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An Extension of the Newton-Kantorovič Method for Solving Nonlinear Equations with an Application to Elasticity*

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PART I

1. INTRODUCTION AND SUMMARY

A. We consider a pair of nonlinear integral equations of Hammerstein type of the form

$$x_j(s) = g_j(s) + \gamma \int_0^1 L_{j1}(s, t)x_1(t)x_2(t)\,dt + \gamma \int_0^1 L_{j2}(s, t)\tfrac{1}{2}x_1^2(t)\,dt,$$
$$0 \leqslant s \leqslant 1, \qquad j = 1, 2, \quad (1.1)$$

for $x_1(s)$ and $x_2(s)$, with $g_j(s)$ and $L_{jk}(s, t)$ given (cf. Section 2, A) and $0 \leqslant \gamma < \infty$. The problem is from nonlinear elasticity theory. It pertains to the buckling of a thin shallow spherical shell clamped at the edge and under uniform external pressure. The functions x_1 and x_2 represent, respectively, the ratios of the slope of the displacement function and of the radial stress to the pressure; γ is a loading parameter proportional to the pressure. Specific definitions are given in the Appendix. The overall discussion of the physical problem, derivations of equations, detailed numerical results and their physical significance are given in [1]. This paper is devoted mainly to the theoretical analysis and practical solution of (1.1). Some new techniques are used for these purposes.

This mathematical study was motivated by the gross disagreement between various theoretically predicted and experimentally observed buckling loads. Indeed, part of the problem is to establish a suitable theoretical definition of buckling. Experimentally, buckling is said to occur when a "small" increase in load causes a "sudden large" increase in displacement.

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Theoretically, the "classical" critical load has been taken to mean the smallest γ for which the derivative with respect to γ of some parameter representing displacement becomes infinite [2]. For example, the parameter[1]

$$\rho := \gamma \int_0^1 x_1(t)t^2\,dt \tag{1.2}$$

represents the volume displacement of the shell. Mathematically, the critical value of γ in integral equation formulations of similar stability problems has been taken as the first bifurcation point [3, p. 161].

The results of this study suggest that certain energy comparisons may be important in determining buckling loads (Section 7, E and [1]). Let

$$\sigma := \frac{1}{4}\gamma^2 \int_0^1 x_1^2(t)x_2(t)\,dt. \tag{1.3}$$

Then (in suitable units) the elastic strain energy of the deformed shell is

$$U = \gamma(\rho - \sigma) \tag{1.4}$$

and the potential energy of the applied load is

$$V = -2\gamma\rho. \tag{1.5}$$

Thus, the total energy of the system is

$$U + V = -\gamma(\rho + \sigma). \tag{1.6}$$

For the application of energy criteria for buckling, solutions of (1.1) are needed for all $\gamma \geqslant 0$. For sufficiently small γ, the contractive mapping principle and the Schauder fixed point theorem can be used to solve (1.1). For arbitrary γ, the continuation method described below was employed. (Cf. [1] for a description of the method in more classical terms.)

B. Abstractly, Eq. (1.1) can be posed in a Banach space (cf. Section 3) in the form

$$P(\gamma, x) := (I - \gamma LF)x - g = 0, \tag{1.7}$$

where $x = (x_1, x_2)$, $g = (g_1, g_2)$, $Fx = (x_1x_2, \frac{1}{2}x_1^2)$ and the operator L is determined by the kernels $L_{jk}(s, t)$. If $P(\gamma_0, x_0) = 0$ and the Fréchet derivative (cf. Section 5, A) of P with respect to x at (γ_0, x_0) is nonsingular, then the Hildebrandt-Graves implicit function theorem (cf. Section 5, B) guarantees the *existence* of a locally unique solution $x = u(\gamma)$ such that $x_0 = u(\gamma_0)$. Moreover, $u(\gamma)$ is continuous. Kantorovič's generalization of Newton's method (cf. Section 5, C) yields successive approximations to these solutions

[1] We use $A := B$ to mean "the symbol A is by *definition* equal to the quantity or expression B."

as well as error bounds. Thus, we can *construct* a "curve" of solutions $x = u(\gamma)$ for γ in some interval, beginning at $\gamma = 0$ where $x = g$ and incrementing γ in small steps.

This continuation method breaks down at the first (critical) point where the x-derivative of P is singular. For γ near such a point, a change of parameter $\gamma = \gamma(\rho, x)$ was introduced to obtain an equivalent relation

$$Q(\rho, x) := P(\gamma(\rho, x), x) = 0, \tag{1.8}$$

for which the x-derivative of Q is nonsingular. In fact, (1.2) was used for this purpose. Now the solution curve can be continued in the form $x = v(\rho)$ with ρ as the independent parameter and γ as a function of both ρ and the solution. Once beyond the critical point, γ can be used again as the independent parameter.

By alternate use of γ and ρ as the independent parameter, a "global" solution curve can be constructed. A typical γ versus ρ graph is given in Fig. 1. For each point on the graph there is a locally unique solution of (1.1)

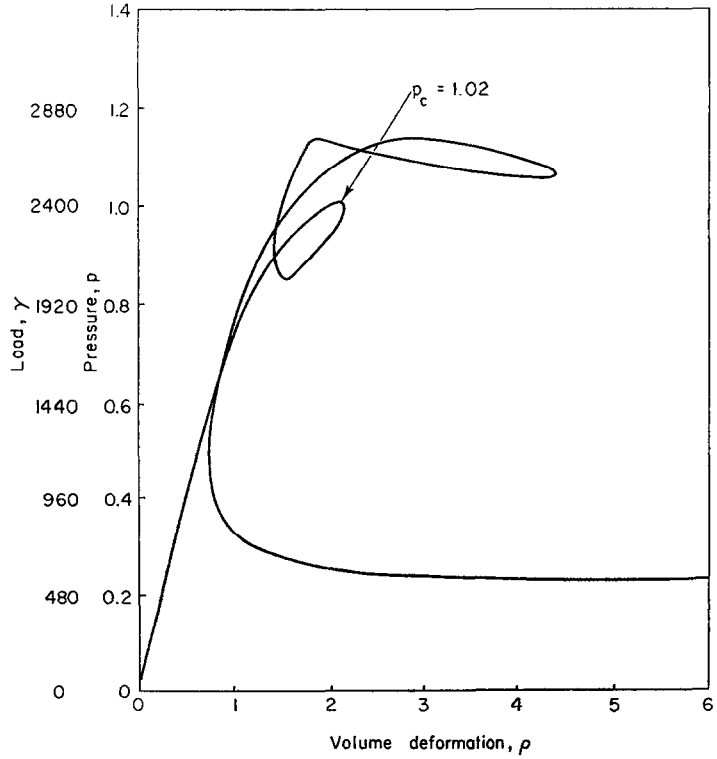

FIG. 1. Pressure vs. volume deformation. $\nu = 1/3$, $\mu = 7$, 20 pt. parabolic rule. $\gamma = p\mu^4 = 2401p$. (For $\rho > 6$, p decreases slowly to 0.19 near $\rho = 12$, thereafter increases monotonically, with $p = 1$ near $\rho = 20.2$.)

which varies continuously with γ and ρ. The local maxima and minima of γ are critical points in the sense used above; they are also bifurcation points of a rather simple type.

C. A discrete numerical analog was obtained by replacing the integrals in (1.1) by sums. The extended Newton's method described above was used on the discrete problem. The computations were carried out on the CDC 1604 at the University of Wisconsin. Questions of convergence and errors of the approximate solutions are considered in Section 9.

The programming and computing for the extended Newton's method were the responsibility of Allen Reiter of the Mathematics Research Center. We take this opportunity to acknowledge his aid and considerable skill.

2. Derivation of the Integral Equations

A. The equilibrium state of the shell is described mathematically by the differential equations

$$\left.\begin{aligned} x_1''(t) + \frac{1}{t}x_1'(t) - \frac{1}{t^2}x_1(t) + \mu^2 x_2(t) &= \frac{\gamma}{t}x_1(t)x_2(t) - 2t, \\ x_2''(t) + \frac{1}{t}x_2'(t) - \frac{1}{t^2}x_2(t) - \mu^2 x_1(t) &= -\frac{\gamma}{2t}x_1^2(t), \end{aligned}\right\} \quad 0 < t < 1, \tag{2.1}$$

and the boundary conditions

$$\begin{aligned} x_1(0) &= 0, \qquad x_1(1) = 0, \\ x_2(0) &= 0, \qquad x_2'(1) - \nu x_2(1) = 0, \end{aligned} \tag{2.2}$$

where (cf. Appendix) μ is a geometric parameter which increases with the ratio of the center height of the shell to the thickness and ν is an elastic parameter (Poisson's ratio). The parameter γ is related to the external (dimensionless) pressure p by

$$\gamma = p\mu^4. \tag{2.3}$$

Ranges of most physical interest are

$$0 \leqslant p \leqslant 2, \qquad 3 \leqslant \mu \leqslant 10, \qquad \tfrac{1}{4} \leqslant \nu \leqslant \tfrac{1}{2}. \tag{2.4}$$

We outline here the derivation of the integral equation formulation of the above boundary value problem. For further details see [1].

With the right members in (2.1) replaced by arbitrary but fixed continuous functions $y_1(t)$ and $y_2(t)$, one obtains an associated linear system. In this

system and in (2.2) let $x = x_1 + ix_2$ and $y = y_1 + iy_2$ to obtain the complex formulation

$$\mathscr{L}x := x''(t) + \frac{1}{t}x'(t) - \left(\frac{1}{t^2} + i\mu^2\right)x(t) = y(t), \tag{2.5}$$

$$x(0) = 0, \qquad x'(1) - \overline{x'(1)} - 2\nu x(1) = 0. \tag{2.6}$$

(The complex conjugation in (2.6) makes this a nonlinear problem.)

Note that $\mathscr{L}x = 0$ becomes Bessel's equation of order 1 upon making the change of variable $\tau = mt$ where $m^2 = -i\mu^2$. Thus, $\mathscr{L}x = 0$ has as fundamental solutions the modified Bessel and Neumann functions $J_1(mt)$ and $N_1(mt)$. The general solution of (2.5) is obtained by means of the method of variation of parameters. The (two) arbitrary constants are determined by (2.6).

Finally, y_1 and y_2 are replaced by the right members of (2.1) to obtain

$$x_j(s) = g_j(s) + \gamma\int_0^1 L_{j1}(s,t)x_1(t)x_2(t)\,dt + \gamma\int_0^1 L_{j2}(s,t)\tfrac{1}{2}x_1^2(t)\,dt,$$
$$0 \leqslant s \leqslant 1, \qquad j = 1, 2, \tag{2.7}$$

where

$$g_1(s) = \operatorname{Re} g(s), \qquad g_2(s) = \operatorname{Im} g(s), \tag{2.8}$$

$$g(s) = -\frac{2s}{m^2} - \left\{\frac{\pi}{m}N_2(m) + \frac{4\nu}{D}\left[C\frac{J_2(m)}{m} - \frac{\overline{J_2(m)}}{\bar{m}}\right]\right\}J_1(ms); \tag{2.9}$$

$$\begin{aligned} L_{11}(s,t) &= \operatorname{Re}[K(s,t) + R(s,t) + S(s,t)], \\ L_{21}(s,t) &= \operatorname{Im}[K(s,t) + R(s,t) + S(s,t)], \\ L_{12}(s,t) &= \operatorname{Im}[K(s,t) + R(s,t) - S(s,t)], \\ L_{22}(s,t) &= -\operatorname{Re}[K(s,t) + R(s,t) - S(s,t)]; \end{aligned} \tag{2.10}$$

$$K(s,t) = K(t,s) = \frac{\pi}{2}N_1(ms)J_1(mt), \qquad \begin{matrix} 0 \leqslant t \leqslant s \leqslant 1, \\ (s,t) \neq (0,0), \end{matrix} \tag{2.11}$$

$$R(s,t) = \frac{2\nu C}{D}J_1(ms)J_1(mt), \qquad 0 \leqslant s, t \leqslant 1, \tag{2.12}$$

$$S(s,t) = -\frac{2\nu}{D}J_1(ms)\overline{J_1(mt)}, \qquad 0 \leqslant s, t \leqslant 1; \tag{2.13}$$

$$C = \frac{\pi}{2}\{m\overline{J_1(m)}N_1'(m) + [\overline{mJ_1'(m) - 2\nu J_1(m)}]N_1(m)\}; \tag{2.14}$$

$$D = |2\nu J_1(m) - mJ_1'(m)|^2 - |mJ_1'(m)|^2; \tag{2.15}$$

$$m = \mu e^{3\pi i/4}. \tag{2.16}$$

The right member of (2.11) is indeterminate for $(s, t) = (0, 0)$. However, since [4, pp. 134–138]

$$J_1(z) = \frac{z}{2} + z^3 E_J(z), \quad E_J(z) \to -\frac{1}{16} \quad \text{as} \quad z \to 0,$$
$$N_1(z) = -\frac{2}{\pi z} + E_N(z), \quad E_N(z) \to 0 \quad \text{as} \quad z \to 0, \tag{2.17}$$

$K(s, t)$ is bounded in the neighborhood of (0,0) and, hence, is uniformly bounded in the unit square. It follows that the functions $L_{jk}(s, t)$ are continuous except at (0, 0) and are bounded uniformly in the unit square.

B. In order to display the system in its real formulation we introduce the Kelvin functions

$$ber_1(\mu s) := \operatorname{Re} J_1(ms), \quad bei_1(\mu s) := \operatorname{Im} J_1(ms),$$
$$ner_1(\mu s) := \operatorname{Re} N_1(ms), \quad nei_1(\mu s) := \operatorname{Im} N_1(ms). \tag{2.18}$$

For brevity we write (for fixed μ)

$$br\ s := ber_1(\mu s), \quad bi\ s := bei_1(\mu s),$$
$$nr\ s := ner_1(\mu s), \quad ni\ s := nei_1(\mu s). \tag{2.19}$$

Then

$$K(s, t) = K(t, s) = \frac{\pi}{2}(nr\ s\ br\ t - ni\ s\ bi\ t) + i\frac{\pi}{2}(nr\ s\ bi\ t + ni\ s\ br\ t),$$
$$0 \leqslant t \leqslant s \leqslant 1, \qquad (s, t) \neq (0, 0). \tag{2.20}$$

Let

$$C = \frac{\pi}{2}(C_1 + iC_2) \tag{2.21}$$

and note that D is real. Then

$$R(s, t) = \frac{\pi\nu}{D}\{C_1(br\ s\ br\ t - bi\ s\ bi\ t) - C_2(br\ s\ bi\ t + bi\ s\ br\ t)$$
$$+ i[C_1(br\ s\ bi\ t + bi\ s\ br\ t) + C_2(br\ s\ br\ t - bi\ s\ bi\ t)]\}; \tag{2.22}$$
$$S(s, t) = -\frac{2\nu}{D}[br\ s\ br\ t + bi\ s\ bi\ t) - i(br\ s\ bi\ t - bi\ s\ br\ t)]. \tag{2.23}$$

The kernels $L_{jk}(s, t)$ are obtained directly from these equations and (2.10). They comprise a (matrix) Green's function for the linear system associated with (2.1) and (2.2).

PART II

3. Abstract Formulation

Let $\mathscr{C} = \mathscr{C}[0, 1]$ be the Banach space of real continuous functions on $[0, 1]$ with the uniform norm $\|f\| = \max_{0 \leqslant t \leqslant 1} |f(t)|$. Define the linear operators $L_{jk} : \mathscr{C} \to \mathscr{C}$ by

$$(L_{jk}f)(s) = \int_0^1 L_{jk}(s, t)f(t)\, dt, \qquad \begin{matrix} 0 \leqslant s \leqslant 1, \\ j, k = 1, 2, \end{matrix} \tag{3.1}$$

where the kernels $L_{jk}(s, t)$ are given by (2.10). The operators L_{jk} are ***bounded*** (equivalently, ***continuous***) and

$$\|L_{jk}\| = \sup_{0 \leqslant s \leqslant 1} \int_0^1 |L_{jk}(s, t)|\, dt, \qquad j, k = 1, 2. \tag{3.2}$$

Moreover, the operators L_{jk} are *compact*, i.e., they transform bounded sets into sets with compact closure. This follows by the usual argument based on the Arzelà-Ascoli lemma on bounded equicontinuous functions.

Let $\mathscr{X} = \mathscr{C} \times \mathscr{C}$ be the Banach space with elements denoted either by[2] $x = (x_1, x_2)$ or by $x = \binom{x_1}{x_2}$, as convenience dictates, and the norm $\|x\| = \max(\|x_1\|, \|x_2\|)$. Define the linear operator $L : \mathscr{X} \to \mathscr{X}$ by

$$Lx = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{pmatrix} L_{11}x_1 + L_{12}x_2 \\ L_{21}x_1 + L_{22}x_2 \end{pmatrix}. \tag{3.3}$$

Then L is bounded and

$$\|L\| \leqslant \max_{j=1,2} (\|L_{j1}\| + \|L_{j2}\|). \tag{3.4}$$

Moreover, L is compact.

Define $F : \mathscr{X} \to \mathscr{X}$ by (cf. (2.7))

$$Fx = (x_1x_2, \tfrac{1}{2}x_1^2). \tag{3.5}$$

Easy calculations yield

$$\|Fx\| \leqslant \|x\|^2, \tag{3.6}$$

$$\|Fx - Fy\| \leqslant (\|x\| + \|y\|)\|x - y\|. \tag{3.7}$$

By (3.6), F is ***bounded*** (it transforms bounded sets into bounded sets). By (3.7), F is continuous (moreover, uniformly continuous on each bounded subset of $\mathscr{X}$).

[2] In Section 2, x was used for $x_1 + ix_2$. It will not be used in that sense again.

Let I denote the identity operator on $\mathscr{X}$.

Let $\mathscr{R}$ denote the Banach space of real numbers with the absolute value for the norm. Consider $\mathscr{R} \times \mathscr{X}$ as a Banach space with elements (γ, x) and norm $\|(\gamma, x)\| = \max(|\gamma|, \|x\|)$. Define the nonlinear operator $P : \mathscr{R} \times \mathscr{X} \to \mathscr{X}$ by

$$P(\gamma, x) : = (I - \gamma LF)x - g, \tag{3.8}$$

where $g = (g_1, g_2)$ is determined by (2.8) and (2.9). Then the desired abstract formulation of (2.7) is

$$P(\gamma, x): = (I - \gamma LF)x - g = 0. \tag{3.9}$$

4. Application of the Contractive Mapping and Schauder Fixed Point Theorems

A. Define the operator $A : \mathscr{R} + \mathscr{X} \to \mathscr{X}$ by

$$A(\gamma, x): = g + \gamma LFx. \tag{4.1}$$

Then (3.9) is equivalent to

$$A(\gamma, x): = g + \gamma LFx = x. \tag{4.2}$$

Thus, for γ fixed, solutions x of (3.9) are fixed points of $A(\gamma, x)$.

Let $\bar{\mathscr{B}}_r (r \geqslant 0)$ denote the closed ball in $\mathscr{X}$ with center $x = 0$ and radius r : $\bar{\mathscr{B}}_r = \{x \in \mathscr{X} : \|x\| \leqslant r\}$.

Lemma 4.1. *Let $r \geqslant \|g\|$. Then $A(\gamma, x) \in \bar{\mathscr{B}}_r$ for all $x \in \bar{\mathscr{B}}_r$ if*

$$0 \leqslant \gamma \leqslant \frac{r - \|g\|}{\|L\| r^2}. \tag{4.3}$$

Proof. Apply (3.6) and (4.1).

Lemma 4.2. *Let $r > 0$ and fix γ such that*

$$0 \leqslant \gamma < \frac{1}{2r\|L\|}. \tag{4.4}$$

Then for $x, y \in \bar{\mathscr{B}}_r$

$$\|A(\gamma, x) - A(\gamma, y)\| < 2\gamma r \|L\| \|x - y\| \tag{4.5}$$

with $2\gamma r\|L\| < 1$, so that $A(\gamma, x)$ represents a contractive *operator on $\bar{\mathscr{B}}_r$.*

Proof. Apply (3.7) and (4.1).

We can now apply the *contractive mapping principle:* if a contractive operator T maps a complete metric space M into itself, then T has a unique fixed point $x^* \in M$; moreover, the sequence defined by $x_n = Tx_{n-1}$, with any $x_0 \in M$, converges to x^*.

Theorem 4.1. *Let* $r > 0$. *If* γ *satisfies* (4.3) *and* (4.4), *then there is a unique* $x \in \bar{\mathscr{B}}_r$ *such that* $A(\gamma, x) = x$ *and, hence,* $P(\gamma, x) = 0$. *Moreover, if* $x_n = A(\gamma, x_{n-1})$ *and* $x_0 \in \bar{\mathscr{B}}_r$, *then* $\| x_n - x \| \to 0$ *as* $n \to \infty$.

What are the limitations of this result? The right member of (4.3) attains the maximum value $1/4\|L\|\|g\|$ for $r = 2\|g\|$. The right member of (4.4) attains the same value for $r = 2\|g\|$. Hence, Theorem 4.1 is applicable in some ball $\bar{\mathscr{B}}_r$ only for $\gamma < 1/4\|L\|\|g\|$.

B. The Schauder fixed point theorem [5] yields another existence result. Consider (4.1) for $A(\gamma, x)$. Since L and F are continuous, LF is continuous. Since L is compact and F is bounded, LF is compact. Hence, LF is *completely continuous*, i.e., continuous and compact. It follows that, for each fixed γ, $A(\gamma, x)$ represents a completely continuous operator on $\mathscr{X}$.

The Schauder theorem asserts that if a completely continuous operator in a Banach space maps a closed, bounded, convex set into itself, then the operator has at least one fixed point in the set. Note that for each $r \geqslant 0$, $\bar{\mathscr{B}}_r$ is a closed bounded convex set. Hence, by Lemma 4.1, we have the following result.

Theorem 4.2. *Let* $r > 0$. *If* γ *satisfies* (4.3), *then there is at least one* $x \in \bar{\mathscr{B}}_r$ *such that* $A(\gamma, x) = x$ *and, hence,* $P(\gamma, x) = 0$.

This theorem applies only for $\gamma \leqslant 1/(4\|L\|\|g\|)$. Thus, the applications of both the contractive mapping and Schauder theorems are limited to γ sufficiently small. The methods given below do not have this limitation.

PART III

5. Implicit Function Theorem and Newton's Method

A. For the present, let $\mathscr{X}$ and $\mathscr{Y}$ be arbitrary Banach spaces. Let $\mathscr{L}(\mathscr{X}; \mathscr{Y})$ be the Banach space of bounded (continuous) linear operators on $\mathscr{X}$ into $\mathscr{Y}$. An operator T on an open subset $\mathscr{S}$ of $\mathscr{X}$ into $\mathscr{Y}$ is said to be (Fréchet) *differentiable* at x_0 in $\mathscr{S}$ if there is an operator $T_x(x_0) \in \mathscr{L}(\mathscr{X}; \mathscr{Y})$ such that

$$\lim_{\|h\| \to 0} \frac{1}{\|h\|} \| T(x_0 + h) - Tx_0 - T_x(x_0)h \| = 0.$$

The operator $T_x(x_0)$ is called the (Fréchet) *derivative* of T at x_0. It is the appropriate local linear approximation of T (the Jacobian matrix in Euclidean n-space).

The operator T is differentiable *on* a set $\mathscr{S}_1$ if it is differentiable at each point of $\mathscr{S}_1$; T is *continuously* differentiable on $\mathscr{S}_1$ if the map $T_x : \mathscr{S}_1 \to \mathscr{L}(\mathscr{X}; \mathscr{Y})$ is continuous at each point of $\mathscr{S}_1$. If T_x is differentiable at x_0 with derivative $T_{xx}(x_0) \in \mathscr{L}(\mathscr{X}; \mathscr{L}(\mathscr{X}; \mathscr{Y}))$, then T is *twice differentiable* at x_0 with second derivative $T_{xx}(x_0)$. Thus, for $h \in \mathscr{X}$, $T_{xx}(x_0)h \in \mathscr{L}(\mathscr{X}; \mathscr{Y})$ and so acts on elements $k \in \mathscr{X} : T_{xx}(x_0)hk \in \mathscr{Y}$. The norm of the second derivative is that in $\mathscr{L}(\mathscr{X}; \mathscr{L}(\mathscr{X}; \mathscr{Y}))$. Therefore

$$\| T_{xx}(x_0)hk \| \leqslant \| T_{xx}(x_0)h \| \| k \| \leqslant \| T_{xx}(x_0) \| \| h \| \| k \|.$$

In a natural manner $T_{xx}(x_0)$ may be viewed as an element of $\mathscr{L}(\mathscr{X}, \mathscr{X}; \mathscr{Y})$, the space of continuous bilinear maps from $\mathscr{X} \times \mathscr{X}$ into $\mathscr{Y}$. Higher order derivatives are defined analogously.

We note the following properties of the derivative.

(i) If T is linear, then $T_x(x_0) = T$ and $T_{xx}(x_0) = 0$ for all $x_0 \in \mathscr{X}$.

(ii) If $T = U + V$, then $T_x(x_0) = U_x(x_0) + V_x(x_0)$.

(iii) If $T = VU$ where $U : \mathscr{X} \to \mathscr{Y}$ and $V : \mathscr{Y} \to \mathscr{Z}$, then

$$T_x(x_0) = V_y(Ux_0)U_x(x_0).$$

(iv) If T is continuously differentiable on the segment $\{x = x_0 + th, 0 \leqslant t \leqslant 1\}$, then

$$\| T(x_0 + h) - Tx_0 \| \leqslant \sup_{0 \leqslant t \leqslant 1} \| T_x(x_0 + th) \| \| h \|.$$

B. Now consider an operator T defined on an open subset of $\mathscr{R} \times \mathscr{X}$ with values $T(\alpha, x)$ in a Banach space $\mathscr{Y}$. The *partial* (Fréchet) *derivatives* $T_\alpha(\alpha_0, x_0)$ and $T_x(\alpha_0, x_0)$ are defined in the usual manner by fixing x and α, respectively.

The Hildebrandt-Graves implicit function theorem is as follows [6, 7].

THEOREM 5.1. *Suppose* $T(\alpha_0, x_0) = 0$, T *is continuously differentiable with respect to* x *at* (α_0, x_0), *and* $[T_x(\alpha_0, x_0)]^{-1}$ *exists.*[3] *Then there are constants* $\epsilon > 0$ *and* $\delta > 0$ *such that for* $| \alpha - \alpha_0 | < \epsilon$ *the equation* $T(\alpha, x) = 0$ *has a unique solution* $x = w(\alpha)$ *with* $\| w(\alpha) - x_0 \| < \epsilon$. *Moreover,* $w(\alpha)$ *is continuous.*

[3] Here and elsewhere, it is understood that the inverse operator is defined on all of the range space. The phrase, "$T_x(\alpha_0, x_0)$ is nonsingular" will have the same meaning.

C. Again, consider an operator $T : \mathscr{X} \to \mathscr{Y}$, (e.g., $T(\alpha, x)$ with α fixed, as above). The Newton-Kantorovič method for solving

$$Tx = 0 \tag{5.1}$$

is based on the iterative scheme

$$x_{n+1} = x_n - [T_x(x_n)]^{-1} T x_n , \tag{5.2}$$

where x_0 is a given initial approximation. Let $\mathscr{B}(x_0 , r) = \{x : \| x - x_0 \| < r\}$ and $\bar{\mathscr{B}}(x_0 , r) = \{x : \| x - x_0 \| \leqslant r\}$. The fundamental theorem is as follows [8–10].

THEOREM 5.2. *Suppose that*

(1) T *is twice differentiable and* $\| T_{xx}(x) \| \leqslant \kappa$ *on* $\mathscr{B}(x_0 , r_0)$, *where* r_0 *is defined by* (5.3);

(2) $\Gamma_0 : = [T_x(x_0)]^{-1}$ *exists and* $\| \Gamma_0 \| \leqslant \beta_0$;

(3) $\| x_1 - x_0 \| = \| \Gamma_0 T x_0 \| \leqslant \eta_0$;

(4) $\tau_0 : = \beta_0 \eta_0 \kappa \leqslant \frac{1}{2}$;

where

$$r_0 : = N(\tau_0)\eta_0 , \qquad N(\tau) : = \frac{1 - \sqrt{1 - 2\tau}}{\tau} . \tag{5.3}$$

Then $Tx = 0$ *has a solution* x^* *in* $\bar{\mathscr{B}}(x_0 , r_0)$, (5.2) *defines a sequence* $\{x_n\}$ *which converges to* x^*, *and*

$$\| x_n - x^* \| \leqslant \frac{1}{2^{n-1}} (2h_0)^{2^{n-1}} \eta_0 . \tag{5.4}$$

If (1) *holds in* $\mathscr{B}(x_0 , q_0)$, *where*

$$q_0 : = M(\tau_0)\eta_0 , \qquad M(\tau) : = \frac{1 + \sqrt{1 - 2\tau}}{\tau} , \tag{5.5}$$

then the solution x^* *is unique in* $\mathscr{B}(x_0 , q_0)$.

Remarks. 1°. If $\| Tx_0 \| \leqslant \zeta_0$, then (3) holds with $\eta_0 \leqslant \beta_0 \zeta_0$ and (4) holds if $\tau_0' : = \beta_0^2 \zeta_0 \kappa \leqslant \frac{1}{2}$.

2°. By (5.3), $N(\tau_0) \leqslant 2$ and $r_0 \leqslant 2\eta_0$ if $\tau_0 \leqslant \frac{1}{2}$. Hence by (4) it suffices to require (1) on $\mathscr{B}(x_0 , 2\eta_0)$.

3°. In $0 < \tau \leqslant \frac{1}{2}$, $N(\tau)$ is an increasing and $M(\tau)$ a decreasing function of τ.

4°. Suppose $\tau_0 < \frac{1}{2}$. From $\| x^* - x_0 \| \leqslant r_0 = (1 - \sqrt{1 - 2\tau_0})/\beta_0\kappa$, we have by (iv) in Section 5, A

$$\| T_x(x^*) - T_x(x_0)\| \leqslant \kappa r_0 = \frac{1 - \sqrt{1 - 2\tau_0}}{\beta_0} < \frac{1}{\beta_0} = \|[T_x(x_0)]^{-1}\|^{-1}$$

which implies that $[T_x(x^*)]^{-1}$ exists and

$$\|[T_x(x^*)]^{-1}\| \leqslant \frac{\beta_0}{1 - [(1 - \sqrt{1 - 2\tau_0})/\beta_0]\beta_0} = \frac{\beta_0}{\sqrt{1 - 2\tau_0}} =: \beta^*. \quad (5.6)$$

6. Derivative Operators and Their Norms. Change of Parameter

A. The implicit function theorem and Newton's method will be applied to solve $P(\gamma, x) := (I - \gamma LF)x - g = 0$. By (3.5),

$$F_x(x_0) = \begin{pmatrix} x_{0,2} & x_{0,1} \\ x_{0,1} & 0 \end{pmatrix}, \qquad \text{where} \quad x_0 = \begin{pmatrix} x_{0,1} \\ x_{0,2} \end{pmatrix}, \quad (6.1)$$

(This is an ordinary Jacobian matrix.) Thus,

$$F_x(x_0)h = \begin{pmatrix} x_{0,2} & x_{0,1} \\ x_{0,1} & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} x_{0,2}h_1 + x_{0,1}h_2 \\ x_{0,1}h_2 \end{pmatrix}. \quad (6.2)$$

It is not difficult to show that

$$\| F_x(x_0)\| = 2\| x_0 \|. \quad (6.3)$$

By (6.1), the map $F_x : \mathscr{X} \to \mathscr{L}(\mathscr{X}; \mathscr{X})$ is linear. Therefore, its derivative $F_{xx}(x_0)$, which is in $\mathscr{L}(\mathscr{X}; \mathscr{L}(\mathscr{X}; \mathscr{X}))$, is constant (independent of x_0):

$$F_{xx}(x_0)h = F_x(h) = \begin{pmatrix} h_2 & h_1 \\ h_1 & 0 \end{pmatrix}. \quad (6.4)$$

By (6.3) and (6.4),

$$\| F_{xx}(x_0)\| = 2. \quad (6.5)$$

For $P(\gamma, x) := (I - \gamma LF)x - g$, we obtain

$$\begin{aligned} P_x(\gamma_0, x_0) &= I - \gamma_0 LF_x(x_0), & P_{xx}(\gamma_0, x_0) &= -\gamma_0 LF_{xx}(x_0), \\ P_\gamma(\gamma_0, x_0) &= -LFx_0, & P_{x\gamma}(\gamma_0, x_0) &= -LF_x(x_0). \end{aligned} \quad (6.6)$$

In particular, for $\gamma_0 \geqslant 0$,

$$\| P_{xx}(\gamma_0, x_0)\| \leqslant 2\gamma_0\| L \|. \quad (6.7)$$

This furnishes a value for κ in condition (1) of Theorem 5.1 which is independent of x_0 and r_0. Corresponding estimates can be obtained for the norms of the other derivatives in (6.6).

B. Both the implicit function theorem and Newton's method fail at points on the solution curve where $P_x(\gamma_0, x_0)$ is singular. To deal with this situation, we employ a suitable change of parameter

$$\gamma = \gamma(\rho, x) \tag{6.8}$$

defined on some open set in $\mathscr{R} \times \mathscr{X}$. Let

$$Q(\rho, x) := P(\gamma(\rho, x), x) \tag{6.9}$$

on this set. If $\gamma(\rho, x)$ is differentiable with respect to x at (ρ_0, x_0) and $\gamma_0 = \gamma(\rho_0, x_0)$, then

$$Q_x(\rho_0, x_0) = P_x(\gamma_0, x_0) + P_\gamma(\gamma_0, x_0)\gamma_x(\rho_0, x_0), \tag{6.10}$$

where $\gamma_x(\rho_0, x_0) \in \mathscr{X}^* := \mathscr{L}(\mathscr{X}; \mathscr{R})$ and $P_\gamma(\gamma_0, x_0) \in \mathscr{L}(\mathscr{R}; \mathscr{X})$, which is isometrically, isomorphic to $\mathscr{X}$ (an operator $Y \in \mathscr{L}(\mathscr{R}; \mathscr{X})$ corresponds to an element $y \in \mathscr{X}$ if and only if $Y\alpha = \alpha y$ for all $\alpha \in \mathscr{R}$). Identifying $\mathscr{L}(\mathscr{R}; \mathscr{X})$ with $\mathscr{X}$, we have $P_\gamma(\gamma_0, x_0) \in \mathscr{X}$. Thus, $Q_x(\rho_0, x_0)$ is an operator in $\mathscr{L}(\mathscr{X}; \mathscr{X})$ of the form $B + y\varphi$, where $B \in \mathscr{L}(\mathscr{X}; \mathscr{X})$, $y \in \mathscr{X}$ and $\varphi \in \mathscr{X}^*$.

It may happen that $Q_x(\rho_0, x_0)$ is nonsingular even though $P_x(\gamma_0, x_0)$ is singular. Thus, the change of parameter may permit the further use of Theorems 5.1 and 5.2. This situation is examined in Section 6, C. But first, a special case is considered.

In the buckling problem, (1.2) was used to define the change of parameter

$$\gamma(\rho, x) := \rho c(x), \tag{6.11}$$

$$c(x) = [b(x)]^{-1}, \qquad b(x) = \int_0^1 x_1(t)t^2\, dt, \tag{6.12}$$

for all $\rho \in \mathscr{R}$ and $x = (x_1, x_2) \in \mathscr{X}$ such that $b(x) \neq 0$. Then

$$Q(\rho, x) := P(\gamma(\rho, x), x) = x - \rho c(x)LFx - g \tag{6.13}$$

on this set. Assume $b(x_0) \neq 0$. Since b is a linear functional

$$b_x(x_0) = b, \qquad b_{xx}(x_0) = 0, \tag{6.14}$$

$$c_x(x_0) = -[c(x_0)]^2 b, \qquad c_{xx}(x_0)hk = 2[c(x_0)]^3 b(k)b(h). \tag{6.15}$$

Therefore,

$$Q_x(\rho_0, x_0)h = [I - \rho_0 c(x_0)LF_x(x_0)]h + \rho_0[c(x_0)]^2 b(h)LFx_0, \quad (6.16)$$

$$Q_\rho(\rho_0, x_0) = -c(x_0)LFx_0, \quad (6.17)$$

$$Q_{xx}(\rho_0, x_0)hk = -\rho_0 c(x_0)LF_{xx}(x_0)hk + \rho_0[c(x_0)]^2 b(k)LF_x(x_0)h + \rho_0[c(x_0)]^2 b(h)LF_x(x_0)k - 2\rho_0[c(x_0)]^3 b(k)b(h)LFx_0. \quad (6.18)$$

By (6.12), $\| b \| := \sup_{\|x\|=1} |b(x)| = \frac{1}{3}$. Hence, by (3.6), (6.3) and (6.5),

$$\| Q_{xx}(\rho_0, x_0)\| \leqslant |\rho_0 c(x_0)| \| L \| \{\| F_{xx}(x_0)\| + 2|c(x_0)| \| b \| \| F_x(x_0)\| + 2|c(x_0)|^2 \| b \|^2 \| Fx_0 \|\},$$

$$\| Q_{xx}(\rho_0, x_0)\| \leqslant 2|\rho_0 c(x_0)| \| L \| \{1 + \tfrac{2}{3}|c(x_0)| \| x_0 \| + \tfrac{1}{9}|c(x_0)|^2 \| x_0 \|^2\},$$

$$\| Q_{xx}(\rho_0, x_0)\| \leqslant 2|\rho_0 c(x_0)| \| L \| \{1 + \tfrac{1}{3}|c(x_0)| \| x_0 \|\}^2. \quad (6.19)$$

C. Suppose that $P_x(\gamma_0, x_0)$ is singular. We seek changes of variable (6.8) and (6.9) such that the operator $Q_x(\rho_0, x_0)$ given by (6.10) is nonsingular. Note that $P_\gamma(\gamma_0, x_0)$ is independent of $\gamma(\rho, x)$. Note also that for any $\varphi \in \mathscr{X}^*$ there exists $\gamma(\rho, x)$ such that $\gamma_x(\rho_0, x_0) = \varphi$, e.g., let $\gamma(\rho, x) := (\rho - \rho_0 + 1)\varphi(x)$.

The addition of an operator to a singular operator to make the sum nonsingular is a fairly common device in linear algebra and functional analysis. For example, it is used to reduce the second case of the Fredholm alternative to the first case. Other examples occur in connection with bordered matrices.

Before stating the results, some additional notation is introduced. For each $x \in \mathscr{X}$, let $[x] := \{\alpha x : \alpha \in \mathscr{R}\}$. For each $B \in \mathscr{L}(\mathscr{X}, \mathscr{X})$, denote its null space by $\mathscr{N}[B]$ and its range by $B\mathscr{X}$. Recall that the codimension of a subspace $\mathscr{X}_1$ of $\mathscr{X}$ is defined (uniquely) by codim. $[\mathscr{X}_1]$ = dim. $[\mathscr{X}_2]$, where $\mathscr{X}_2$ is any subspace of $\mathscr{X}$ such that

$$\mathscr{X} = \mathscr{X}_1 + \mathscr{X}_2, \qquad \mathscr{X}_1 \cap \mathscr{X}_2 = [0].$$

LEMMA 6.1. *Assume $P_x(\gamma_0, x_0)$ is singular. Then there exist changes of variable* (6.8) *and* (6.9) *such that $Q_x(\rho_0, x_0)$ is nonsingular if and only if*

(1) dim $\mathscr{N}[P_x(\gamma_0, x_0)] = 1$, i.e., *for some nonzero $h_0 \in \mathscr{X}$,* $\mathscr{N}[P_x(\gamma_0, x_0)] = [h_0]$;

(2) codim.$[P_x(\gamma_0, x_0)\mathscr{X}] = 1$;

(3) $P_\gamma(\gamma_0, x_0) \notin P_x(\gamma_0, x_0)\mathscr{X}$.

Under these conditions, $Q_x(\rho_0, x_0)$ is nonsingular if and only if

(4) $\gamma_x(\rho_0, x_0)h_0 \neq 0$.

Since the proof is not difficult, it is omitted.

In the buckling problem, (6.6) gives $P_x(\gamma_0, x_0) = I - \gamma LF_x(x_0)$, where $LF_x(x_0)$ is compact. Therefore, (1) and (2) of Lemma 6.1 are equivalent to each other and to the condition that γ_0 is a simple characteristic value (reciprocal eigenvalue) of $LF_x(x_0)$. In (1), h_0 is an associated eigenvector. Condition (3) now becomes

$$LFx_0 \notin [I - \gamma_0 LF_x(x_0)]\mathscr{X}. \tag{6.20}$$

If $\gamma(\rho, x)$ is defined by (6.11), then (4) is equivalent to

$$\rho_0 \neq 0, \qquad b(x_0) \neq 0, \qquad b(h_0) \neq 0. \tag{6.21}$$

7. A Global Curve of Solutions by Continuation with Newton's Method

A. In this section we seek to obtain the curve discussed in the Introduction. Heuristically, let us suppose that the solutions of $P(\gamma, x) = 0$, and of the equivalent $Q(\rho, x) = 0$, together form a one-dimensional manifold $\mathscr{G}_0$ given locally by suitable functions $x = u(\gamma)$ or $x = v(\rho)$. Let x_0 be an approximate solution for $\gamma = \gamma_0$. (The statements below will have obvious analogs for $\rho = \rho_0$.) In general, x_0 will be the finally accepted approximation to $u(\gamma_0)$ obtained by Newton's method applied to $P(\gamma_0, x) = 0$. We wish to apply Newton's method to $P(\gamma_1, x) = 0$. As an initial approximation for $u(\gamma_1)$ it is natural to consider

$$x_0 + u_\gamma(\gamma_0)(\gamma_1 - \gamma_0) \tag{7.1}$$

where u_γ is obtained by differentiating $P(\gamma, u(\gamma)) \equiv 0$ and solving:

$$u_\gamma(\gamma_0) = -[P_x(\gamma_0, u(\gamma_0))]^{-1} P_\gamma(\gamma_0, u(\gamma_0)). \tag{7.2}$$

Since x_0 is an approximation to $u(\gamma_0)$, in practice one uses in (7.2)

$$G_0 := [P_x(\gamma_0, x_0)]^{-1}$$

which is available from the Newton method computations at $\gamma = \gamma_0$. Thus, as an initial approximation for $u(\gamma_1)$, we take instead of (7.1)

$$x_0 - G_0 P_\gamma(\gamma_0, x_0)(\gamma_1 - \gamma_0). \tag{7.3}$$

Newton's method then yields a sequence of points which one hopes will converge to a solution of $P(\gamma_1, x) = 0$.

For the case of $P(\gamma, x)$ a second order initial approximation is easily available (assuming sufficient differentiability). By (6.6)

$$P_x(\gamma, u(\gamma))u_\gamma(\gamma) \equiv -P_\gamma(\gamma, u(\gamma)) \equiv LFu(\gamma).$$

Differentiate with respect to γ and use $F_{xx}(x)h = F_x(h)$ to obtain

$$\begin{aligned} P_x(\gamma, u(\gamma))u_{\gamma\gamma}(\gamma) &= -[P_{x\gamma}(\gamma, u(\gamma)) + P_{xx}(\gamma, u(\gamma))u_\gamma(\gamma)]u_\gamma(\gamma) + LF_x(u(\gamma))u_\gamma(\gamma) \\ &= [LF_x(u(\gamma)) + \gamma LF_{xx}(u(\gamma))u_\gamma(\gamma)]u_\gamma(\gamma) + LF_x(u(\gamma))u_\gamma(\gamma) \\ &= [2LF_x(u(\gamma)) + \gamma LF_x(u_\gamma(\gamma))]u_\gamma(\gamma). \end{aligned}$$

This can be solved for $u_{\gamma\gamma}$. As a initial approximation for $u(\gamma_1)$ we then may take

$$x_0 + u_\gamma(\gamma_0)(\gamma_1 - \gamma_0) + \tfrac{1}{2}u_{\gamma\gamma}(\gamma_0)(\gamma_1 - \gamma_0)^2. \tag{7.4}$$

Higher order extrapolations can be obtained similarly.

B. We find in the next theorem that if the initial approximation (7.3) is used in the case $P(\gamma, x) = 0$ then the Newton iterates do converge if the increment in γ is small enough. We remark however that a much larger step than that given in Theorem 7.1 may work, especially if a higher order extrapolation is used, e.g. (7.4).

The continuous curve of solutions thus obtained by varying γ continuously is approximated by linear segments, with error estimates and uniqueness regions given by this theorem.

THEOREM 7.1. *For $\gamma = \gamma_0$, let x_0 be an approximate solution of $P(\gamma_0, x) = 0$ such that $[P_x(\gamma_0, x_0)]^{-1}$ exists and the $\kappa_0, \beta_0, \zeta_0$ of Theorem 5.2 satisfy $\tau_0' := \beta_0^2\zeta_0\kappa_0 < \frac{1}{2}$ with $\kappa_0 := 2\gamma_0\ell$ where $\|L\| \leqslant \ell$ (cf. Remark 1° after Theorem 5.2). Let the quantities d, e, f be defined by (7.10), (7.12), (7.14) below. Suppose $0 < \delta < 1/\beta_0 d$ and*

$$\tau_{10}' := \frac{\tau_0' + f\delta}{(1 - \beta_0 d\delta)^2} < \frac{1}{2}. \tag{7.5}$$

If Newton's method is applied for γ with $|\gamma - \gamma_0| < \delta$ using as initial approximation

$$x_0(\gamma) := x_0 - G_0P_\gamma(\gamma_0, x_0)(\gamma - \gamma_0), \quad G_0 := [P_x(\gamma_0, x_0)]^{-1}, \tag{7.6}$$

then the Newton iterates at γ converge to a solution $x = u(\gamma)$. Moreover, these $u(\gamma)$ form a continuous arc of solutions, $P(\gamma, u(\gamma)) \equiv 0$; this arc is approximated by the linear segment $x_0(\gamma)$ defined by (7.6); the error of approximation is uniformly bounded by

$$\|u(\gamma) - x_0(\gamma)\| \leqslant \frac{1}{\beta_{10}\kappa_{10}}(1 - \sqrt{1 - 2\tau_{10}'}) \tag{7.7}$$

where β_{10} and κ_{10} are defined by (7.11) *and* (7.13); *and the $u(\gamma)$ are unique in the balls $\mathscr{B}(x_0(\gamma), r)$ of uniform radius*

$$r = \frac{1}{\beta_{10}\kappa_{10}}(1 + \sqrt{1 - 2\tau'_{10}}). \tag{7.8}$$

PROOF. With γ fixed, for brevity let $x_{10} := x_0(\gamma)$ in (7.6). We show that the conditions of Theorem 5.2 (and Remark 1° following it) hold for (γ, x_{10}) with $\beta_{10}^2\zeta_{10}\kappa_{10} < \frac{1}{2}$. First, for β_{10}, recall that if S is an invertible operator and $\| R - S \| < 1/\| S^{-1} \|$ then R is invertible and

$$\| R^{-1} \| \leqslant \| S^{-1} \|/(1 - \| S^{-1} \| \, \| R - S \|).$$

From (7.6),

$$\| x_{10} - x_0 \| \leqslant \| G_0 \| \, \| P_\gamma(\gamma_0, x_0) \| \, | \gamma - \gamma_0 | \leqslant \beta_0\ell \| x_0 \|^2\delta, \tag{7.9}$$

$$\| x_{10} \| \leqslant (1 + \beta_0\ell \| x_0 \|\delta) \| x_0 \|.$$

Using (iv) of Section 5, A together with the bounds in Section 6, A and (3.6) we have

$$\begin{aligned} \| P_x(\gamma, x_{10}) - P_x(\gamma_0, x_0) \| &\leqslant \| P_x(\gamma, x_{10}) - P_x(\gamma_0, x_{10}) \| \\ &\quad + \| P_x(\gamma_0, x_{10}) - P_x(\gamma_0, x_{10}) \| \\ &\leqslant \sup_{0\leqslant t\leqslant 1} \| P_{x\gamma}(\gamma_0 + t(\gamma - \gamma_0), x_{10}) \| \, | \gamma - \gamma_0 | \\ &\quad + \sup_{0\leqslant t\leqslant 1} \| P_{xx}(\gamma_0, x_0 + t(x_{10} - x_0)) \| \, \| x_{10} - x_0 \| \\ &\leqslant 2\ell \| x_{10} \| \, \delta + \kappa_0\beta_0\ell \| x_0 \|^2 \, \delta \leqslant d\delta \end{aligned}$$

where

$$d := [2(1 + \beta_0\ell \| x_0 \|\delta) + \beta_0\kappa_0 \| x_0 \|]\ell \| x_0 \|. \tag{7.10}$$

Thus, for $\delta < 1/\beta_0 d$, $P_x(\gamma, x_{10})$ has an inverse G_{10} and

$$\| G_{10} \| \leqslant \beta_{10} := \frac{\beta_0}{1 - \beta_0 d\delta}. \tag{7.11}$$

Similarly, for ζ_{10} we have

$$\begin{aligned} \| P(\gamma, x_{10}) \| &\leqslant \sup_{0\leqslant t\leqslant 1} \| P_x(\gamma, x_0 + t(x_{10} - x_0)) \| \, \| x_{10} - x_0 \| \\ &\quad + \sup_{0\leqslant t\leqslant 1} \| P_\gamma(\gamma_0 + t(\gamma - \gamma_0), x_0) \| \, \delta + \| P(\gamma_0, x_0) \| \\ &\leqslant [1 + 2\gamma\ell(\| x_0 \| + \| x_{10} - x_0 \|)] \| x_{10} - x_0 \| + \ell \| x_0 \|^2 \, \delta + \zeta_0 \\ &\leqslant \zeta_0 + e\,\delta =: \zeta_{10} \end{aligned}$$

where

$$e := \{1 + [1 + 2(\gamma_0 + \delta)\ell \| x_0 \|(1 + \beta_0\ell \| x_0 \|\delta)]\beta_0\}\ell \| x_0 \|^2. \tag{7.12}$$

For κ_{10}, we note $\| P_{xx}(\gamma_0, x_0)\| \leqslant 2\gamma_0\ell = :\kappa_0$ independent of x, so that

$$\kappa_{10} : = 2(\gamma_0 + \delta)\ell = \kappa_0 + 2\ell\delta \tag{7.13}$$

will serve to bound $P_{xx}(\gamma, x)$ for $\gamma \leqslant \gamma_0 + \delta$ and for all x.

Thus, letting

$$f : = \beta_0^2(2\zeta_0\ell + e\kappa_0 + 2e\ell\delta), \tag{7.14}$$

we have

$$\beta_{10}^2\zeta_{10}\kappa_{10} \leqslant \left(\frac{\beta_0}{1 - \beta_0 d\delta}\right)^2 (\zeta_0 + e\,\delta)(\kappa_0 + 2\ell\,\delta) \leqslant \frac{\beta_0^2\zeta_0\kappa_0 + f\,\delta}{(1 - \beta_0 d\delta)^2} = : \tau_{10}' . \tag{7.15}$$

This yields the convergence part of the theorem by (7.5).

The estimates (7.7) and (7.8) are applications of (5.3), (5.5), and Remark 3° following Theorem 5.2. Finally, Remark 4° implies $[P_x(\gamma, u(\gamma))]^{-1}$ exists so that, in view of the uniqueness, $u(\gamma)$ is continuous by the implicit function theorem. q.e.d.

Remarks. It is clear that for $\| x \| \leqslant B$ and $\gamma \leqslant C$ the quantities d, e, f as well as $P_{xx}(\gamma, x)$ can be bounded uniformly in terms of β_0, ℓ, B, C. Consequently, if the $P_x(\gamma, u(\gamma))$ have uniformly bounded inverses (uniform β_0) for $0 \leqslant \gamma \leqslant C$, then a uniform step δ can be employed. (Cf. also [11].) Conversely, as $P_x(\gamma, u(\gamma))$ becomes singular, the steps in γ become smaller. Indeed, for successive $\gamma_0, \gamma_1, \gamma_2, \ldots$ the steps may decrease so rapidly that $\sum_j (\gamma_{j+1} - \gamma_j)$ is less than the desired upper limit of γ. (Something like this is to be expected in view of the buckling behavior.)

C. It is not hard to see that using Section 7, B, essentially the same results can be derived for $Q(\rho, x)$: If x_0 is an approximate solution for $\rho = \rho_0$ such that $[Q_x(\rho_0, x_0)]^{-1}$ exists and $b(x_0) \neq 0$ (cf. (6.12)), then with a sufficiently small increment in ρ, an extrapolation analogous to (7.3) starts a Newton sequence converging to a solution $v(\rho)$; the $v(\rho)$ form a continuous arc approximated by a linear segment and surrounded by balls of uniqueness with uniform radius. (One may argue on physical grounds that, in view of (1.2), $b(x) \neq 0$ for any displacements of interest.)

D. Now view the points (γ, x) as lying in the product space $\mathscr{X} : = \mathscr{R} \times \mathscr{X}$. Let $\mathscr{S}$ denote the set of points $(\gamma, x) \in \mathscr{X}$ satisfying $P(\gamma, x) = 0$. By the foregoing $\mathscr{S}$ contains a continuous curve $\mathscr{G}$ through $(0, g)$, defined locally by $(\gamma, u(\gamma))$, which can be approximated by a stepwise procedure beginning at $(0, g)$. The balls of uniqueness in Theorem 7.1 then appear as horizontal sections of a "tube" of uniqueness surrounding $\mathscr{G}$. See Fig. 2. If as γ approaches some γ_c, $u(\gamma)$ approaches u_c and $P_x(\gamma, u(\gamma))$ becomes singular (so $\beta_0 \to \infty$), then the tube constricts to zero diameter. But if $Q_x(\rho, x)$ is

not singular near such "critical points", then $\mathcal{G}$ can be continued, now locally defined by $(\rho, v(\rho))$ as in Section 7, C, so (γ_c, u_c) is obtained from $(\rho_c, v(\rho_c))$ for some ρ_c. Also, $\mathcal{G}$ has further stepwise linear approximations, and the

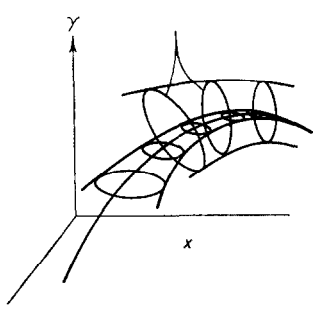

FIG. 2.

corresponding balls of uniqueness (now "bent disks" on the hypersurfaces of constant ρ in $\mathcal{Z}$) comprise a new tube of uniqueness about $\mathcal{G}$ in the very region where the previous tube constricted to zero radius. With ρ as independent parameter, one computes the corresponding $\gamma = \gamma(\rho, v(\rho))$ from (6.11). If $(\rho, v(\rho))$ approaches a point where $Q_x(\rho, x)$ becomes singular, then one returns to γ as independent parameter (providing $P_x(\gamma, u(\gamma))$ is not simultaneously nearly singular). In so doing the steps in γ are taken consistent with the current sign of $d\gamma/d\rho$ (actually one decrements or increments according as the successive computed values of γ were decreasing or increasing).

Alternate use of γ and ρ as needed may yield a continuous global curve $\mathcal{G}$ of solutions in $\mathcal{Z}$ such that along $\mathcal{G}$, γ passes continuously (but not necessarily monotonically) from 0 to ∞. One obtains with $\mathcal{G}$ a tube of uniqueness surrounding it. In particular this precludes geometrical branch points (as in Fig. 3). Furthermore, no other branch of $\mathcal{S}$ can pass through $(0, g)$; in fact, other branches must tend to infinity as $\gamma \to 0$ since in Theorem 4.1 the smaller γ is, the larger the balls of uniqueness $\mathcal{B}_r$ may be taken.

We do not investigate here the conditions under which $\mathcal{G}$ constitutes all

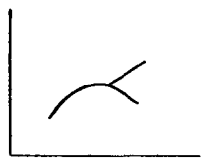

FIG. 3.

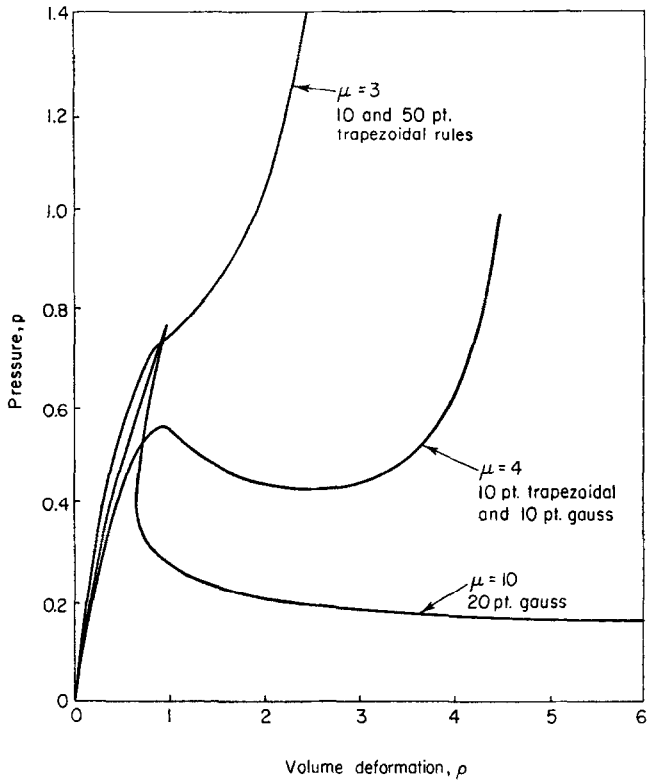

FIG. 4. Pressure vs. volume deformation, $\nu = 1/3$.

of $\mathscr{S}$, nor when geometric branching may occur, nor when γ must eventually tend monotonically to infinity on $\mathscr{G}$. The desirable properties were found by computation to be present in the problem at hand. In particular, use of γ and the ρ defined in (1.2) as parameters proved wholly adequate for all cases computed. The values of γ were advanced through the range given in (2.3), (2.4), in particular exceeding the ranges of applicability of the contractive mapping and fixed point theorems of Section 4.

Selected graphs are shown in Figs. 1 and 4. We comment that ρ may be viewed as a component of x in $\mathscr{X}$ (cf. 1.2). Thus, with other parameters, one may obtain other components, and hence vastly different graphs (as in the case of a helix in space projected onto various planes).

E. The "classical" critical loads of Section 1, A were found in the present case merely to be points where $\mathscr{G}$ has a "horizontal tangent" in $\mathscr{R} \times \mathscr{X}$, i.e. (cf. (7.2)) where

$$P_\gamma(\gamma_c, x_c)(\gamma - \gamma_c) + P_x(\gamma_c, x_c)h = 0$$

can be satisfied with $\gamma = \gamma_c$ and h in the null space of $P_x(\gamma_c, x_c)$. Perhaps more than warranted importance has been attached to these points in the classical theory of buckling. By contrast, intrinsic physical significance is attached to the energies. These can be estimated using (1.2)–(1.5). If there is more than one solution for any γ or ρ then "preferred" solutions are distinguishable on the basis of relative energies. (Cf. [1, 12].) If $\mathscr{G}$ constitutes all of $\mathscr{S}$ then this permits a complete discussion of the physical possibilities. (See Fig. 5.)

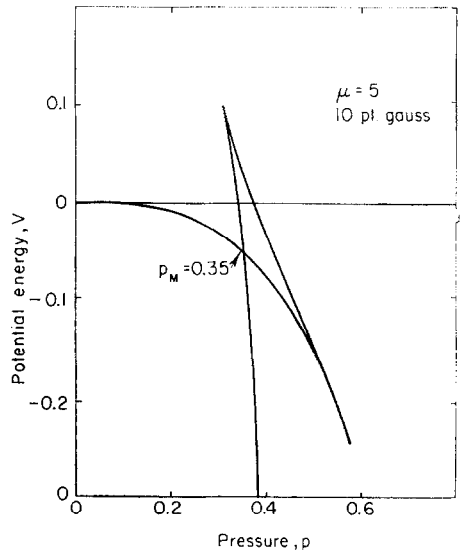

FIG. 5. Potential energy vs. pressure, $\nu = 1/3$.

PART IV

8. Discrete Numerical Representation

A. For numerical purposes, the integrals in (1.1) are replaced by quadratures with abscissas s_k, $k = 1, \ldots, m$, $0 \leqslant s_1 < s_2 < \cdots < s_m \leqslant 1$, and corresponding weights $W_1, \ldots, W_m$. The two equations (1.1) are thereby replaced by analogous equations for vectors $\mathbf{x}_j := \mathrm{col}(x_{j1}, \ldots, x_{jm})$, $j = 1, 2$:

$$x_{jk} = g_j(s_k) + \gamma \sum_{l=1}^{m} L_{j1}(s_k, s_l) x_{1l} x_{2l} W_l + \gamma \sum_{l=1}^{m} L_{j2}(s_k, s_l) \tfrac{1}{2}(x_{1l})^2 W_l, \quad (j = 1, 2). \tag{8.1}$$

For the abstract setting (cf. Section 3), let $\mathscr{R}_j^m$ denote the m-dimensional real Banach space with elements $\mathbf{x}_j$ and norm

$$\| \mathbf{x}_j \| = \max(| x_{j1} |, ..., | x_{jm} |), \qquad (j = 1, 2).$$

Let $\mathscr{X}_m := \mathscr{R}_1^m \times \mathscr{R}_2^m$ be the Banach space with elements $\mathbf{x} = \operatorname{col}(\mathbf{x}_1, \mathbf{x}_2)$ and norm $\| \mathbf{x} \| = \max(\| \mathbf{x}_1 \|, \| \mathbf{x}_2 \|)$. Then (8.1) has the abstract formulation (cf. (3.9))

$$\mathbf{P}(\gamma, \mathbf{x}) := (\mathbf{I} - \gamma \mathbf{L}\mathbf{F})\mathbf{x} - \mathbf{g} = \mathbf{0} \tag{8.2}$$

where

$$\mathbf{g} = \begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{pmatrix}, \ \mathbf{g}_j = \operatorname{col}(g_j(s_1), ..., g_j(s_m)), \qquad j = 1, 2,$$

$$\mathbf{F}\mathbf{x} = \operatorname{col}(\mathbf{x}_1\mathbf{x}_2, \tfrac{1}{2}(\mathbf{x}_1)^2) = \operatorname{col}(x_{11}x_{21}, ..., x_{1m}x_{2m}, \tfrac{1}{2}(x_{11})^2, ..., \tfrac{1}{2}(x_{1m})^2),$$

$$\mathbf{L} = \begin{pmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{pmatrix}, \qquad \mathbf{L}_{jk} = \begin{pmatrix} L_{jk}(s_1, s_1) \cdots L_{jk}(s_1, s_m) \\ \vdots \qquad\qquad \vdots \\ L_{jk}(s_m, s_1) \cdots L_{jk}(s_m, s_m) \end{pmatrix}$$

and $\mathbf{I}$ is the identity matrix. Also,

$$\| \mathbf{L} \| = \max_{j=1,2} (\| \mathbf{L}_{j1} \| + \| \mathbf{L}_{j2} \|), \| \mathbf{L}_{jk} \| = \max_{1 \leqslant i \leqslant m} \sum_{l=1}^{m} | L_{jk}(s_i, s_l) | \, | W_l |.$$

Given a solution, or approximate solution, $(\mathbf{x}_1, \mathbf{x}_2)$ of (8.1) we obtain naturally an approximate solution $(\tilde{x}_1, \tilde{x}_2)$ of (1.1) by allowing the s_k in (8.1) to vary continuously:

$$\tilde{x}_j(s) := g_j(s) + \gamma \sum_{l=1}^{m} L_{j1}(s, s_l) x_{1l} x_{2l} W_l + \gamma \sum_{l=1}^{m} L_{j2}(s, s_l) \tfrac{1}{2}(x_{1l})^2 W_l,$$

$$(j = 1, 2). \tag{8.3}$$

B. The program of solution described in Section 7 is now applied to (8.2), beginning with the solution $\mathbf{x} = \mathbf{g}$ when $\gamma = 0$. The formulas of the previous chapter can be used merely by replacing vectors and operators in $\mathscr{X}$ by their analogs in $\mathscr{X}_m$. For example $\mathbf{F}_x(\mathbf{x}_0)$ is just the Jacobian matrix. (Here and below we write subscript x instead of subscript $\mathbf{x}$.) Of course, for b in (6.12) we now use

$$\tilde{b}(\mathbf{x}) := \sum_{k=1}^{m} x_{1k} s_k^2 W_k \tag{8.4}$$

to change to the parameter ρ. Thus, when γ is the parameter, one approximates ρ by $\tilde{\rho} := \gamma \tilde{b}(\mathbf{x})$. As $\mathbf{P}_x(\gamma, \mathbf{x})$ becomes singular, ρ is raised to the status of independent parameter, and γ, or rather the approximation $\tilde{\gamma}$, is computed from $\tilde{\gamma} := \rho \cdot [\tilde{b}(\mathbf{x})]^{-1}$. Upon such change of parameter, care must be taken as to the sign of the new parameter increments.

Initial approximations for Newton's method are obtained from the analog of (7.3) (or (7.4)) for $\mathbf{P}(\gamma, \mathbf{x})$ or $\mathbf{Q}(\rho, \mathbf{x})$ as appropriate. Denote these extrapolations by $\mathbf{x}_{(0)}(\gamma_1)$ or $\mathbf{x}_{(0)}(\rho_1)$.

To obtain the inverse $\mathbf{G}_{10}$ of $\mathbf{P}_x(\gamma_1, \mathbf{x}_{(0)}(\gamma_1))$ it is convenient to use an iterative procedure with $\mathbf{G}_0 := [\mathbf{P}_x(\gamma_0, \mathbf{x}_0)]^{-1}$ as first approximation. (Similar statements apply to $\mathbf{Q}$.) Specifically, the inverse of $\mathbf{B}$ is obtained as the limit of $\mathbf{A}_k$ given by (cf. [13], p. 100)

$$\mathbf{A}_{k+1} = \mathbf{A}_k + \mathbf{A}_k[\mathbf{I} - \mathbf{B}\mathbf{A}_k].$$

The same iteration starting with $\mathbf{G}_{10}$ provides $\mathbf{G}_{11} = [\mathbf{P}_x(\gamma_1, \mathbf{x}_{(1)}(\gamma_1))]^{-1}$ for the second cycle of the Newton iteration; similarly for $\mathbf{G}_{1n}$.

For computational economy, the increments in the independent parameter should be as large as possible while still permitting a usable first approximate solution to be obtained by extrapolation. On the other hand, in the neighborhood of singularities of the derivative operators these increments will necessarily be small. Thus, a "variable step size" is called for in the actual computation.

C. *Remarks on Computation.* The smallness of step needed in some cases is illustrated in Fig. 4 for $\mu = 10$. In the neighborhood of the (actually, blunt) tip of the "spike" it was necessary to take $\Delta p = 0.0001$ and $\Delta\rho = 0.000125$; in other regions $\Delta p = 0.1$ and $\Delta\rho = 0.1$, even $\Delta\rho = 1.0$, were applicable. (A different change of parameter might have been more efficient.) The computer program provided for halving the current step width when the initial approximation obtained by extrapolation failed to yield a convergent sequence in the Newton iteration; conversely, the step was doubled when the extrapolation was successful for the preceding step.

With the iterative procedures described, most of the computer time is spent multiplying matrices, so that economy demands a highly efficient machine language matrix multiply routine.

Some computations were performed using the series mentioned after (7.4). With up to six terms expanded about $\gamma = 0$, where $\mathbf{G}_0 = \mathbf{I}$, the series provided successful initial approximations for Newton's method for values of p as large as $p = 0.6$ when $\mu = 3$, and $p = 0.5$ when $\mu = 4$ (cf. (2.3)). Such a procedure may be appropriate in other problems.

9. Justification of the Numerical Results

The statements of Section 7 apply to P and Q in $\mathscr{R} \times \mathscr{X}$ or to $\mathbf{P}$ and $\mathbf{Q}$ in $\mathscr{R} \times \mathscr{X}_m$. In [14], Anselone and Moore have shown that for linear equations

$$(I - \gamma L)x - g = 0, \tag{9.1}$$

if m is sufficiently large then whenever there is a solution of

$$(\mathbf{I} - \gamma\mathbf{L})\mathbf{x} - \mathbf{g} = 0 \tag{9.2}$$

there is also a solution of (9.1) given approximately as in (8.3), and one can obtain an error estimate.

Baluev [15] considered Newton's method for Urysohn equations

$$x(s) + \int_0^1 H(s, t, x(t))\, dt = g(s). \tag{9.3}$$

He showed that, under suitable smoothness assumptions, if (9.3) has a solution then the discrete analog (cf. (8.1)) using Gaussian quadrature also has a solution, obtainable by Newton's method (using as first approximation the discrete representation of the solution of (9.3)!)

Using [14] one can show that for differentiable kernels $L(s, t)$ the Hammerstein equation

$$x(s) - \int_0^1 L(s, t)F(x(t))\, dt = g(s)$$

has a solution (approximated as in (8.3)) whenever criteria related to those of Kantorovič's Theorem 5.2 can be satisfied for the discrete analog (as in (8.1)); of course, m must be sufficiently large. This result was not available at the time the data reported here were computed, so the necessary criteria related to the discretization were not verified. The Kantorovič criteria for the discrete equation (8.1) were satisfied with typical terminal value of $\tau = \beta\eta\kappa$ on the order of 10^{-4}.

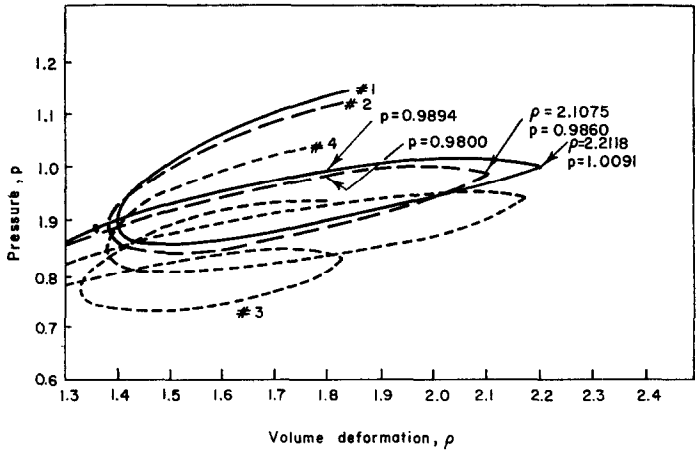

FIG. 6. Comparison of different integration methods.

Incidentally, the Newton method, as an iterative procedure, mitigates against accumulation of round-off error.

Lastly, to test the sensitivity of results with regard to order and type of quadrature, comparison in the case $\mu = 7$ was made for

#1 $m = 20$ parabolic quadrature

#2 $m = 20$ Gaussian quadrature

#3 $m = 10$ Gaussian quadrature

#4 $m = 10$ trapezoidal quadrature.

The results are indicated by the graphs in Fig. 6, all pertaining to the first loop in Fig. 1 (but to a different scale.)

Appendix

The functions $x_1(t)$ and $x_2(t)$ are defined as follows:

$$x_1(t) = \frac{Eh^3}{3qa^3(1-\nu^2)}\beta(r), \qquad x_2(t) = \frac{2h}{qa^3\sqrt{3(1-\nu^2)}}\psi(r)$$

where

$t = r/a$, a = base radius of shell, r = radial coordinate,

h = thickness of shell

E = Young's modulus, ν = Poisson's ratio,

q = normal pressure,

$\beta(r) = dw/dr$ where $w(r)$ = vertical displacement from unloaded position,

$\psi(r) = rN_r$ where N_r = longitudinal (radial) stress in shell.

The dimensionless load is defined by $p = q/q_0$ where q_0 is the buckling pressure (from the linear theory) of the complete spherical shell of the same radius of curvature $R = a^2/2H$ where H = center height of shell:

$$q_0 = \frac{2E}{\sqrt{3(1-\nu)^2}}\left(\frac{h}{R}\right)^2 = \frac{8Eh^2H^2}{a^4\sqrt{3(1-\nu^2)}}.$$

The geometric parameter μ defining the shape of the shell is given by

$$\mu^2 = \frac{8H}{h}\sqrt{3(1-\nu^2)}.$$

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