding to the ansatz is compatible (i.e. in involution) with Eq. (2). ‘Anti-reduction’ of Eq. (2) by the ansatz containing two new unknown functions of one argument to the system of two ODEs means that

$$(8u^2 + 8xuu_x + 5x^2u_x^2 - x^2uu_{xx} + 6xu^{7/2})\partial_u$$

is a generalized conditional symmetry operator of Eq. (2).

6. On nonclassical symmetries

We also study nonclassical (conditional) symmetries of Eq. (2). (See e.g. [26,29] for necessary definitions and properties of nonclassical symmetries.) Reduction operators of Eq. (2) have the general form $Q = \tau\partial_t + \xi\partial_x + \eta\partial_u$, where τ, ξ and η are functions of t, x and u, and $(\tau, \xi) \neq 0$. Since (2) is an evolution equation, there are two principally different cases for finding Q: $\tau \neq 0$ and $\tau = 0$.

We derive the system of determining equations in the case of $\tau \neq 0$ and integrate it completely. As a result, we obtain the following statement. *Any conditional symmetry operator of* Eq. (2) *in the case of* $\tau \neq 0$ *is equivalent to a Lie symmetry operator.*

As is well known, the operators with the vanishing coefficient of ∂_t form the so-called 'no-go' case in the study of conditional symmetries of an arbitrary (1 + 1)-dimensional evolution equation since the problem of finding them is reduced to the problem of solving a single equation which is equivalent to the initial one (see e.g. [7,25,28]). Note that "no-go" has to be treated as the impossibility just of exhaustively solving the problem. A number of particular examples of reduction operators with $\tau = 0$ can be constructed under additional constraints and then applied for finding exact solutions of the initial equation. Since the determining equation has more independent variables and, therefore, more degrees of freedom, it is more convenient often to guess a simple solution or a simple ansatz for the determining equation, which can give a parametric set of complicated solutions of the initial equation. Namely, in the case $\tau = 0$ we have $\xi \neq 0$. Up to usual equivalence of reduction operators, ξ can be assumed equal to 1, i.e. $Q = \partial_x + \eta\partial_u$. The conditional invariance criterion implies the determining equation for the coefficient η

$$\eta_t = \frac{\eta_{xx} + 2\eta\eta_{xu} + \eta^2\eta_{uu}}{u^{3/2}} - \frac{9\eta\eta_x + 6\eta^2\eta_u}{2u^{5/2}} + \frac{15\eta^3}{4u^{7/2}} + \frac{\eta - u\eta_u}{x} - \frac{u}{x^2}$$

which is reduced with a non-point transformation to Eq. (2), where η becomes a parameter. We have found partial solutions of the determining equation but all of them result in the above solutions of Eq. (2). In particular, the operator

$$\partial_x - 2\frac{u}{x}\left(1 \pm \sqrt{C_1 u - x u^{3/2}}\right)\partial_u$$

gives solutions (1), (6) and (7) in the case of the values $C_1 = 0$, $C_1 = -1$ and $C_1 = 1$ respectively. The ansatz of Section 5 is associated under the condition $\varphi^1 = C_1$ with the operator

$$\partial_x - \frac{u}{6x}\left(4C_1 x^3 u^{\frac{3}{2}} + 9xu^{\frac{3}{2}} + 8\right)\partial_u$$

which results, therefore, in the solutions (1)–(5), depending on values of C_1.

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