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Structure of thin irreducible modules of a Q-polynomial distance-regular graph

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ABSTRACT

Let Γ be a Q-polynomial distance-regular graph with vertex set X, diameter $D \geqslant 3$ and adjacency matrix A. Fix $x \in X$ and let $A^* = A^*(x)$ be the corresponding dual adjacency matrix. Recall that the Terwilliger algebra $T = T(x)$ is the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A and A^*. Let W denote a thin irreducible T-module. It is known that the action of A and A^* on W induces a linear algebraic object known as a Leonard pair. Over the past decade, many results have been obtained concerning Leonard pairs. In this paper, we apply these results to obtain a detailed description of W. In our description, we do not assume that the reader is familiar with Leonard pairs. Everything will be proved from the point of view of Γ. Our results are summarized as follows. Let $\{E_i\}_{i=0}^D$ be a Q-polynomial ordering of the primitive idempotents of Γ and let $\{E_i^*\}_{i=0}^D$ be the dual primitive idempotents of Γ with respect to x. Let r, t and d be the endpoint, dual endpoint and diameter of W, respectively. Let u and v be nonzero vectors in E_tW and E_r^*W, respectively. We show that $\{E_{r+i}^*A^iv\}_{i=0}^d$ and $\{E_{t+i}A^{*i}u\}_{i=0}^d$ are bases for W that are orthogonal with respect to the standard Hermitian dot product. We display the matrix representations of A and A^* with respect to these bases. We associate with W two sequences of polynomials $\{p_i\}_{i=0}^d$ and $\{p_i^*\}_{i=0}^d$. We show that for $0 \leqslant i \leqslant d$, $p_i(A)v = E_{r+i}^*A^iv$ and $p_i^*(A^*)u = E_{t+i}A^{*i}u$. Next, we show that $\{E_{r+i}^*u\}_{i=0}^d$ and $\{E_{t+i}v\}_{i=0}^d$ are orthogonal bases for W; we call these the standard basis and dual standard basis for W, respectively. We display the matrix representations of A and A^* with respect to these bases. The entries in these matrices will play an important role in our theory.

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We call these the intersection numbers and dual intersection numbers of W. Using these numbers, we compute all inner products involving the standard and dual standard bases. We also use these numbers to define two normalizations u_i, v_i (resp., u_i^*, v_i^*) for p_i (resp., p_i^*). Using the orthogonality of the standard and dual standard bases, we show that for each of the sequences $\{p_i\}_{i=0}^d, \{p_i^*\}_{i=0}^d, \{u_i\}_{i=0}^d, \{u_i^*\}_{i=0}^d, \{v_i\}_{i=0}^d, \{v_i^*\}_{i=0}^d$ the polynomials involved are orthogonal and we display the orthogonality relations. We also show that each of the sequences satisfy a three-term recurrence and a relation known as the Askey–Wilson duality. We then turn our attention to two more bases for W. We find the matrix representations of A and A^* with respect to these bases. From the entries of these matrices we obtain two sequences of scalars known as the first split sequence and second split sequence of W. We associate with W a sequence of scalars called the parameter array. This sequence consists of the eigenvalues of the restriction of A to W, the eigenvalues of the restriction of A^* to W, the first split sequence of W and the second split sequence of W. We express all the scalars and polynomials associated with W in terms of its parameter array. We show that the parameter array of W is determined by r, t, d and one more free parameter. From this we conclude that the isomorphism class of W is determined by these four parameters. Finally, we apply our results to the case in which Γ has q-Racah type or classical parameters.

1. Introduction

The Terwilliger algebra T of a distance-regular graph was first introduced in [8]. This algebra has been used extensively to study the Q-polynomial property [4,5,7]. In this paper, we continue this study focusing on the structure of thin irreducible T-modules.

Let Γ be a Q-polynomial distance-regular graph with vertex set X, diameter $D \geqslant 3$, and adjacency matrix A (see Section 2 for formal definitions). Fix $x \in X$ and let $A^* = A^*(x)$ be the corresponding dual adjacency matrix. Recall that the Terwilliger algebra $T = T(x)$ is the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A and A^*. Let W be a thin irreducible T-module. It is known that the action of A and A^* on W induces a linear algebraic object called a Leonard pair; this was first introduced by Terwilliger in [10]. The theory of Leonard pairs has been developed over the past decade. We apply these results to obtain a detailed description of W. In our description, we do not assume that the reader is familiar with Leonard pairs. The results will be proved from the point of view of Γ.

Our results are summarized as follows. Let $\{E_i\}_{i=0}^D$ be a Q-polynomial ordering of the primitive idempotents of Γ and let $\{E_i^*\}_{i=0}^D$ be the dual primitive idempotents of Γ with respect to x. Let r, t and d be the endpoint, dual endpoint and diameter of W, respectively. Let u and v be nonzero vectors in E_tW and E_r^*W, respectively. We show that $\{E_{r+i}^*A^iv\}_{i=0}^d$ and $\{E_{t+i}A^{*i}u\}_{i=0}^d$ are bases for W that are orthogonal with respect to the standard Hermitian dot product. We display the matrix representations of A and A^* with respect to these bases. We associate with W two sequences of polynomials $\{p_i\}_{i=0}^d$ and $\{p_i^*\}_{i=0}^d$. We show that for $0 \leqslant i \leqslant d$, $p_i(A)v = E_{r+i}^*A^iv$ and $p_i^*(A^*)u = E_{t+i}A^{*i}u$. Next, we show that $\{E_{r+i}^*u\}_{i=0}^d$ and $\{E_{t+i}v\}_{i=0}^d$ are orthogonal bases for W; we call these the standard basis and dual standard basis for W, respectively. We display the matrix representations of A and A^* with respect to these bases. The entries in these matrices will play an important role in our theory. We call these the intersection numbers and dual intersection numbers of W. Using these numbers, we compute all inner products involving the standard and dual standard bases. We also use these numbers to define two normalizations u_i, v_i (resp., u_i^*, v_i^*) for p_i (resp., p_i^*). Using the orthogonality of the standard and dual standard bases, we show that for each of the sequences $\{p_i\}_{i=0}^d, \{p_i^*\}_{i=0}^d, \{u_i\}_{i=0}^d, \{u_i^*\}_{i=0}^d, \{v_i\}_{i=0}^d, \{v_i^*\}_{i=0}^d$ the polynomials involved are orthogonal and we display the orthogonality relations. We also show that each of the sequences satisfy a three-term recurrence and a relation known as the Askey–Wilson duality. We then turn our attention to two more bases for W. We find the matrix representations of A and A^* with respect to these bases. From the entries of these matrices we obtain two sequences of scalars known as the first split sequence and second split sequence of W. We associate with W a sequence of scalars called the parameter array. This sequence consists of the eigenvalues of the restriction of A to W, the eigenvalues of the restriction of A^* to W, the first split sequence of W and the second split sequence of W. We express all the scalars and polynomials associated with W in terms of its parameter array. We show that the parameter array of W is determined by r, t, d and one more free parameter. From this we conclude that the isomorphism class of W is determined by these four parameters. Finally, we apply our results to the case in which Γ has q-Racah type or classical parameters.

2. Preliminaries

In this section, we recall some basic concepts concerning Q-polynomial distance-regular graphs. For more background information see [2,3].

Let X be a non-empty finite set. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra of matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. We let I (resp., J) denote the identity matrix (resp., all 1's matrix) in $\mathrm{Mat}_X(\mathbb{C})$. Let $V = \mathbb{C}X$ be the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. Observe that $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. For $u, v \in V$, define $\langle u, v\rangle := u^t\bar{v}$, where u^t is the transpose of u and $\bar{v}$ is the complex conjugate of v. Observe that $\langle , \rangle$ is a positive definite Hermitian form on V. Note that $\langle Bu, v\rangle = \langle u, \overline{B}^t v\rangle$ for all $B \in \mathrm{Mat}_X(\mathbb{C})$ and $u, v \in V$. For $y \in X$, let $\hat{y}$ denote the element in V with a 1 in the y coordinate and 0 in all other coordinates. Observe that $\{\hat{y} | y \in X\}$ is an orthonormal basis for V.

Let $\Gamma = (X, R)$ be a finite undirected connected graph without loops or multiple edges, with vertex set X and edge set R. Let ∂ denote the path-length distance function for Γ. Set $D = \max\{\partial(x, y) | x, y \in X\}$. We refer to D as the *diameter* of Γ. For $x \in X$ and an integer $i \geqslant 0$, let $\Gamma_i(x) = \{y | y \in X, \ \partial(x, y) = i\}$. Abbreviate $\Gamma(x) := \Gamma_1(x)$. For an integer $k \geqslant 0$, we say that Γ is *regular* with *valency* k whenever $k = |\Gamma(x)|$ for all $x \in X$. We say that Γ is *distance-regular* whenever there exists scalars p_{ij}^h $(0 \leqslant h, i, j \leqslant D)$ such that $p_{ij}^h = |\Gamma_i(x) \cap \Gamma_j(y)|$ for all $x, y \in X$ with $\partial(x, y) = h$. We refer to the p_{ij}^h as the *intersection numbers* of Γ. For the rest of this paper, assume that Γ is distance-regular with diameter $D \geqslant 3$. Note that by the triangle inequality, we have (i) $p_{ij}^h = 0$ if one of h, i, j is greater than the sum of the other two; (ii) $p_{ij}^h \neq 0$ if one of h, i, j is equal to the sum of the other two. We abbreviate $c_i := p_{1i-1}^i$ $(1 \leqslant i \leqslant D)$, $a_i := p_{1i}^i$ $(0 \leqslant i \leqslant D)$, $b_i := p_{1i+1}^i$ $(0 \leqslant i \leqslant D-1)$. For notational convenience, define $b_D = 0$, $c_0 = 0$. Observe that Γ is regular with valency $k = b_0$. To avoid trivialities, we always assume that $k \geqslant 3$. Note that $c_i + a_i + b_i = k$ for $0 \leqslant i \leqslant D$. For $0 \leqslant i \leqslant D$, let $k_i = p_{ii}^0$. Observe that $k_i = |\Gamma_i(x)|$ for all $x \in X$. By [2, p. 195],

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \quad (0 \leqslant i \leqslant D). \tag{1}$$

We refer to k_i as the ith*valency* of Γ.

We now recall the Bose–Mesner algebra of Γ. For $0 \leqslant i \leqslant D$, define $A_i \in \mathrm{Mat}_X(\mathbb{C})$ to have (x, y)-entry equal to 1 if $\partial(x, y) = i$, and 0 otherwise. We refer to A_i as the ith*distance matrix* of Γ. Note that (i) $A_0 = I$; (ii) $\sum_{i=0}^D A_i = J$; (iii) $A_i^t = A_i$ $(0 \leqslant i \leqslant D)$; (iv) $A_iA_j = \sum_{h=0}^D p_{ij}^h A_h$ $(0 \leqslant i, j \leqslant D)$. Observe that $\{A_i\}_{i=0}^D$ are linearly independent. Thus, they form a basis for a subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$; M is called the *Bose–Mesner algebra of* Γ. Abbreviate $A := A_1$ and call this the *adjacency matrix* of Γ. By [2, p. 190], M is generated by A. By [2, p. 59], M has a second basis $\{E_i\}_{i=0}^D$ which satisfies the following: (i) $E_0 = |X|^{-1}J$; (ii) $\sum_{i=0}^D E_i = I$; (iii) $E_i^t = E_i = \overline{E_i}$ $(0 \leqslant i \leqslant D)$; (iv) $E_iE_j = \delta_{ij}E_i$ $(0 \leqslant i, j \leqslant D)$. For notational convenience, define $E_{-1} = 0$, $E_{D+1} = 0$. For $0 \leqslant i \leqslant D$, let m_i denote the rank of E_i; we call m_i the *multiplicity* of Γ associated with E_i. Since $\{E_i\}_{i=0}^D$ is a basis for M, there exist complex scalars $\{\theta_i\}_{i=0}^D$ such that $A = \sum_{i=0}^D \theta_iE_i$. Note that for $0 \leqslant i \leqslant D$, $AE_i = E_iA = \theta_iE_i$. Thus, E_iV is an eigenspace for A, and

θ_i is the corresponding eigenvalue. Since A is symmetric, $\theta_i \in \mathbb{R}$. Since A generates M, the $\{\theta_i\}_{i=0}^{D}$ are mutually distinct. Note that

$$V = \sum_{i=0}^{D} E_i V \quad \text{(orthogonal direct sum)}, \tag{2}$$

and that

$$E_i = \prod_{\substack{0 \leqslant j \leqslant D \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j} \quad (0 \leqslant i \leqslant D). \tag{3}$$

We call θ_i the *eigenvalue* of Γ associated with E_i.

We now recall the Krein parameters of Γ. Observe that $A_i \circ A_j = \delta_{ij} A_i$ for $0 \leqslant i, j \leqslant D$, where $\circ$ is the entry-wise multiplication. Thus, M is closed under $\circ$. Consequently, there exist complex scalars q_{ij}^h $(0 \leqslant h, i, j \leqslant D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h \quad (0 \leqslant i, j \leqslant D).$$

The q_{ij}^h are known as the *Krein parameters* or *dual intersection numbers* of Γ. By [2, p. 69], the q_{ij}^h are real and nonnegative.

We now consider the Q-polynomial property. The graph Γ is said to be Q-*polynomial* (with respect to the given ordering $\{E_i\}_{i=0}^{D}$ of primitive idempotents) whenever both: (i) $q_{ij}^h = 0$ if one of h, i, j is greater than the sum of the other two; (ii) $q_{ij}^h \neq 0$ if one of h, i, j is equal to the sum of the other two. For the rest of this paper, we assume that Γ is Q-polynomial with respect to $\{E_i\}_{i=0}^{D}$. We abbreviate $c_i^* := q_{1i-1}^i$ $(1 \leqslant i \leqslant D)$, $a_i^* := q_{1i}^i$ $(0 \leqslant i \leqslant D)$, $b_i^* := q_{1i+1}^i$ $(0 \leqslant i \leqslant D-1)$. For notational convenience, define $b_D^* = 0$, $c_0^* = 0$. By [2, p. 67], $m_i = q_{ii}^0$ $(0 \leqslant i \leqslant D)$. By [2, p. 196],

$$m_i = \frac{b_0^* b_1^* \cdots b_{i-1}^*}{c_1^* c_2^* \cdots c_i^*} \quad (0 \leqslant i \leqslant D). \tag{4}$$

We now recall the dual Bose–Mesner algebra of Γ. For the rest of this paper, fix $x \in X$. For $0 \leqslant i \leqslant D$, define $E_i^* = E_i^*(x)$ to be the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{otherwise} \end{cases} \quad (y \in X). \tag{5}$$

We refer to E_i^* as the ith *dual primitive idempotent* of Γ with respect to x. For notational convenience, define $E_{-1}^* = 0$, $E_{D+1}^* = 0$. Note that (i) $\sum_{i=0}^{D} E_i^* = I$; (ii) $E_i^{*t} = E_i^* = \overline{E_i^*}$ $(0 \leqslant i \leqslant D)$; (iii) $E_i^* E_j^* = \delta_{ij} E_i^*$ $(0 \leqslant i, j \leqslant D)$. Observe that $\{E_i^*\}_{i=0}^{D}$ are linearly independent. Thus, they form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$; M^* is called the *dual Bose–Mesner algebra* of Γ with respect to x. For $0 \leqslant i \leqslant D$, define $A_i^* = A_i^*(x)$ to be the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ such that $(A_i^*)_{yy} = |X|(E_i)_{xy}$ for $y \in X$. By [8, p. 379], $\{A_i^*\}_{i=0}^{D}$ is a basis for M^* and satisfies the following properties: (i) $A_0^* = I$; (ii) $\sum_{i=0}^{D} A_i^* = |X| E_0^*$; (iii) $A_i^{*t} = A_i^* = \overline{A_i^*}$ $(0 \leqslant i \leqslant D)$; (iv) $A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^*$ $(0 \leqslant i, j \leqslant D)$. We refer to A_i^* as the ith *dual distance matrix* of Γ with respect to x. Abbreviate $A^* := A_1^*$ and call this the *dual adjacency matrix* of Γ with respect ot x. By [8, Lemma 3.11], M^* is generated by A^*. Since $\{E_i^*\}_{i=0}^{D}$ is a basis for M^*, there exist complex scalars $\{\theta_i^*\}_{i=0}^{D}$ such that $A^* = \sum_{i=0}^{D} \theta_i^* E_i^*$. Note that for $0 \leqslant i \leqslant D$, $A^* E_i^* = E_i^* A^* = \theta_i^* E_i^*$. Since A^* is real, $\theta_i^* \in \mathbb{R}$. Since A^* generates M^*, the $\{\theta_i^*\}_{i=0}^{D}$ are mutually distinct. Observe that

$$E_i^* V = \mathrm{Span}\,\{\hat{y} | y \in X,\ \partial(x, y) = i\} \quad (0 \leqslant i \leqslant D).$$

Moreover,

$$V = \sum_{i=0}^{D} E_i^* V \quad \text{(orthogonal direct sum)} \tag{6}$$

and

$$E_i^* = \prod_{\substack{0 \leqslant j \leqslant D \\ j \neq i}} \frac{A^* - \theta_j^* I}{\theta_i^* - \theta_j^*} \quad (0 \leqslant i \leqslant D). \tag{7}$$

We call θ_i^* the *dual eigenvalue* of Γ associated with E_i^*.

We now recall the Terwilliger algebra of Γ. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*. We refer to T as the *Terwilliger algebra of Γ with respect to x*. Observe that T is generated by A, A^*. Moreover, T is semi-simple. By [8, Lemma 3.2],

$$E_i^* A_h E_j^* = 0 \text{ if and only if } p_{ij}^h = 0 \qquad (0 \leqslant h, i, j \leqslant D), \tag{8}$$

$$E_i A_h^* E_j = 0 \text{ if and only if } q_{ij}^h = 0 \qquad (0 \leqslant h, i, j \leqslant D). \tag{9}$$

It follows from (8) and (9) that

$$\begin{aligned} AE_i^*V &\subseteq E_{i-1}^*V + E_i^*V + E_{i+1}^*V \quad (0 \leqslant i \leqslant D), \\ A^*E_iV &\subseteq E_{i-1}V + E_iV + E_{i+1}V \quad (0 \leqslant i \leqslant D). \end{aligned}$$

Moreover,

$$E_i^* A^h E_j^* = \begin{cases} 0, & h < |\, i - j| \\ \neq 0, & h = |\, i - j| \end{cases} \quad (0 \leqslant h, i, j \leqslant D), \tag{10}$$

$$E_i A^{*h} E_j = \begin{cases} 0, & h < |\, i - j| \\ \neq 0, & h = |\, i - j| \end{cases} \quad (0 \leqslant h, i, j \leqslant D). \tag{11}$$

Lemma 2.1. *For $0 \leqslant i, j, k, l \leqslant D$ with $i + j = |k - l|$,*

$$E_l^* A^{i+j} E_k^* = \begin{cases} E_l^* A^i E_{l+i}^* A^j E_k^*, & i + j = k - l \\ E_l^* A^i E_{k+j}^* A^j E_k^*, & i + j = l - k \end{cases}.$$

Proof. In $E_l^* A^{i+j} E_k^*$, write A^{i+j} as $A^i I A^j$ with $I = \sum_{m=0}^{D} E_m^*$. Evaluate the result using (10). $\square$

Lemma 2.2. *For $0 \leqslant i, j, k, l \leqslant d$ with $i + j = |k - l|$,*

$$E_l (A^*)^{i+j} E_k = \begin{cases} E_l A^{*i} E_{l+i} A^{*j} E_k, & i + j = k - l \\ E_l A^{*i} E_{k+j} A^{*j} E_k, & i + j = l - k \end{cases}.$$

Proof. Similar to the proof of Lemma 2.1. $\square$

3. *T*-Modules

In this section, we recall some basic facts concerning the T-modules of Γ.

Let W be a subspace of V. We say that W is a *T-module* whenever $TW \subseteq W$. Note that V is a T-module. We refer to V as the *standard module*. Let W and W' be T-modules. By a *T-module isomorphism* from W to W', we mean a vector space isomorphism $\sigma : W \to W'$ such that $(\sigma B - B\sigma)W = 0$ for all $B \in T$. If such a map exists, we say that W and W' are *isomorphic* as T-modules. A T-module W is said to be *irreducible* whenever $W \neq 0$ and W contains no T-modules besides 0 and W. W is said to be *thin* whenever $\dim E_i^*W \leqslant 1$ for $0 \leqslant i \leqslant D$. Similarly, W is said to be *dual thin* whenever $\dim E_iW \leqslant 1$ for $0 \leqslant i \leqslant D$.

We now recall the notion of endpoint, dual endpoint, diameter and dual diameter. Observe that $W = \sum E_i^*W$ (orthogonal direct sum) where the sum is taken over all indices $i (0 \leqslant i \leqslant D)$ such that $E_i^*W \neq 0$. Similarly, $W = \sum E_iW$ (orthogonal direct sum) where the sum is taken over all indices $i (0 \leqslant i \leqslant D)$ such that $E_iW \neq 0$. Let $r = \min\{i | 0 \leqslant i \leqslant D, E_i^*W \neq 0\}$ and $t = \min\{i | 0 \leqslant i \leqslant D, E_iW \neq 0\}$. We call r and t the *endpoint* and *dual endpoint* of W, respectively. Let $d = |\{i | 0 \leqslant i \leqslant D, E_i^*W \neq 0\}| - 1$ and $d^* = |\{i | 0 \leqslant i \leqslant D, E_iW \neq 0\}| - 1$. We refer to d and d^* as the *diameter* and *dual diameter* of W, respectively.

Lemma 3.1 [8, Lemma 3.9]**.** *Let W be an irreducible T-module with endpoint r, dual endpoint t, diameter d and dual diameter d^*. Then (i) and (v) below hold:*

(i) $AE_i^*W \subseteq E_{i-1}^*W + E_i^*W + E_{i+1}^*W$ $(0 \leqslant i \leqslant D)$.
(ii) $E_i^*W \neq 0$ *if and only if* $r \leqslant i \leqslant r + d$ $(0 \leqslant i \leqslant D)$.
(iii) $E_i^*AE_j^*W \neq 0$ *if* $|i - j| = 1$ $(0 \leqslant i, j \leqslant D)$.
(iv) $W = \sum_{i=0}^{d} E_{r+i}^*W$ (*orthogonal direct sum*).
(v) *Suppose W is thin. Then $E_iW = E_iE_r^*W$ for $0 \leqslant i \leqslant D$. Moreover, W is dual thin and $d = d^*$.*

Lemma 3.2 [8, Lemma 3.12]. *Let W be as in Lemma 3.1. Then (i)–(v) below hold:*

(i) $A^*E_iW \subseteq E_{i-1}W + E_iW + E_{i+1}W$ $(0 \leqslant i \leqslant D)$.
(ii) $E_iW \neq 0$ *if and only if* $t \leqslant i \leqslant t + d$ $(0 \leqslant i \leqslant D)$.
(iii) $E_iA^*E_jW \neq 0$ *if* $|i - j| = 1$ $(0 \leqslant i, j \leqslant D)$.
(iv) $W = \sum_{i=0}^{d} E_{t+i}W$ (*orthogonal direct sum*).
(v) *Suppose W is dual thin. Then $E_i^*W = E_i^*E_tW$ for $0 \leqslant i \leqslant D$. Moreover, W is thin and $d^* = d$.*

Lemma 3.3 [8, Lemma 3.6]**.** *There exists a unique irreducible T-module of endpoint* 0, *dual endpoint* 0 *and diameter D. Moreover, it is thin and dual thin. We refer to this module as the trivial T-module.*

For the rest of this paper, we will have the following assumption on W.

Assumption 3.4. From now on, W will denote a thin irreducible T-module with endpoint r, dual endpoint t and diameter d. Unless otherwise stated, we assume that $d > 0$.

4. Generators for End(W)

With reference to Assumption 3.4, let $\mathrm{End}(W) = \mathrm{End}_{\mathbb{C}}(W)$ denote the $\mathbb{C}$-algebra of all $\mathbb{C}$-linear transformations from W to W. In this section, we will look at bases and generators of End(W). We begin with two lemmas whose proofs are routine and left to the reader.

Lemma 4.1. *For $0 \leqslant i \leqslant d$, let w_i be a nonzero vector in E_{r+i}^*W. Note that $\{w_i\}_{i=0}^{d}$ is a basis for W. With respect to this basis:*

(i) *the matrix representation of E_{r+i}^* has (i, i)-entry* 1 *and all other entries* 0 $(0 \leqslant i \leqslant d)$;
(ii) *the matrix representation of A^* is diag* $\left(\theta_r^*, \theta_{r+1}^*, \ldots, \theta_{r+d}^*\right)$;
(iii) *the matrix representation of A is tridiagonal with each entry nonzero on the superdiagonal and subdiagonal.*

Lemma 4.2. *For $0 \leqslant i \leqslant d$, let w_i^* be a nonzero vector in $E_{t+i}W$. Note that $\{w_i^*\}_{i=0}^{d}$ is a basis for W. With respect to this basis:*

(i) *the matrix representation of* E_{t+i} *has* (i,i)*-entry* 1 *and all other entries* 0 $(0 \leqslant i \leqslant d)$;
(ii) *the matrix representation of* A *is* $\mathrm{diag}(\theta_t, \theta_{t+1}, \ldots, \theta_{t+d})$;
(iii) *the matrix representation of* A^* *is tridiagonal with each entry nonzero on the superdiagonal and subdiagonal*.

Definition 4.3. We refer to the sequence $\{\theta_{t+i}\}_{i=0}^d$ (resp., $\{\theta^*_{r+i}\}_{i=0}^d$) as the *eigenvalue sequence* (resp., *dual eigenvalue sequence*) of W.

Lemma 4.4. *On* W,

$$\prod_{i=0}^{d}(A - \theta_{t+i}I) = 0, \quad \prod_{i=0}^{d}\left(A^* - \theta^*_{r+i}I\right) = 0.$$

Proof. Immediate from Lemmas 4.1(ii) and 4.2(ii). $\square$

Lemma 4.5. *Let* B (*resp.*, B^*) *denote the matrix representation of* A (*resp.*, A^*) *with respect to the basis given in Lemma* 4.1 (*resp.*, *Lemma* 4.2). *Then*

$$(B^h)_{ij} = \begin{cases} 0, & h < |i-j| \\ \neq 0, & h = |i-j| \end{cases} \quad (0 \leqslant h,i,j \leqslant d),$$

$$(B^{*h})_{ij} = \begin{cases} 0, & h < |i-j| \\ \neq 0, & h = |i-j| \end{cases} \quad (0 \leqslant h,i,j \leqslant d).$$

Proof. Routine using Lemmas 4.1(iii) and 4.2(iii). $\square$

Using Lemma 4.5 we obtain the following strengthening of (10) and (11).

Lemma 4.6. *For* $0 \leqslant h,i,j \leqslant d$, *the following hold on* W:

$$E^*_{r+i}A^hE^*_{r+j} = \begin{cases} 0, & h < |i-j| \\ \neq 0, & h = |i-j| \end{cases}, \tag{12}$$

$$E_{t+i}A^{*h}E_{t+j} = \begin{cases} 0, & h < |i-j| \\ \neq 0, & h = |i-j| \end{cases}. \tag{13}$$

Proof. Let B denote matrix representation of A with respect to the basis given in Lemma 4.1. By construction, the matrix representation of $E^*_{r+i}A^hE^*_{r+j}$ with respect to this basis has (i,j)-entry $(B^h)_{ij}$ and all other entries are 0. Line (12) follows from this and Lemma 4.5. The proof of (13) is similar. $\square$

Theorem 4.7. *Each of the following forms a basis for the* $\mathbb{C}$*-vector space* $\mathrm{End}(W)$:

(i) *the actions of* $\{A^mE^*_rA^n \mid 0 \leqslant m,n \leqslant d\}$ *on* W,
(ii) *the actions of* $\{A^{*m}E_tA^{*n} \mid 0 \leqslant m,n \leqslant d\}$ *on* W.

Proof. Let S denote $\{A^mE^*_rA^n \mid 0 \leqslant m,n \leqslant d\}$. Observe that $|S| = (d+1)^2$ and this is equal to the dimension of $\mathrm{End}(W)$. It suffices to show that the actions of the elements of S on W are linearly independent. Let $\{w_i\}_{i=0}^d$ be the basis for W in Lemma 4.1. With respect to this basis, let B and F^*_r be the matrix representations of A and E^*_r. We claim that for $0 \leqslant m,n \leqslant d$, $B^mF^*_rB^n$ has entries

$$(B^mF^*_rB^n)_{ij} = \begin{cases} 0, & i > m \text{ or } j > n \\ \neq 0, & i = m \text{ and } j = n \end{cases} \quad (0 \leqslant i,j \leqslant d). \tag{14}$$

By Lemma 4.1(i), F^*_r has $(0,0)$-entry 1 and all other entries are 0. Thus,

$$(B^mF^*_rB^n)_{ij} = (B^m)_{i0}(B^n)_{0j} \quad (0 \leqslant i,j \leqslant d).$$

Combining this with Lemma 4.5, we obtain (14). It follows from (14) that actions of the elements of S on W are linearly independent and hence form a basis for $\mathrm{End}(W)$. Similarly, (ii) can be shown to be a basis for $\mathrm{End}(W)$. $\square$

Theorem 4.8. *Each of the following is a generating set for the $\mathbb{C}$-algebra* $\mathrm{End}(W)$*:*

(i) *the actions of A, E^*_r on W,*
(ii) *the actions of A^*, E_t on W,*
(iii) *the actions of A, A^* on W.*

Proof. By Theorem 4.7, (i) and (ii) are generating sets for $\mathrm{End}(W)$. The set (iii) is a generating set for $\mathrm{End}(W)$ by (i) and since E^*_r is a polynomial in A^*. $\square$

Definition 4.9. Define $\mathcal{D}$ (resp., $\mathcal{D}^*$) to be the subalgebra of $\mathrm{End}(W)$ generated by the action of A (resp., A^*) on W.

Lemma 4.10. *Each of the following forms a basis for the $\mathbb{C}$-vector space $\mathcal{D}$:*

(i) *the actions of $\{A^i\}_{i=0}^d$ on W,*
(ii) *the actions of $\{E_{t+i}\}_{i=0}^d$ on W.*

Proof. (i) By Lemma 4.5, $\{A^i\}_{i=0}^d$ are linearly independent on W. Combining this with Lemma 4.4, we obtain the result.

(ii) Immediate from (3) and (i). $\square$

Lemma 4.11. *Each of the following forms a basis for the $\mathbb{C}$-vector space $\mathcal{D}^*$:*

(i) *the actions of $\{A^{*i}\}_{i=0}^d$ on W,*
(ii) *the actions of $\{E^*_{r+i}\}_{i=0}^d$ on W.*

Proof. Similar to the proof of Lemma 4.10. $\square$

Corollary 4.12. *Each of the following forms a basis for the $\mathbb{C}$-vector space* $\mathrm{End}(W)$:

(i) *the actions of $\{E_{t+i}E^*_rE_{t+j} | 0 \leqslant i,j \leqslant d\}$ on W,*
(ii) *the actions of $\{E^*_{r+i}E_tE^*_{r+j} | 0 \leqslant i,j \leqslant d\}$ on W.*

Proof. Immediate from Theorem 4.7 and Lemmas 4.10 and 4.11. $\square$

5. The scalars $a_i(W)$ and $x_i(W)$

Let W be as in Assumption 3.4. In this section, we associate with W two sequences of scalars called the $a_i(W)$ and $x_i(W)$. We will then describe the algebraic properties of these scalars.

Notation 5.1. For any $Y \in T$, $\mathrm{tr}_W Y$ denotes the trace of the action of Y on W.

Definition 5.2. Define

$$a_i(W) = \mathrm{tr}_W\left(E^*_{r+i}A\right) \quad a^*_i(W) = \mathrm{tr}_W\left(E_{t+i}A^*\right) \quad (0 \leqslant i \leqslant d), \tag{15}$$

$$x_i(W) = \mathrm{tr}_W\left(E^*_{r+i}AE^*_{r+i-1}A\right) \quad x^*_i(W) = \mathrm{tr}_W\left(E_{t+i}A^*E_{t+i-1}A^*\right) \quad (1 \leqslant i \leqslant d). \tag{16}$$

For notational convenience, define $x_0(W) = 0$ and $x^*_0(W) = 0$.

Lemma 5.3. *For $0 \leqslant i \leqslant d$, let w_i be a nonzero vector in $E^*_{r+i}W$. Let B denote the matrix representation of A with respect to $\{w_i\}_{i=0}^d$. Then (i)–(iii) below hold:*

(i) $B_{ii} = a_i(W) \quad (0 \leqslant i \leqslant d)$.
(ii) $B_{i,i-1}B_{i-1,i} = x_i(W) \quad (1 \leqslant i \leqslant d)$.
(iii) $x_i(W) \neq 0 \quad (1 \leqslant i \leqslant d)$.

Proof. (i) By Lemma 4.1(i), (iii), the (j,j)-entry of the matrix representation of $E^*_{r+i}A$ with respect to $\{w_i\}_{i=0}^d$ is B_{ii} if $j = i$ and 0 otherwise $(0 \leqslant j \leqslant d)$. Taking the trace of this matrix and using (15), we obtain the desired result.

(ii) By Lemma 4.1(i), (iii), the (j,j)-entry of the matrix representation of $E^*_{r+i}AE^*_{r+i-1}A$ with respect to $\{w_i\}_{i=0}^d$ is $B_{i,i-1}B_{i-1,i}$ if $j = i$ and 0 otherwise $(0 \leqslant j \leqslant d)$. Taking the trace of this matrix and using (16), we obtain the desired result.

(iii) Immediate from (ii) and Lemma 4.1(iii). □

Lemma 5.4. *For $0 \leqslant i \leqslant d$, let w^*_i be a nonzero vector in $E_{t+i}W$. Let B^* denote the matrix representation of A^* with respect to $\{w^*_i\}_{i=0}^d$. Then (i)–(iii) below hold:*

(i) $B^*_{ii} = a^*_i(W) \quad (0 \leqslant i \leqslant d)$.
(ii) $B^*_{i,i-1}B^*_{i-1,i} = x^*_i(W) \quad (1 \leqslant i \leqslant d)$.
(iii) $x^*_i(W) \neq 0 \quad (1 \leqslant i \leqslant d)$.

Proof. Similar to the proof of Lemma 5.3. □

Theorem 5.5. *Let v be a nonzero vector in E^*_rW. Then for $0 \leqslant i \leqslant d$, $E^*_{r+i}A^iv$ is nonzero and hence is a basis for $E^*_{r+i}W$. Moreover, $\{E^*_{r+i}A^iv\}_{i=0}^d$ is a basis for W.*

Proof. Since v spans E^*_rW, $E^*_{r+i}A^iv$ spans $E^*_{r+i}A^iE^*_rW$. By Lemma 4.6, $E^*_{r+i}A^iE^*_rW \neq 0$. Hence, $E^*_{r+i}A^iv \neq 0$. The rest of the assertion follows. □

Theorem 5.6. *Let u be a nonzero vector in E_tW. Then for $0 \leqslant i \leqslant d$, $E_{t+i}A^{*i}u$ is nonzero and hence is a basis for $E_{t+i}W$. Moreover, $\{E_{t+i}A^{*i}u\}_{i=0}^d$ is a basis for W.*

Proof. Similar to the proof of Theorem 5.5. □

Theorem 5.7. *With respect to the basis given in Theorem 5.5, the matrix representation of A is*

$$\begin{pmatrix} a_0(W) & x_1(W) & & & & 0 \\ 1 & a_1(W) & x_2(W) & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & a_{d-1}(W) & x_d(W) \\ 0 & & & & 1 & a_d(W) \end{pmatrix}. \tag{17}$$

Proof. Let $\{w_i\}_{i=0}^d$ be the basis for W in Theorem 5.5. Let B denote the matrix representation of A with respect to this basis. Note that for $0 \leqslant i \leqslant d-1$, $E^*_{r+i+1}Aw_i = B_{i+1,i}w_{i+1}$. By Lemma 2.1,

$$E^*_{r+i+1}Aw_i = E^*_{r+i+1}AE^*_{r+i}A^iE^*_rv = E^*_{r+i+1}A^{i+1}E^*_rv = w_{i+1}.$$

Thus, $B_{i+1,i} = 1$ for $0 \leqslant i \leqslant d-1$. The rest of the assertion follows from Lemma 5.3(i), (ii). □

Theorem 5.8. *With respect to the basis given in Theorem 5.6, the matrix representation of A^* is*

$$\begin{pmatrix} a_0^*(W) & x_1^*(W) & & & & 0 \\ 1 & a_1^*(W) & x_2^*(W) & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & a_{d-1}^*(W) & x_d^*(W) \\ 0 & & & & 1 & a_d^*(W) \end{pmatrix}.$$

Proof. Similar to the proof of Theorem 5.7. □

Lemma 5.9. *The following hold on W:*

(i) $E^*_{r+i}AE^*_{r+i} = a_i(W)E^*_{r+i} \quad (0 \leqslant i \leqslant d)$.
(ii) $E^*_{r+i}AE^*_{r+i-1}AE^*_{r+i} = x_i(W)E^*_{r+i} \quad (1 \leqslant i \leqslant d)$.
(iii) $E^*_{r+i-1}AE^*_{r+i}AE^*_{r+i-1} = x_i(W)E^*_{r+i-1} \quad (1 \leqslant i \leqslant d)$.
(iv) $E_{t+i}A^*E_{t+i} = a_i^*(W)E_{t+i} \quad (0 \leqslant i \leqslant d)$.
(v) $E_{t+i}A^*E_{t+i-1}A^*E_{t+i} = x_i^*(W)E_{t+i} \quad (1 \leqslant i \leqslant d)$.
(vi) $E_{t+i-1}A^*E_{t+i}A^*E_{t+i-1} = x_i^*(W)E_{t+i-1} \quad (1 \leqslant i \leqslant d)$.

Proof. (i) Let $\{w_j\}_{j=0}^d$ be the basis for W in Theorem 5.5. By (17), $E^*_{r+i}AE^*_{r+i}w_j = \delta_{ij}a_i(W)E^*_{r+i}w_j$. The result follows.

(ii) Let G_i denote the action of E^*_{r+i} on W. Since W is thin, $G_i \operatorname{End}(W)G_i$ has dimension 1. Observe that G_i is a nonzero element of $G_i \operatorname{End}(W)G_i$. Thus there exists $\alpha \in \mathbb{C}$ such that $E^*_{r+i}AE^*_{r+i-1}AE^*_{r+i} = \alpha E^*_{r+i}$ on W. Take the trace of both sides of this equation. Evaluating this using Definition 5.2 and the fact that $\operatorname{tr}_W(E^*_{r+i}) = 1$, we find that $\alpha = x_i(W)$.

(iii) Similar to (ii).

(iv)–(vi) Similar to the proofs of (i)–(iii). □

Lemma 5.10. *The following hold:*

(i) $\sum_{i=0}^d a_i(W) = \sum_{i=0}^d \theta_{t+i}$.
(ii) $\sum_{i=0}^d a_i^*(W) = \sum_{i=0}^d \theta^*_{r+i}$.
(iii) $a_i(W) \in \mathbb{R},\ a_i^*(W) \in \mathbb{R} \quad (0 \leqslant i \leqslant d)$.
(iv) $x_i(W) \in \mathbb{R},\ x_i(W) > 0 \quad (1 \leqslant i \leqslant d)$.
(v) $x_i^*(W) \in \mathbb{R},\ x_i^*(W) > 0 \quad (1 \leqslant i \leqslant d)$.

Proof. (i) Immediate from Theorem 5.7 and the fact that $\{\theta_{t+i}\}_{i=0}^d$ are the eigenvalues of the action of A on W.

(ii) Similar to the proof of (i).

(iii) By Lemma 5.9(i), $a_i(W)$ is an eigenvalue of the real symmetric matrix $E^*_{r+i}AE^*_{r+i}$. Thus, $a_i(W) \in \mathbb{R}$. Similarly, $a_i^*(W) \in \mathbb{R}$.

(iv) By Lemma 5.9(ii), $x_i(W)$ is an eigenvalue of the real symmetric matrix $E^*_{r+i}AE^*_{r+i-1}AE^*_{r+i}$. Thus, $x_i(W) \in \mathbb{R}$. Since

$$E^*_{r+i}AE^*_{r+i-1}AE^*_{r+i} = \left(E^*_{r+i-1}AE^*_{r+i}\right)^t \left(E^*_{r+i-1}AE^*_{r+i}\right)$$

is positive definite, $x_i(W) > 0$.

(v) Similar to the proof of (iv). □

6. The polynomial p_i

Let W be as in Assumption 3.4. In the previous section, we defined two bases for W. In this section, we will use these bases to obtain two sequences of polynomials. We will investigate some properties of these polynomials. Let $\mathbb{C}[\lambda]$ denote the $\mathbb{C}$-algebra of polynomials in λ with coefficients in $\mathbb{C}$.

Definition 6.1. For $0 \leqslant i \leqslant d+1$, define $p_i = p_i^W$ in $\mathbb{C}[\lambda]$ by $p_0 = 1$,

$$\lambda p_i = p_{i+1} + a_i(W)p_i + x_i(W)p_{i-1} \quad (0 \leqslant i \leqslant d), \tag{18}$$

where $x_i(W)$ and $a_i(W)$ are as in Definition 5.2 and $p_{-1} = 0$.

Definition 6.2. For $0 \leqslant i \leqslant d+1$, define $p_i^* = p_i^{*W}$ in $\mathbb{C}[\lambda]$ by $p_0^* = 1$,

$$\lambda p_i^* = p_{i+1}^* + a_i^*(W)p_i^* + x_i^*(W)p_{i-1}^* \quad (0 \leqslant i \leqslant d), \tag{19}$$

where $x_i^*(W)$ and $a_i^*(W)$ are as in Definition 5.2 and $p_{-1}^* = 0$.

Lemma 6.3. *For any nonzero $u \in E_tW$ and nonzero $v \in E_r^*W$,*

$$p_i(A)v = E_{r+i}^*A^i v \quad (0 \leqslant i \leqslant d), \tag{20}$$

$$p_i^*(A^*)u = E_{t+i}A^{*i}u \quad (0 \leqslant i \leqslant d). \tag{21}$$

Moreover, $p_{d+1}(A)v = 0$ and $p_{d+1}^*(A^*)u = 0$.

Proof. For $0 \leqslant i \leqslant d+1$, let $w_i = E_{r+i}^*A^i v$ and $w_i' = p_i(A)v$. Recall that by Theorem 5.5, $\{w_i\}_{i=0}^d$ is a basis for W. By (17),

$$Aw_i = w_{i+1} + a_i(W)w_i + x_i(W)w_{i-1} \quad (0 \leqslant i \leqslant d). \tag{22}$$

By (18),

$$Aw_i' = w_{i+1}' + a_i(W)w_i' + x_i(W)w_{i-1}' \quad (0 \leqslant i \leqslant d). \tag{23}$$

Comparing (22) and (23) and using the fact that $w_0 = w_0'$, we find that $w_i = w_i'$ for $0 \leqslant i \leqslant d+1$. Hence (20) holds. Since $w_{d+1} = 0$, $p_{d+1}(A)v = 0$. The rest of the assertion is proved similary. $\square$

Theorem 6.4. *For $0 \leqslant i \leqslant d$,*

$$p_i(A)E_r^*W = E_{r+i}^*W, \tag{24}$$

$$p_i^*(A^*)E_tW = E_{t+i}W. \tag{25}$$

Proof. Let v be a nonzero vector in E_r^*W. By (20), $E_{r+i}^*A^i v$ spans $p_i(A)E_r^*W$. By Theorem 5.5, $E_{r+i}^*A^i v$ spans E_{r+i}^*W. From these comments, we obtain (24). The proof for (25) is similar. $\square$

Theorem 6.5. *For $0 \leqslant i \leqslant d$, the following hold on W:*

$$p_i(A)E_r^* = E_{r+i}^*A^iE_r^*,$$

$$p_i^*(A^*)E_t = E_{t+i}A^{*i}E_t.$$

Proof. Abbreviate $\Delta := p_i(A)E_r^* - E_{r+i}^*A^iE_r^*$. We will show that $\Delta = 0$ on W. For $0 \leqslant j \leqslant d$, let w_j be a nonzero vector in E_{r+j}^*W. Note that $\Delta w_j = 0$ for $1 \leqslant j \leqslant d$. By Lemma 6.3, $\Delta w_0 = 0$. Therefore, $\Delta = 0$ on W. The second assertion is proved similary. $\square$

Theorem 6.6. *The following hold:*

(i) *p_{d+1} is both the minimal polynomial and the characteristic polynomial of the action of A on W.*
(ii) $p_{d+1} = \prod_{i=0}^d (\lambda - \theta_{t+i})$.

(iii) p^*_{d+1} *is both the minimal polynomial and characteristic polynomial of the action of* A^* *on* W.
(iv) $p^*_{d+1} = \prod_{i=0}^{d} \left(\lambda - \theta^*_{r+i}\right)$.

Proof. (i) By Lemma 6.3, $p_{d+1}(A)E^*_r W = 0$. For $1 \leqslant i \leqslant d$,

$$\begin{aligned} p_{d+1}(A)E^*_{r+i}W &= p_{d+1}(A)p_i(A)E^*_r W \quad \text{by (24)} \\ &= p_i(A)p_{d+1}(A)E^*_r W \\ &= 0. \end{aligned}$$

Therefore, $p_{d+1}(A)E^*_{r+i}W = 0$ for $0 \leqslant i \leqslant d$. Hence by Lemma 3.1(iv), $p_{d+1}(A) = 0$ on W. By Theorem 5.5 and (20), $p_i(A) \neq 0$ on W for $0 \leqslant i \leqslant d$. From these comments, p_{d+1} is the minimal polynomial of the action of A on W. Since the characteristic polynomial of the action of A on W has degree $d+1$, it follows that p_{d+1} is also the characteristic polynomial of this action.

(ii) Immediate from (i) and the fact that $\{\theta_{t+i}\}_{i=0}^{d}$ are the eigenvalues of the action of A on W.

(iii) and (iv) Similar to the proofs of (i) and (ii). $\square$

7. The scalars ν, m_i

Let W be as in Assumption 3.4. In this section, we will investigate the algebraic properties of two more scalars associated with W, called the $m_i(W)$ and the $\nu(W)$.

Definition 7.1. For $0 \leqslant i \leqslant d$, define

$$m_i(W) = \mathrm{tr}_W \left(E_{t+i}E^*_r\right), \tag{26}$$

$$m^*_i(W) = \mathrm{tr}_W \left(E^*_{r+i}E_t\right). \tag{27}$$

Lemma 7.2. *For* $0 \leqslant i \leqslant d$, *the following* (i)–(iv) *hold on* W:

(i) $E_{t+i}E^*_r E_{t+i} = m_i(W)E_{t+i}$.
(ii) $E^*_r E_{t+i}E^*_r = m_i(W)E^*_r$.
(iii) $E^*_{r+i}E_t E^*_{r+i} = m^*_i(W)E^*_{r+i}$.
(iv) $E_t E^*_{r+i}E_t = m^*_i(W)E_t$.

Proof. (i) Let H_i denote the action of E_{t+i} on W. Since W is thin, $H_i\,\mathrm{End}(W)H_i$ has dimension 1. Note that H_i is a nonzero element of $H_i\,\mathrm{End}(W)H_i$, hence a basis for $H_i\,\mathrm{End}(W)H_i$. Thus there exists $\alpha \in \mathbb{C}$ such that $E_{t+i}E^*_r E_{t+i} = \alpha E_{t+i}$ on W. Taking the trace of both sides of this equation and using Definition 7.1 and the fact that $\mathrm{tr}_W(E^*_r) = 1$, we find that $\alpha = m_i(W)$.

(ii) Let L_r denote the action of E^*_r on W. Since W is thin, $L_r\,\mathrm{End}(W)L_r$ has dimension 1. Note that L_r is a nonzero element of $L_r\,\mathrm{End}(W)L_r$, hence a basis for $L_r\,\mathrm{End}(W)L_r$. Thus there exists $\alpha \in \mathbb{C}$ such that $E^*_r E_{t+i}E^*_r = \alpha E^*_r$ on W. Arguing as in the proof of (i), we find that $\alpha = m_i(W)$.

(iii) and (iv) Similar to the proofs of (i) and (ii). $\square$

Lemma 7.3. *The following hold:*

(i) $\sum_{i=0}^{d} m_i(W) = 1$.
(ii) $\sum_{i=0}^{d} m^*_i(W) = 1$.
(iii) $m_i(W) \in \mathbb{R},\ m_i(W) > 0 \quad (0 \leqslant i \leqslant d)$.
(iv) $m^*_i(W) \in \mathbb{R},\ m^*_i(W) > 0 \quad (0 \leqslant i \leqslant d)$.

Proof. (i) Observe that on W, $\sum_{i=0}^{d} E_{t+i} = I$. In this equation, multiply each term on the right by E^*_r, take the trace and use Definition 7.1 to obtain $\sum_{i=0}^{d} m_i(W) = 1$.

(ii) Similar to the proof of (i).

(iii) By Lemma 7.2(i), $m_i(W)$ is an eigenvalue of the real symmetric matrix $E_{t+i}E^*_rE_{t+i}$. Hence $m_i(W) \in \mathbb{R}$. Since $E_{t+i}E^*_rE_{t+i} = (E^*_rE_{t+i})^t(E^*_rE_{t+i})$ is positive definite, $m_i(W) > 0$.

(iv) Similar to the proof of (iii). □

Definition 7.4. Note that $m_0(W) = m^*_0(W)$. We denote the multiplicative inverse of this common value to be $\nu(W)$.

The following is an immediate consequence of Lemma 7.2 and Definition 7.4.

Lemma 7.5. *The following hold on* W*:*

(i) $\nu(W)E_tE^*_rE_t = E_t$.
(ii) $\nu(W)E^*_rE_tE^*_r = E^*_r$.

8. Two bases for W

Let W be as in Assumption 3.4. In this section, we will look at two bases for W called the standard basis and dual standard basis.

Theorem 8.1. *Let* u *and* v *be nonzero vectors in* E_tW *and* E^*_rW*, respectively. Then* (*i*) *and* (*ii*) *below hold:*

(i) $\{E^*_{r+i}u\}_{i=0}^d$ *is a basis for* W.
(ii) $\{E_{t+i}v\}_{i=0}^d$ *is a basis for* W.

Proof. (i) By Lemma 3.1(iv), it suffices to show that $E^*_{r+i}u \neq 0$ for $0 \leqslant i \leqslant d$. By Lemmas 3.1(ii) and 3.2(v), $E^*_{r+i}E_tW = E^*_{r+i}W \neq 0$. Since u spans E_tW, $E^*_{r+i}u$ spans $E^*_{r+i}E_tW$. Therefore, $E^*_{r+i}u \neq 0$.

(ii) Similar to (i). □

Definition 8.2. Let u and v be nonzero vectors in E_tW and E^*_rW, respectively. We call $\{E^*_{r+i}u\}_{i=0}^d$ (resp., $\{E_{t+i}v\}_{i=0}^d$) a *standard* (resp., *dual standard*) basis for W.

Theorem 8.3. *Let* $\{w_i\}_{i=0}^d$ *be a standard basis for* W *and* $\{w'_i\}_{i=0}^d$ *be a sequence of vectors in* W*. Then the following are equivalent:*

(i) $\{w'_i\}_{i=0}^d$ *is a standard basis for* W.
(ii) *There exists a nonzero* $\alpha \in \mathbb{C}$ *such that* $w'_i = \alpha w_i$ *for* $0 \leqslant i \leqslant d$.

Proof. By Definition 8.2, there exists a nonzero $u \in E_tW$ such that $w_i = E^*_{r+i}u$ for $0 \leqslant i \leqslant d$. Note that $\{w'_i\}$ is a standard basis for W if and only if there exists a nonzero $u' \in E_tW$ such that $w'_i = E^*_{r+i}u'$ for $0 \leqslant i \leqslant d$. Since u spans E_tW, $u' = \alpha u$ for some nonzero $\alpha \in \mathbb{C}$. The conclusion follows. □

Theorem 8.4. *Let* $\{v_i\}_{i=0}^d$ *be a dual standard basis for* W *and* $\{v'_i\}_{i=0}^d$ *be a sequence of vectors in* W. *Then the following are equivalent*.

(i) $\{v'_i\}_{i=0}^d$ *is a dual standard basis for* W.
(ii) *There exists a nonzero* $\alpha \in \mathbb{C}$ *such that* $v'_i = \alpha v_i$ *for* $0 \leqslant i \leqslant d$.

Proof. Similar to the proof of Theorem 8.3. □

We now give various characterizations of a standard basis and dual standard basis.

Theorem 8.5. *Let* $\{w_i\}_{i=0}^d$ *be a sequence of vectors in* W*, not all* 0. *Then* $\{w_i\}_{i=0}^d$ *is a standard basis for* W *if and only if both* (*i*) *and* (*ii*) *below hold:*

(i) $w_i \in E^*_{r+i}W (0 \leqslant i \leqslant d)$.
(ii) $\sum_{i=0}^{d} w_i \in E_t W$.

Proof. Suppose that $\{w_i\}_{i=0}^d$ is a standard basis for W. By Definition 8.2, there exists a nonzero $u \in E_tW$ such that $w_i = E^*_{r+i}u$ for $0 \leqslant i \leqslant d$. Thus, (i) holds. Combining Lemma 3.1(ii) and the fact that $\sum_{j=0}^{D} E^*_j = I$, we have $u = \sum_{i=0}^{d} E^*_{r+i}u$. From this comment, we find that $\sum_{i=0}^{d} w_i = \sum_{i=0}^{d} E^*_{r+i}u = u \in E_tW$. Hence, (ii) holds. Conversely, suppose that $\{w_i\}_{i=0}^d$ satisfies (i) and (ii). Let $u = \sum_{i=0}^{d} w_i$. By (ii) and the fact that not all of $\{w_i\}_{i=0}^d$ are 0, u is a nonzero vector in E_tW. By (i), $E^*_{r+i}u = w_i$ for $0 \leqslant i \leqslant d$. Therefore, $\{w_i\}_{i=0}^d$ is a standard basis for W. $\square$

Theorem 8.6. *Let $\{v_i\}_{i=0}^d$ be a sequence of vectors in W, not all 0. Then $\{v_i\}_{i=0}^d$ is a dual standard basis for W if and only if both (i) and (ii) below hold:*

(i) $v_i \in E_{t+i}W (0 \leqslant i \leqslant d)$.
(ii) $\sum_{i=0}^{d} v_i \in E^*_r W$.

Proof. Similar to the proof of Theorem 8.5. $\square$

Lemma 8.7. *Let $\{w_i\}_{i=0}^d$ be a basis for W. With respect to this basis, let B and B^* denote the matrix representations of A and A^*, respectively. Then $\{w_i\}_{i=0}^d$ is a standard basis for W if and only if both (i) and (ii) below hold:*

(i) *B has constant row sum θ_t.*
(ii) $B^* = \mathrm{diag}\left(\theta^*_r, \theta^*_{r+1}, \ldots, \theta^*_{r+d}\right)$.

Proof. Let $w = \sum_{i=0}^{d} w_i$. Note that $Aw = \sum_{i=0}^{d}\sum_{j=0}^{d} B_{ji}w_j$. Since E_tW is the eigenspace of A corresponding to θ_t, by the previous statement, B has constant row sum equal to θ_t if and only if $w \in E_tW$. Observe also that $w_i \in E^*_{r+i}W$ if and only if $B^* = \mathrm{diag}(\theta^*_r, \theta^*_{r+1}, \ldots, \theta^*_{r+d})$. The result follows from these comments and Theorem 8.5. $\square$

Lemma 8.8. *Let $\{v_i\}_{i=0}^d$ be a basis for W. With respect to this basis, let B and B^* be the matrix representations of A and A^*, respectively. Then $\{v_i\}_{i=0}^d$ is a dual standard basis for W if and only if both (i) and (ii) below hold:*

(i) *B^* has constant row sum θ^*_r.*
(ii) $B = \mathrm{diag}(\theta_t, \theta_{t+1}, \ldots, \theta_{t+d})$.

Proof. Similar to the proof of Lemma 8.7. $\square$

Definition 8.9. Define the two maps $\flat : \mathrm{End}(W) \to \mathrm{Mat}_{d+1}(\mathbb{C})$ and $\sharp : \mathrm{End}(W) \to \mathrm{Mat}_{d+1}(\mathbb{C})$ as follows: For every $Y \in \mathrm{End}(W)$, $Y^\flat$ (resp., $Y^\sharp$) is the matrix representation of Y with respect to a standard basis (resp., dual standard basis) for W. Note that $Y^\flat$ (resp., $Y^\sharp$) is independent of the choice of standard basis (resp., dual standard basis) by Theorem 8.3 (resp., Theorem 8.4).

Theorem 8.10. *With reference to Definition 8.9, the following hold:*

(i) *$A^\flat$ has constant row sum θ_t.*
(ii) $A^{*\flat} = \mathrm{diag}\left(\theta^*_r, \theta^*_{r+1}, \ldots, \theta^*_{r+d}\right)$.
(iii) *$A^{*\sharp}$ has constant row sum θ^*_r.*
(iv) $A^\sharp = \mathrm{diag}(\theta_t, \theta_{t+1}, \ldots, \theta_{t+d})$.

Proof. Immediate from Lemmas 8.7 and 8.8. $\square$

9. The scalars $b_i(W)$, $c_i(W)$

Let W be as in Assumption 3.4 and let $\flat$, $\sharp$ be the maps in Definition 8.9. In this section, we will take a close look at the entries of $A^{\flat}$ and $A^{*\sharp}$.

By Lemmas 4.1 and 4.2, the matrices $A^{\flat}$ and $A^{*\sharp}$ are tridiagonal. Moreover, by Lemma 5.3, the (i,i)-entry of these matrices are $a_i(W)$ and $a_i^*(W)$, respectively. We now take a close look at the superdiagonal and subdiagonal entries of these matrices.

Definition 9.1. Define

$$b_i(W) = (A^{\flat})_{i,i+1}, \quad b_i^*(W) = (A^{*\sharp})_{i,i+1} \quad (0 \leqslant i \leqslant d-1),$$

$$c_i(W) = (A^{\flat})_{i,i-1}, \quad c_i^*(W) = (A^{*\sharp})_{i,i-1} \quad (1 \leqslant i \leqslant d).$$

Thus,

$$A^{\flat} = \begin{pmatrix} a_0(W) & b_0(W) & & & & \mathbf{0} \\ c_1(W) & a_1(W) & b_1(W) & & & \\ & c_2(W) & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & a_{d-1}(W) & b_{d-1}(W) \\ \mathbf{0} & & & & c_d(W) & a_d(W) \end{pmatrix}, \tag{28}$$

$$A^{*\sharp} = \begin{pmatrix} a_0^*(W) & b_0^*(W) & & & & \mathbf{0} \\ c_1^*(W) & a_1^*(W) & b_1^*(W) & & & \\ & c_2^*(W) & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & a_{d-1}^*(W) & b_{d-1}^*(W) \\ \mathbf{0} & & & & c_d^*(W) & a_d^*(W) \end{pmatrix}. \tag{29}$$

For notational convenience, define $b_d(W) = 0$, $c_0(W) = 0$ (resp., $b_d^*(W) = 0$, $c_0^*(W) = 0$). Observe that by Lemmas 4.1(iii) and 4.2(iii), $b_i(W)$, $b_i^*(W)$ $(0 \leqslant i \leqslant d-1)$, $c_i(W)$, $c_i^*(W)$ $(1 \leqslant i \leqslant d)$ are all nonzero.

Definition 9.2. By the *intersection numbers* (resp., *dual intersection numbers*) of W, we mean the $a_i(W)$, $b_i(W)$, $c_i(W)$ (resp., $a_i^*(W)$, $b_i^*(W)$, $c_i^*(W)$).

Lemma 9.3. *The following hold:*

(i) $b_{i-1}(W)c_i(W) = x_i(W) \quad (1 \leqslant i \leqslant d)$.
(ii) $c_i(W) + a_i(W) + b_i(W) = \theta_t \quad (0 \leqslant i \leqslant d)$.
(iii) $b_{i-1}^*(W)c_i^*(W) = x_i^*(W) \quad (1 \leqslant i \leqslant d)$.
(iv) $c_i^*(W) + a_i^*(W) + b_i^*(W) = \theta_r^* \quad (0 \leqslant i \leqslant d)$.
(v) $b_i(W) \in \mathbb{R}$, $c_i(W) \in \mathbb{R} \quad (0 \leqslant i \leqslant d)$.
(vi) $b_i^*(W) \in \mathbb{R}$, $c_i^*(W) \in \mathbb{R} \quad (0 \leqslant i \leqslant d)$.

Proof. (i) Immediate from Lemma 5.3(ii).

(ii) Immediate from Theorem 8.10(i).

(iii) and (iv) Similar to the proofs of (i) and (ii).

(v) Recall that $a_0(W) \in \mathbb{R}$ by Lemma 5.10(iii). Since $\theta_t \in \mathbb{R}$ and $a_0(W) + b_0(W) = \theta_t$, we have $b_0(W) \in \mathbb{R}$. By Lemma 5.10(iii) and (iv), we obtain $a_i(W) \in \mathbb{R}$ and $x_i(W) \in \mathbb{R}$ for $0 \leqslant i \leqslant d$. Combining this with (i), (ii) and the fact that $b_0(W) \in \mathbb{R}$ and $b_i(W) \neq 0$ for $0 \leqslant i \leqslant d-1$, we find that $b_i(W) \in \mathbb{R}$ and $c_i(W) \in \mathbb{R}$ for $0 \leqslant i \leqslant d$.

(vi) Similar to the proof of (v). $\square$

Lemma 9.4. *For $0 \leqslant i \leqslant d$,*

$$b_0(W)b_1(W)\cdots b_{i-1}(W) = p_i(\theta_t), \tag{30}$$
$$b_0^*(W)b_1^*(W)\cdots b_{i-1}^*(W) = p_i^*\left(\theta_r^*\right), \tag{31}$$

where $p_i = p_i^W$, $p_i^* = p_i^{*W}$ are from Definitions 6.1 and 6.2.

Proof. We prove (30) by induction on i. It can be verified that (30) is true for $i = 0, 1$. Fix $2 \leqslant i \leqslant d$. By (18),

$$p_i(\theta_t) = (\theta_t - a_{i-1}(W))p_{i-1}(\theta_t) - x_{i-1}(W)p_{i-2}(\theta_t). \tag{32}$$

Eliminate $x_{i-1}(W)$ and $a_{i-1}(W)$ in (32) using Lemma 9.3(i), (ii). Evaluate the result using the inductive hypothesis to obtain the desired result. Eq. (31) is proved similarly. □

Theorem 9.5. *The following (i)–(iv) hold:*

(i) $b_i(W) = \frac{p_{i+1}(\theta_t)}{p_i(\theta_t)} \quad (0 \leqslant i \leqslant d-1)$.
(ii) $c_i(W) = \frac{x_i(W)p_{i-1}(\theta_t)}{p_i(\theta_t)} \quad (1 \leqslant i \leqslant d)$.
(iii) $b_i^*(W) = \frac{p_{i+1}^*\left(\theta_r^*\right)}{p_i^*\left(\theta_r^*\right)} \quad (0 \leqslant i \leqslant d-1)$.
(iv) $c_i^*(W) = \frac{x_i^*(W)p_{i-1}^*\left(\theta_r^*\right)}{p_i^*\left(\theta_r^*\right)} \quad (1 \leqslant i \leqslant d)$.

In the above lines, $p_j = p_j^W$, $p_j^* = p_j^{*W}$ are from Definitions 6.1 and 6.2.

Proof. (i) Immediate from Lemma 9.4.
(ii) Immediate from (i) and Lemma 9.3(i).
(iii) and (iv) Similar to the proofs of (i) and (ii). □

Lemma 9.6 [9, Theorem 4.1(vi)]. *Let W be the trivial T-module. For $0 \leqslant i \leqslant D$, let a_i, b_i, c_i (resp., a_i^*, b_i^*, c_i^*) be the intersection (resp., dual intersection) numbers of Γ. Then*

(i) $a_i(W) = a_i$, $b_i(W) = b_i$, $c_i(W) = c_i$,
(ii) $a_i^*(W) = a_i^*$, $b_i^*(W) = b_i^*$, $c_i^*(W) = c_i^*$.

We finish this section with a few comments.

Lemma 9.7. *Let W, W' be thin irreducible T-modules. The following are equivalent.*

(i) *W and W' are isomorphic T-modules.*
(ii) *W and W' have the same endpoint, dual endpoint, diameter and intersection numbers.*
(iii) *W and W' have the same endpoint, dual endpoint, diameter and dual intersection numbers.*

Proof. (i) ⇒ (ii) Suppose that W and W' are isomorphic T-modules. Let $\phi : W \to W'$ be an isomorphism of T-modules. Thus, $\phi(E_iW) = E_iW'$. Hence $E_iW \neq 0$ if and only if $E_iW' \neq 0$. Similarly, $E_i^*W \neq 0$ if and only if $E_i^*W' \neq 0$. Therefore, W and W' have the same endpoint, dual endpoint and diameter. Since W and W' are isomorphic, the matrices representing the action of A on W and W' are the same. Hence they have the same intersection numbers.

(ii) ⇐ (i) Suppose that W and W' have the same endpoint r, dual endpoint t and diameter d. Suppose also that they have the same intersection numbers. For $0 \leqslant i \leqslant d$, let $w_i = E_{r+i}^*u$ and $w_i' = E_{r+i}^*u'$, where

u and u' are nonzero vectors in E_tW and E_tW', respectively. Since W and W' both have dimension $d+1$, there exists a vector space isomorphism $\phi : W \to W'$ such that $\phi(w_i) = w_i'$. Since $w_i \in E^*_{r+i}W$, it can be easily verified that $(\phi A^* - A^*\phi)w_i = 0$ for $0 \leqslant i \leqslant D$. By (28) and the fact that W and W' have the same intersection numbers, $(\phi A - A\phi)w_i = 0$ for $0 \leqslant i \leqslant d$. From these comments, $(\phi A - A\phi)W = 0$ and $(\phi A^* - A^*\phi)W = 0$. Since T is generated by A, A^*, we find that ϕ is a T-module isomorphism. Therefore, W and W' are isomorphic T-modules.

(i) $\Leftrightarrow$ (iii) Similar to the proof of (i) $\Leftrightarrow$ (ii). $\square$

10. The scalar $k_i(W)$

Let W be as in Assumption 3.4. In this section, we will look at a sequence of scalars closely related with the $m_i(W)$.

Definition 10.1. For $0 \leqslant i \leqslant d$, define

$$k_i(W) = m_i^*(W)\nu(W),$$
$$k_i^*(W) = m_i(W)\nu(W),$$

where $m_i(W),\ m_i^*(W),\ \nu(W)$ are from Definitions 7.1 and 7.4.

Lemma 10.2. *The following (i)–(iii) hold:*

(i) $k_0(W) = 1,\ k_0^*(W) = 1.$
(ii) $\sum_{i=0}^{d} k_i(W) = \nu(W).$
(iii) $\sum_{i=0}^{d} k_i^*(W) = \nu(W).$
(iv) $k_i(W) > 0,\ k_i^*(W) > 0 \quad (0 \leqslant i \leqslant d).$

In the above lines, $\nu(W)$ is from Definition 7.4.

Proof. (i) Immediate from Definition 10.1.
(ii) Immediate from Lemma 7.3(i) and Definition 10.1.
(iii) Similar to the proof of (ii).
(iv) Immediate from Lemma 7.3(iii), (iv) and Definitions 7.4 and 10.1. $\square$

We now relate $k_i(W)$ (resp., $k_i^*(W)$) and the intersection (resp., dual intersection) numbers of W.

Lemma 10.3. *For $0 \leqslant i \leqslant d$,*

$$k_i(W)c_i(W) = k_{i-1}(W)b_{i-1}(W), \tag{33}$$
$$k_i^*(W)c_i^*(W) = k_{i-1}^*(W)b_{i-1}^*(W), \tag{34}$$

where $b_j(W),\ b_j^*(W),\ c_j(W),\ c_j^*(W)$ are from Definition 9.1 and $b_{-1}(W) = 0,\ b_{-1}^*(W) = 0$.

Proof. We proceed by induction on i. Since $c_0(W) = 0$, Eq. (33) holds for $i = 0$. Assume $1 \leqslant i \leqslant d$. By Definition 9.1, on W

$$AE^*_{r+i}E_t = b_{i-1}(W)E^*_{r+i-1}E_t + a_i(W)E^*_{r+i}E_t + c_{i+1}(W)E^*_{r+i+1}E_t, \tag{35}$$

where $c_{d+1}(W) = 0$. Take the trace of both sides of (35). Evaluate this using Definition 7.1 and the fact that $E_tA = \theta_tE_t$. Multiplying $\nu(W)$ on both sides of the resulting equation and using Definition 10.1 we obtain

$$\theta_t k_i(W) = b_{i-1}(W)k_{i-1}(W) + a_i(W)k_i(W) + c_{i+1}(W)k_{i+1}(W). \tag{36}$$

Solving for $c_{i+1}(W)k_{i+1}(W)$ in (36) using the inductive hypothesis and Lemma 9.3(ii), we find that (33) holds for $i+1$. The proof of (34) is similar. $\square$

Theorem 10.4. *For $0 \leqslant i \leqslant d$,*

$$k_i(W) = \frac{b_0(W)b_1(W)\cdots b_{i-1}(W)}{c_1(W)c_2(W)\cdots c_i(W)}, \tag{37}$$

$$k_i^*(W) = \frac{b_0^*(W)b_1^*(W)\cdots b_{i-1}^*(W)}{c_1^*(W)c_2^*(W)\cdots c_i^*(W)}, \tag{38}$$

where $b_j(W),\ b_j^*(W),\ c_j(W),\ c_j^*(W)$ are from Definition 9.1.

Proof. Solve for $k_i(W)$ and $k_i^*(W)$ in Lemma 10.3 recursively to obtain the desired result. $\square$

Corollary 10.5. *Let W be the trivial T-module. Then for $0 \leqslant i \leqslant D$,*

$$k_i(W) = k_i, \quad k_i^*(W) = m_i,$$

where k_i is the ith valency of Γ and m_i is the multiplicity of Γ associated with E_i.

Proof. Immediate from (1), (4), Lemma 9.6 and Theorem 10.4. $\square$

11. The polynomials u_i and v_i

Let W be as in Assumption 3.4. In this section, we will look at two normalizations of the polynomials p_i and p_i^* in Definitions 6.1 and 6.2.

Definition 11.1. Define $v_i = v_i^W$ and $v_i^* = v_i^{*W}$ in $\mathbb{C}[\lambda]$ by

$$v_i = \frac{p_i}{c_1(W)c_2(W)\cdots c_i(W)} \quad (0 \leqslant i \leqslant d), \tag{39}$$

$$v_i^* = \frac{p_i^*}{c_1^*(W)c_2^*(W)\cdots c_i^*(W)} \quad (0 \leqslant i \leqslant d), \tag{40}$$

where $p_i = p_i^W,\ p_i^* = p_i^{*W}$ are from Definitions 6.1 and 6.2 and $c_j(W),\ c_j^*(W)$ are from Definition 9.1. For notational convenience, define $v_{-1} = 0,\ v_{-1}^* = 0$.

Lemma 11.2. *For $0 \leqslant i \leqslant d$,*

$$v_i(\theta_t) = k_i(W), \quad v_i^*(\theta_r^*) = k_i^*(W),$$

where $v_i = v_i^W$, $v_i^* = v_i^{*W}$ are from Definition 11.1.

Proof. Immediate from Lemma 9.4, Theorem 10.4 and Definition 11.1. $\square$

Lemma 11.3. *With reference to Definition 11.1, for $0 \leqslant i \leqslant d-1$,*

$$\lambda v_i = b_{i-1}(W)v_{i-1} + a_i(W)v_i + c_{i+1}(W)v_{i+1}, \tag{41}$$

$$\lambda v_i^* = b_{i-1}^*(W)v_{i-1}^* + a_i^*(W)v_i^* + c_{i+1}^*(W)v_{i+1}^*, \tag{42}$$

where $b_{-1}(W) = 0,\ b_{-1}^*(W) = 0$. Moreover,

$$\lambda v_d - a_d(W)v_d - b_{d-1}(W)v_{d-1} = c^{-1}p_{d+1},$$

$$\lambda v_d^* - a_d^*(W)v_d^* - b_{d-1}^*(W)v_{d-1}^* = c^{*-1}p_{d+1}^*,$$

where

$$c = c_1(W)c_2(W)\cdots c_d(W),$$
$$c^* = c_1^*(W)c_2^*(W)\cdots c_d^*(W).$$

Proof. To obtain (41), divide both sides of (18) by $c_1(W)c_2(W)\cdots c_i(W)$ and eliminate $x_i(W)$ using Lemma 9.3(i). The proof of (42) is similar. □

Theorem 11.4. *With reference to Definition 11.1, for* $0 \leqslant i \leqslant d$,

$$v_i(A)E_r^*u = E_{r+i}^*u, \quad v_i^*(A^*)E_tv = E_{t+i}v, \tag{43}$$

where u and v are nonzero vectors in E_tW and E_r^*W, respectively.

Proof. For $0 \leqslant i \leqslant d$, let $w_i = E_{r+i}^*u$ and $w_i' = v_i(A)E_r^*u$. By (28),

$$Aw_i = b_{i-1}(W)w_{i-1} + a_i(W)w_i + c_{i+1}(W)w_{i+1} \quad (0 \leqslant i \leqslant d-1), \tag{44}$$

where $b_{-1}(W) = 0$. Using (41), we obtain

$$Aw_i' = b_{i-1}(W)w_{i-1}' + a_i(W)w_i' + c_{i+1}(W)w_{i+1}' \quad (0 \leqslant i \leqslant d-1). \tag{45}$$

Using the fact that $w_0 = w_0'$ and comparing (44) and (45), we obtain the equation on the left of (43). The equation on the right of (43) can be similarly obtained. □

Definition 11.5. For $0 \leqslant i \leqslant d$, define $u_i = u_i^W$ and $u_i^* = u_i^{*W}$ in $\mathbb{C}[\lambda]$ as follows:

$$u_i = \frac{p_i}{p_i(\theta_t)}, \tag{46}$$
$$u_i^* = \frac{p_i^*}{p_i^*(\theta_r^*)}, \tag{47}$$

where $p_i = p_i^W$, $p_i^* = p_i^{*W}$ are from Definitions 6.1 and 6.2. For notational convenience, define $u_{-1} = 0$, $u_{-1}^* = 0$.

Lemma 11.6. *With reference to Definition 11.1, for* $0 \leqslant i \leqslant d$,

$$v_i = k_i(W)u_i, \quad v_i^* = k_i^*(W)u_i^*,$$

where $u_i = u_i^W$, $u_i^* = u_i^{*W}$ are from Definition 11.5 and $k_i(W)$, $k_i^*(W)$ are from Definition 10.1.

Proof. Immediate from Lemma 9.4, Theorem 10.4 and Definitions 11.1 and 11.5. □

Lemma 11.7. *With reference to Definition 11.5, for* $0 \leqslant i \leqslant d-1$,

$$\lambda u_i = c_i(W)u_{i-1} + a_i(W)u_i + b_i(W)u_{i+1}, \tag{48}$$
$$\lambda u_i^* = c_i^*(W)u_{i-1}^* + a_i^*(W)u_i^* + b_i^*(W)u_{i+1}^*. \tag{49}$$

Moreover,

$$\lambda u_d - c_d(W)u_{d-1} - a_d(W)u_d = p_{d+1}/p_d(\theta_t),$$
$$\lambda u_d^* - c_d^*(W)u_{d-1}^* - a_d^*(W)u_d^* = p_{d+1}^*/p_d^*(\theta_r^*).$$

Proof. To obtain (48), divide both sides of (18) by $p_i(\theta_t)$ and eliminate $x_i(W)$ using Lemma 9.3(i). Evaluate the result using Lemma 9.4. The proof of (49) is similar. □

Theorem 11.8. *With reference to Definition 11.5, for* $0 \leqslant i, j \leqslant d$,

$$\theta_{t+j} u_i(\theta_{t+j}) = c_i(W) u_{i-1}(\theta_{t+j}) + a_i(W) u_i(\theta_{t+j}) + b_i(W) u_{i+1}(\theta_{t+j}),$$
$$\theta^*_{r+j} u^*_i\left(\theta^*_{r+j}\right) = c^*_i(W) u^*_{i-1}\left(\theta^*_{r+j}\right) + a^*_i(W) u^*_i\left(\theta^*_{r+j}\right) + b^*_i(W) u^*_{i+1}\left(\theta^*_{r+j}\right),$$

where $u_{d+1} = 0,\ u^*_{d+1} = 0$.

Proof. Immediate from (48) and (49) with $\lambda = \theta_{t+j}$ and $\lambda = \theta^*_{r+j}$. $\square$

12. Some inner products and the Askey–Wilson duality

Let W be as in Assumption 3.4. In this section, we will look at all inner products involving the elements of a standard basis and a dual standard basis for W. Using these inner products, we will show that all the polynomials associated with W satisfy relations known as the Askey–Wilson duality.

Throughout the entire section, u and v are nonzero vectors in $E_t W$ and $E^*_r W$, respectively. Recall that by Definition 8.2, $\{E^*_{r+i}u\}_{i=0}^d$ (resp., $\{E_{t+i}v\}_{i=0}^d$) is a standard basis (resp., dual standard basis) for W. By (2) and (6), each of these bases is orthogonal. We now compute some square norms.

Theorem 12.1. *For* $0 \leqslant i \leqslant d$,

$$\|E^*_{r+i}u\|^2 = \|u\|^2 k_i(W)/\nu(W), \tag{50}$$
$$\|E_{t+i}v\|^2 = \|v\|^2 k^*_i(W)/\nu(W), \tag{51}$$

where $\nu(W)$ *is from Definition 7.4 and* $k_i(W),\ k^*_i(W)$ *are from Definition 10.1.*

Proof. Note that

$$\begin{aligned}
\|E^*_{r+i}u\|^2 &= \left\langle E^*_{r+i}u, E^*_{r+i}u\right\rangle, \\
&= \left\langle u, E^{*2}_{r+i}u\right\rangle, \\
&= \left\langle u, E^*_{r+i}u\right\rangle, \\
&= \langle u, v_i(A)E^*_r u\rangle \quad \text{by Lemma 11.4}, \\
&= \langle v_i(A)u, E^*_r u\rangle, \\
&= \langle v_i(\theta_t)u, E^*_r u\rangle, \\
&= k_i(W)\langle u, E^*_r u\rangle \quad \text{by Lemma 11.2}.
\end{aligned}$$

Since $u \in E_t W$, $u = E_t u$. Using this we find that $\langle u, E^*_r u\rangle = \langle E_t u, E^*_r E_t u\rangle = \langle u, E_t E^*_r E_t u\rangle$. Evaluating $E_t E^*_r E_t$ using Lemma 7.5(i) we find that $\langle u, E^*_r u\rangle = \|u\|^2/\nu(W)$. Thus, we obtain (50). Eq. (51) is proved similarly. $\square$

Our next goal is to compute the inner product between the elements of $\{E^*_{r+i}u\}_{i=0}^d$ and $\{E_{t+i}v\}_{i=0}^d$. We need the following lemma.

Lemma 12.2. *The following hold:*

(i) $\langle E^*_r u, E_t v\rangle = \langle u, v\rangle/\nu(W)$.
(ii) $E^*_r u = \frac{\langle u,v\rangle}{\|v\|^2}v$.
(iii) $E_t v = \frac{\langle v,u\rangle}{\|u\|^2}u$.
(iv) $\langle u, v\rangle \neq 0$.
(v) $\nu(W)|\langle u, v\rangle|^2 = \|u\|^2\|v\|^2$.

In the above lines, $\nu(W)$ *is from Definition 7.4.*

Proof. (i) Since $v \in E_r^*W$, $v = E_r^*v$. Using this we find that $\langle E_r^*u, E_tv\rangle = \langle E_r^*u, E_tE_r^*v\rangle = \langle u, E_r^*E_tE_r^*v\rangle$. Evaluate $E_r^*E_tE_r^*$ using Lemma 7.5(ii) to obtain the desired result.

(ii) Since v spans E_r^*W, $E_r^*u = \alpha v$ for some $\alpha \in \mathbb{C}$. Thus $\langle E_r^*u, v\rangle = \alpha\|v\|^2$. Since $\langle E_r^*u, v\rangle = \langle u, E_r^*v\rangle = \langle u, v\rangle$, we find that $\alpha = \frac{\langle u,v\rangle}{\|v\|^2}$.

(iii) Similar to the proof of (ii).

(iv) Observe that $E_r^*u \neq 0$ since it is an element of a standard basis. It follows from this and (ii) that $\langle u, v\rangle \neq 0$.

(v) Eliminate E_r^*u and E_tv in (i) using (ii) and (iii). $\square$

Theorem 12.3. *For* $0 \leqslant i, j \leqslant d$,

$$\left\langle E_{r+i}^*u, E_{t+j}v\right\rangle = u_i(\theta_{t+j})k_i(W)k_j^*(W)\langle u, v\rangle/\nu(W), \tag{52}$$

$$\left\langle E_{r+i}^*u, E_{t+j}v\right\rangle = u_j^*\left(\theta_{r+i}^*\right)k_i(W)k_j^*(W)\langle u, v\rangle/\nu(W), \tag{53}$$

where $\nu(W)$, $k_i(W)$, $k_j^*(W)$ *are from Definitions 7.4 and 10.1 and* $u_i = u_i^W$, $u_j^* = u_j^{*W}$ *are from Definition 11.5.*

Proof. Note that

$$\begin{aligned}
\langle E_{r+i}^*u, E_{t+j}v\rangle &= \langle v_i(A)E_r^*u, E_{t+j}v\rangle \quad \text{by Theorem 11.4},\\
&= \langle E_r^*u, v_i(A)E_{t+j}v\rangle,\\
&= v_i(\theta_{t+j})\,\langle E_r^*u, E_{t+j}v\rangle,\\
&= v_i(\theta_{t+j})\left\langle E_r^*u, v_j^*(A^*)E_tv\right\rangle \quad \text{by Theorem 11.4},\\
&= v_i(\theta_{t+j})\left\langle v_j^*(A^*)E_r^*u, E_tv\right\rangle,\\
&= v_i(\theta_{t+j})v_j^*\left(\theta_r^*\right)\langle E_r^*u, E_tv\rangle,\\
&= v_i(\theta_{t+j})v_j^*\left(\theta_r^*\right)\langle u, v\rangle/\nu(W) \quad \text{by Lemma 12.2(i)}.
\end{aligned}$$

The result then follows from Lemmas 11.2 and 11.6. Eq. (53) is proved similarly. $\square$

Theorem 12.4. *For* $0 \leqslant i, j \leqslant d$,

$$u_i(\theta_{t+j}) = u_j^*\left(\theta_{r+i}^*\right), \tag{54}$$

where $u_i = u_i^W$ *and* $u_j^* = u_j^{*W}$ *are from Definition 11.5.*

Proof. Compare (52) with (53). $\square$

Theorem 12.5. *For* $0 \leqslant i, j \leqslant d$,

$$\frac{p_i(\theta_{t+j})}{p_i(\theta_t)} = \frac{p_j^*\left(\theta_{r+i}^*\right)}{p_j^*\left(\theta_r^*\right)}, \tag{55}$$

$$\frac{v_i(\theta_{t+j})}{k_i(W)} = \frac{v_j^*\left(\theta_{r+i}^*\right)}{k_j^*(W)}, \tag{56}$$

where $p_i = p_i^W$, $p_i^* = p_i^{*W}$, $v_i = v_i^W$, $v_i^* = v_i^{*W}$ *are from Definitions 6.1, 6.2 and 11.1.*

Proof. Immediate from Definition 11.5 and Theorems 11.6 and 12.4. $\square$

Eqs. (54)–(56) are known as the *Askey–Wilson duality*. Combining Theorem 11.8 and 12.4, we obtain the following result.

Theorem 12.6. *For $0 \leqslant i, j \leqslant d$,*

$$\theta_{t+j} u_j^*\left(\theta^*_{r+i}\right) = b_i(W) u_j^*\left(\theta^*_{r+i+1}\right) + a_i(W) u_j^*\left(\theta^*_{r+i}\right) + c_i(W) u_j^*\left(\theta^*_{r+i-1}\right), \tag{57}$$

$$\theta^*_{r+j} u_j(\theta_{t+i}) = b_i^*(W) u_j(\theta_{t+i+1}) + a_i^*(W) u_j(\theta_{t+i}) + c_i^*(W) u_j(\theta_{t+i-1}), \tag{58}$$

where $u_j = u_j^W$ and $u_j^* = u_j^{*W}$ are from Definition 11.5.

13. The orthogonality relations

Let W be as in Assumption 3.4. In this section, we display the transition matrix relating a standard basis and a dual standard basis. Using this and the results of the previous section, we display the orthogonality relations satisfied by the polynomials we have seen in this paper.

Theorem 13.1. *Let u and v be nonzero vectors in E_tW and E_r^*W, respectively. For $0 \leqslant i \leqslant d$,*

$$E^*_{r+i} u = \frac{\langle u, v\rangle}{\|v\|^2} \sum_{j=0}^{d} v_i(\theta_{t+j}) E_{t+j} v, \tag{59}$$

$$E_{t+i} v = \frac{\langle v, u\rangle}{\|u\|^2} \sum_{j=0}^{d} v_i^*\left(\theta^*_{r+j}\right) E^*_{r+j} u, \tag{60}$$

where $v_i = v_i^W$, $v_i^* = v_i^{*W}$ are from Definition 11.1.

Proof. Combining Lemma 3.2(ii) and the fact that $\sum_{j=0}^{D} E_j = I$, we find that $v = \sum_{j=0}^{d} E_{t+j} v$. By Theorem 11.4 and Lemma 12.2(ii), $E^*_{r+i} u = \frac{\langle u,v\rangle}{\|v\|^2} v_i(A) v$. Therefore,

$$\begin{aligned} E^*_{r+i} u &= \frac{\langle u, v\rangle}{\|v\|^2} v_i(A) v, \\ &= \frac{\langle u, v\rangle}{\|v\|^2} v_i(A) \sum_{j=0}^{d} E_{t+j} v, \\ &= \frac{\langle u, v\rangle}{\|v\|^2} \sum_{j=0}^{d} v_i(\theta_{t+j}) E_{t+j} v. \end{aligned}$$

Hence, (59) holds. Eq. (60) is proved similarly. □

Theorem 13.2. *For $0 \leqslant i, j \leqslant d$,*

$$\sum_{h=0}^{d} v_i(\theta_{t+h}) v_j(\theta_{t+h}) k_h^*(W) = \delta_{ij} \nu(W) k_i(W), \tag{61}$$

$$\sum_{h=0}^{d} v_h(\theta_{t+i}) v_h(\theta_{t+j}) (k_h(W))^{-1} = \delta_{ij} \nu(W) \left(k_i^*(W)\right)^{-1}, \tag{62}$$

and

$$\sum_{h=0}^{d} v_i^*\left(\theta^*_{r+h}\right) v_j^*\left(\theta^*_{r+h}\right) k_h(W) = \delta_{ij} \nu(W) k_i^*(W), \tag{63}$$

$$\sum_{h=0}^{d} v_h^*\left(\theta^*_{r+i}\right) v_h^*\left(\theta^*_{r+j}\right)\left(k_h^*(W)\right)^{-1} = \delta_{ij}\nu(W)(k_i(W))^{-1}, \tag{64}$$

where $\nu(W)$, $k_h(W)$, $k_h^*(W)$ *are from Definitions 7.4 and 10.1 and* $v_h = v_h^W$, $v_h^* = v_h^{*W}$ *are from Definition 11.1.*

Proof. Concerning (61), let u be a nonzero vector in E_tW. We compute $\langle E^*_{r+i}u, E^*_{r+j}u\rangle$ in two ways. First, by (6) and (50), $\langle E^*_{r+i}u, E^*_{r+j}u\rangle = \delta_{ij}\|u\|^2 k_i(W)/\nu(W)$. Secondly, we compute $\langle E^*_{r+i}u, E^*_{r+j}u\rangle$ by evaluating each of $E^*_{r+i}u$ and $E^*_{r+j}u$ using (59). Simplify the result using (51) and Lemma 12.2(v). We find that $\langle E^*_{r+i}u, E^*_{r+j}u\rangle$ is equal to $\|u\|^2/(\nu(W))^2$ times the left side of (61). Eq. (61) follows from these comments. Similarly, we obtain (63). To obtain (62), evaluate (63) using (56). To obtain (64), evaluate (61) using (56). □

Theorem 13.3. *For* $0 \leqslant i, j \leqslant d$,

$$\sum_{h=0}^{d} u_i(\theta_{t+h})u_j(\theta_{t+h})k_h^*(W) = \delta_{ij}\nu(W)(k_i(W))^{-1}, \tag{65}$$

$$\sum_{h=0}^{d} u_h(\theta_{t+i})u_h(\theta_{t+j})k_h(W) = \delta_{ij}\nu(W)\left(k_i^*(W)\right)^{-1}, \tag{66}$$

and

$$\sum_{h=0}^{d} u_i^*\left(\theta^*_{r+h}\right) u_j^*\left(\theta^*_{r+h}\right) k_h(W) = \delta_{ij}\nu(W)\left(k_i^*(W)\right)^{-1}, \tag{67}$$

$$\sum_{h=0}^{d} u_h^*\left(\theta^*_{r+i}\right) u_h^*\left(\theta^*_{r+j}\right) k_h^*(W) = \delta_{ij}\nu(W)(k_i(W))^{-1}, \tag{68}$$

where $\nu(W)$, $k_h(W)$, $k_h^*(W)$ *are from Definitions 7.4 and 10.1 and* $u_h = u_h^W$, $u_h^* = u_h^{*W}$ *are from Definition 11.5.*

Proof. Evaluate each of (61)–(64) using Lemma 11.6. □

Theorem 13.4. *For* $0 \leqslant i, j \leqslant d$,

$$\sum_{h=0}^{d} p_i(\theta_{t+h})p_j(\theta_{t+h})k_h^*(W) = \delta_{ij}\nu(W)x_1(W)x_2(W)\cdots x_i(W), \tag{69}$$

$$\sum_{h=0}^{d} \frac{p_h(\theta_{t+i})p_h(\theta_{t+j})}{x_1(W)x_2(W)\cdots x_h(W)} = \delta_{ij}\nu(W)\left(k_i^*(W)\right)^{-1}, \tag{70}$$

and

$$\sum_{h=0}^{d} p_i^*\left(\theta^*_{r+h}\right) p_j^*\left(\theta^*_{r+h}\right) k_h(W) = \delta_{ij}\nu(W)x_1^*(W)x_2^*(W)\cdots x_i^*(W), \tag{71}$$

$$\sum_{h=0}^{d} \frac{p_h^*\left(\theta^*_{r+i}\right) p_h^*\left(\theta^*_{r+j}\right)}{x_1^*(W)x_2^*(W)\cdots x_h^*(W)} = \delta_{ij}\nu(W)(k_i(W))^{-1}, \tag{72}$$

where $x_h(W)$, $\nu(W)$, $k_h(W)$, $k_h^*(W)$ are from Definitions 5.2, 7.4 and 10.1 and $p_h = p_h^W$, $p_h^* = p_h^{*W}$ are from Definitions 6.1 and 6.2.

Proof. Evaluate each of (61)–(64) using Definition 11.1. Simplify the result using Lemma 9.3(i) and (iii). $\square$

We now present Theorem 13.2 in matrix form.

Definition 13.5. Define matrices $P = P(W)$ and $P^* = P^*(W)$ in $\mathrm{Mat}_{d+1}(\mathbb{C})$ as follows. For $0 \leqslant i, j \leqslant d$, their (i, j)-entries are

$$P_{ij} = v_j(\theta_{t+i}), \quad P^*_{ij} = v^*_j\left(\theta^*_{r+i}\right),$$

where $v_j = v_j^W$, $v_j^* = v_j^{*W}$ are from Definition 11.1.

Theorem 13.6. *With reference to Definition 13.5, $P^*P = \nu(W)I$, where $\nu(W)$ is from Definition 7.4.*

Proof. We compute the (i, j)-entry of P^*P using Definition 13.5 and (56). We find that this is equal to $(k_i(W))^{-1}$ times the left hand side of (61). Using (61), we obtain $P^*P = \nu(W)I$. $\square$

Theorem 13.7. *Let $\flat$ and $\sharp$ be the maps in Definition 8.9. With reference to Definition 13.5, $Y^\sharp P = PY^\flat$ for $Y \in End(W)$.*

Proof. By Lemma 13.1, the transition matrix from a standard basis to a dual standard basis for W is a scalar multiple of P. Therefore, $Y^\sharp P = PY^\flat$. $\square$

14. Two more bases for *W*

Let W be as in Assumption 3.4. In Sections 8 and 9, we found two bases for W with respect to which A and A^* are represented by tridiagonal and diagonal matrices. In this section, we will look at two more bases for W with respect to which A and A^* are represented by lower bidiagonal and upper bidiagonal matrices.

Definition 14.1. For $0 \leqslant i \leqslant d$, define $\tau_i = \tau_i^W$, $\tau_i^* = \tau_i^{*W}$, $\eta_i = \eta_i^W$, $\eta_i^* = \eta_i^W$ in $\mathbb{C}[\lambda]$ as follows:

$$\tau_i = \prod_{h=0}^{i-1}(\lambda - \theta_{t+h}), \quad \tau_i^* = \prod_{h=0}^{i-1}\left(\lambda - \theta^*_{r+h}\right),$$
$$\eta_i = \prod_{h=0}^{i-1}(\lambda - \theta_{t+d-h}), \quad \eta_i^* = \prod_{h=0}^{i-1}\left(\lambda - \theta^*_{r+d-h}\right).$$

Observe that each of τ_i, τ_i^*, η_i, η_i^* is monic of degree i.

Lemma 14.2. *For $0 \leqslant i, j \leqslant d$,*

(i) *each of $\tau_i(\theta_{t+j})$, $\tau_i^*\left(\theta^*_{r+j}\right)$ is 0 if $j < i$ and nonzero if $j = i$;*
(ii) *each of $\eta_i(\theta_{t+j})$, $\eta_i^*\left(\theta^*_{r+j}\right)$ is 0 if $j > d - i$ and nonzero if $j = d - i$.*

Proof. Immediate from Definition 14.1. $\square$

Lemma 14.3. *Let v be a nonzero vector in E_r^*W. Then $\{\tau_i(A)v\}_{i=0}^d$ is a basis for W.*

Proof. By Theorem 5.5 and Lemma 6.3, $\{p_i(A)v\}_{i=0}^d$ is a basis for W. For $0 \leqslant i \leqslant d$, each of τ_i and p_i is a polynomial of degree i. The result follows. $\square$

Definition 14.4. For $0 \leqslant i \leqslant d$, define

$$U_i = \tau_i(A)E_r^*W.$$

For notational convenience, define $U_{-1} = 0$ and $U_{d+1} = 0$.

Lemma 14.5. *With reference to Definition 14.4, U_i has dimension 1 for $0 \leqslant i \leqslant d$. Moreover,*

$$W = \sum_{i=0}^{d} U_i \quad \text{(direct sum)}. \tag{73}$$

Proof. Immediate from Lemma 14.3 and Definition 14.4. $\square$

Lemma 14.6. *For $0 \leqslant i \leqslant d$,*

(i) $\sum_{h=0}^{i} U_h = \sum_{h=0}^{i} E_{r+h}^*W$,
(ii) $\sum_{h=i}^{d} U_h = \sum_{h=i}^{d} E_{t+h}W$.

Proof. Let v be a nonzero vector in E_r^*W.

(i) By Lemma 3.1(i), $\tau_j(A)v$ is contained in $\sum_{h=0}^{i} E_{r+h}^*W$ for $0 \leqslant j \leqslant i$. Hence, $\sum_{h=0}^{i} U_h \subseteq \sum_{h=0}^{i} E_{r+h}^*W$. In this inclusion, equality holds since each side has dimension $i+1$.

(ii) For $i \leqslant j \leqslant d$,

$$\begin{aligned}
\tau_j(A)v &= \sum_{l=0}^{D} E_l\tau_j(A)v, \\
&= \sum_{h=0}^{d} E_{t+h}\tau_j(A)v, \\
&= \sum_{h=0}^{d} \tau_j(\theta_{t+h})E_{t+h}v, \\
&= \sum_{h=j}^{d} \tau_j(\theta_{t+h})E_{t+h}v \quad \text{by Lemma 14.2}.
\end{aligned}$$

Hence $\tau_j(A)v \in \sum_{h=i}^{d} E_{t+h}W$ for $i \leqslant j \leqslant d$. Thus, $\sum_{h=i}^{d} U_h \subseteq \sum_{h=i}^{d} E_{t+h}W$. In this inclusion, equality holds since each side has dimension $i+1$. $\square$

Lemma 14.7. *For $0 \leqslant i \leqslant d$,*

$$U_i = \left(\sum_{h=0}^{i} E_{r+h}^*W\right) \cap \left(\sum_{h=i}^{d} E_{t+h}W\right).$$

Proof. By Lemma 14.5, $U_i = (U_0 + U_1 + \cdots + U_i) \cap (U_i + U_{i+1} + \cdots + U_d)$. Combining this with Lemma 14.6, we obtain the desired result. $\square$

Lemma 14.8. *For $0 \leqslant i \leqslant d$,*

(i) $(A - \theta_{t+i}I)U_i = U_{i+1}$,
(ii) $\left(A^* - \theta_{r+i}^*I\right) U_i = U_{i-1}$.

Proof. (i) Immediate from Definition 14.4.

(ii) Assume $1 \leqslant i \leqslant d$, otherwise, we are done since $U_0 = E^*_r W$. Let v be a nonzero vector in $E^*_r W$. Since $A^* E^*_{r+i} = \theta^*_{r+i} E^*_{r+i}$, we have $(A^* - \theta^*_{r+i} I)(\sum_{h=0}^{i} E^*_{r+h} W) \subseteq \sum_{h=0}^{i-1} E^*_{r+h} W$. By Lemma 3.2(i), we have $(A^* - \theta^*_{r+i} I)(\sum_{h=i}^{d} E_{t+h}) \subseteq \sum_{h=i-1}^{d} E_{t+h} W$. Combining these comments with Lemma 14.7, we find that $(A^* - \theta^*_{r+i} I) U_i \subseteq U_{i-1}$. We now show equality holds. Suppose that $(A^* - \theta^*_{r+i} I) U_i \subsetneq U_{i-1}$. Then $(A^* - \theta^*_{r+i} I) U_i = 0$ since $\dim U_{i-1} = 1$. Let $W' = U_i + U_{i+1} + \cdots + U_d$. Observe that W' is nonzero. By (i), $AW' \subseteq W'$. Since $(A^* - \theta^*_{r+i} I) U_i = 0$ and $(A^* - \theta^*_{r+j} I) U_j \subseteq U_{j-1}$ for $i + 1 \leqslant j \leqslant d$, we find that $A^* W' \subseteq W'$. Hence W' is a nonzero T-submodule of W. Since the T-module W is irreducible, $W' = W$. This contradicts (73) since $i > 0$. Therefore, $(A^* - \theta^*_{r+i} I) U_i = U_{i-1}$. $\square$

By Lemma 14.8, for $1 \leqslant i \leqslant d$, U_i is invariant under $(A - \theta_{t+i-1} I)(A^* - \theta^*_{r+i} I)$ and the corresponding eigenvalue is nonzero.

Definition 14.9. For $1 \leqslant i \leqslant d$, let $\varphi_i = \varphi_i(W)$ be the eigenvalue of $(A - \theta_{t+i-1} I)(A^* - \theta^*_{r+i} I)$ corresponding to U_i. Observe that $\varphi_i \neq 0$. We refer to the sequence $\{\varphi_i\}_{i=1}^{d}$ as the *first split sequence of* W. For notational convenience, define $\varphi_0 = 0$.

Theorem 14.10. *With respect to the basis for W in Lemma 14.3, the matrices representing A, A^* are*

$$\begin{pmatrix} \theta_t & & & & & 0 \\ 1 & \theta_{t+1} & & & & \\ & 1 & \theta_{t+2} & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \theta_{t+d-1} & \\ 0 & & & & 1 & \theta_{t+d} \end{pmatrix}, \quad \begin{pmatrix} \theta^*_r & \varphi_1 & & & & 0 \\ & \theta^*_{r+1} & \varphi_2 & & & \\ & & \theta^*_{r+2} & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \theta^*_{r+d-1} & \varphi_d \\ 0 & & & & & \theta^*_{r+d} \end{pmatrix}.$$

Proof. Immediate from Definitions 14.4 and 14.9 and Lemma 14.8. $\square$

In Lemmas 14.3–14.8 and Theorem 14.10, we replace E_{t+i} with E_{t+d-i} for $0 \leqslant i \leqslant d$ and we routinely obtain the following results.

Lemma 14.11. *Let v be a nonzero vector in $E^*_r W$. Then $\{\eta_i(A)v\}_{i=0}^{d}$ is a basis for W.*

Definition 14.12. For $0 \leqslant i \leqslant d$, define

$$U_i^{\Downarrow} = \eta_i(A) E^*_r W.$$

For notational convenience, define $U_{-1}^{\Downarrow} = 0$ and $U_{d+1}^{\Downarrow} = 0$.

Lemma 14.13. *With reference to Definition 14.12,*

$$W = \sum_{i=0}^{d} U_i^{\Downarrow} \quad \text{(direct sum)}. \tag{74}$$

Lemma 14.14. *For $0 \leqslant i \leqslant d$,*

(i) $\sum_{h=0}^{i} U_h^{\Downarrow} = \sum_{h=0}^{i} E^*_{r+h} W$,
(ii) $\sum_{h=i}^{d} U_h^{\Downarrow} = \sum_{h=0}^{d-i} E_{t+h} W$.

Lemma 14.15. *For* $0 \leqslant i \leqslant d$,

(i) $(A - \theta_{t+d-i}I)U_i^{\Downarrow} = U_{i+1}^{\Downarrow}$,
(ii) $\left(A^* - \theta^*_{r+i}I\right) U_i^{\Downarrow} = U_{i-1}^{\Downarrow}$.

By Lemma 14.15, for $1 \leqslant i \leqslant d$, $U_i^{\Downarrow}$ is invariant under $(A - \theta_{t+d-i+1}I)(A^* - \theta^*_{r+i}I)$ and the corresponding eigenvalue is nonzero.

Definition 14.16. For $1 \leqslant i \leqslant d$, let $\phi_i = \phi_i(W)$ be the eigenvalue of $(A - \theta_{t+d-i+1}I)(A^* - \theta^*_{r+i}I)$ corresponding to $U_i^{\Downarrow}$. Observe that $\phi_i \neq 0$. We refer to the sequence $\{\phi_i\}_{i=1}^d$ as the *second split sequence of W*.

Theorem 14.17. *With respect to the basis for W in Lemma 14.11, the matrices representing* A, A^* *are*

$$\begin{pmatrix} \theta_{t+d} & & & & & 0 \\ 1 & \theta_{t+d-1} & & & & \\ & 1 & \theta_{t+d-2} & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \theta_{t+1} & \\ 0 & & & & 1 & \theta_t \end{pmatrix}, \quad \begin{pmatrix} \theta^*_r & \phi_1 & & & & 0 \\ & \theta^*_{r+1} & \phi_2 & & & \\ & & \theta^*_{r+2} & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \theta^*_{r+d-1} & \phi_d \\ 0 & & & & & \theta^*_{r+d} \end{pmatrix}.$$

In [10, Lemma 12.7], it was shown that $\{\varphi_i\}_{i=1}^d$ and $\{\phi_i\}_{i=1}^d$ are related by the following:

$$\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_{t+h} - \theta_{t+d-h}}{\theta_t - \theta_{t+d}} + \left(\theta^*_{r+i} - \theta^*_r\right)(\theta_{t+i-1} - \theta_{t+d}) \qquad (1 \leqslant i \leqslant d), \tag{75}$$

$$\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_{t+h} - \theta_{t+d-h}}{\theta_t - \theta_{t+d}} + \left(\theta^*_{r+i} - \theta^*_r\right)(\theta_{t+d-i+1} - \theta_t) \qquad (1 \leqslant i \leqslant d). \tag{76}$$

Definition 14.18. By the *parameter array of W*, we mean the sequence of scalars

$$\left(\{\theta_{t+i}\}_{i=0}^d, \{\theta^*_{r+i}\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d\right),$$

where r, t, d are from Assumption 3.4, and the φ_i, ϕ_i are from Definitions 14.9 and 14.16.

15. Describing *W* in terms of its parameter array

Let W be as in Assumption 3.4. Up until now, we have associated with W a number of polynomials and parameters. In this section, we will express all these polynomials and parameters in terms of the parameter array $(\{\theta_{t+i}\}_{i=0}^d, \{\theta^*_{r+i}\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$ of W. Recall the polynomials τ_i, τ_i^*, η_i, η_i^* from Definition 14.1.

Theorem 15.1. *For* $0 \leqslant i \leqslant d$,

$$u_i = \sum_{h=0}^{i} \frac{\tau_h^*\left(\theta^*_{r+i}\right)}{\varphi_1\varphi_2\cdots\varphi_h}\tau_h, \tag{77}$$

$$u_i^* = \sum_{h=0}^{i} \frac{\tau_h(\theta_{t+i})}{\varphi_1\varphi_2\cdots\varphi_h}\tau_h^*, \tag{78}$$

where $u_i = u_i^W$, $u_i^* = u_i^{*W}$ *are from Definition 11.5.*

Proof. We first verify (77). Since u_i has degree i, there exist complex scalars $\{\alpha_h\}_{h=0}^{i}$ such that $u_i = \sum_{h=0}^{i} \alpha_h \tau_h$. By Lemma 14.2(i), $\tau_0(\theta_t) = 1$ and $\tau_i(\theta_t) = 0$ for $1 \leqslant i \leqslant d$. From these comments and since $u_i(\theta_t) = 1$, we have $\alpha_0 = 1$. Now assume $i \geqslant 1$, otherwise we are done. Let v be a nonzero vector in E_r^*W. By Theorem 11.4 and Lemma 11.6, $u_i(A)v \in E_{r+i}^*W$. Thus,

$$
\begin{aligned}
0 &= \left(A^* - \theta_{r+i}^* I\right) u_i(A)v, \\
&= \sum_{h=0}^{i} \alpha_h A^* \tau_h(A)v - \theta_{r+i}^* \sum_{h=0}^{i} \alpha_h \tau_h(A)v, \\
&= \sum_{h=0}^{i} \alpha_h \left(\theta_{r+h}^* \tau_h(A)v + \varphi_h \tau_{h-1}(A)v\right) - \theta_{r+i}^* \sum_{h=0}^{i} \alpha_h \tau_h(A)v \quad \text{by Theorem 14.10}, \\
&= \sum_{h=0}^{i-1} \left(\varphi_{h+1}\alpha_{h+1} + \alpha_h \theta_{r+h}^* - \theta_{r+i}^* \alpha_h\right) \tau_h(A)v.
\end{aligned}
$$

By Lemma 14.3, $\{\tau_h(A)v\}_{h=0}^{i-1}$ are linearly independent. Thus, $\varphi_{h+1}\alpha_{h+1} + \alpha_h\theta_{r+h}^* - \theta_{r+i}^*\alpha_h = 0$ for $0 \leqslant h < i$. From this recursive equation and the fact that $\alpha_0 = 1$, we find that $\alpha_h = \tau_h^*(\theta_{r+i}^*)/(\varphi_1\varphi_2\cdots\varphi_h)$ for $0 \leqslant h \leqslant i$. Therefore, (77) holds. We now prove (78). Let f_i be the polynomial on the right in (78). Using (77), we find that $f_i(\theta_{r+j}^*) = u_j(\theta_{t+i})$ for $0 \leqslant j \leqslant i$. By Theorem 12.4, $u_i^*(\theta_{r+j}^*) = u_j(\theta_{t+i})$. Therefore, $f_i(\theta_{r+j}^*) = u_i^*(\theta_{r+j}^*)$ for $0 \leqslant j \leqslant i$. By this and since u_i^*, f_i have degree i, we find that $u_i^* = f_i$. □

Lemma 15.2. *For* $0 \leqslant i \leqslant d$,

$$
p_i(\theta_t) = \frac{\varphi_1\varphi_2\cdots\varphi_i}{\tau_i^*\left(\theta_{r+i}^*\right)}, \quad p_i^*\left(\theta_r^*\right) = \frac{\varphi_1\varphi_2\cdots\varphi_i}{\tau_i(\theta_{t+i})}, \tag{79}
$$

where $p_i = p_i^W$, $p_i^* = p_i^{*W}$ *are from Definitions 6.1 and 6.2.*

Proof. We first prove the equation on the left in (79). We compute the coefficient of λ^i in u_i in two ways: one way using (77) and another way using Definition 11.5. Comparing the results, we obtain the equation on the left in (79). Argue similarly to obtain the equation on the right in (79). □

Theorem 15.3. *For* $0 \leqslant i \leqslant d-1$,

$$
b_i(W) = \varphi_{i+1} \frac{\tau_i^*\left(\theta_{r+i}^*\right)}{\tau_{i+1}^*\left(\theta_{r+i+1}^*\right)}, \quad b_i^*(W) = \varphi_{i+1} \frac{\tau_i(\theta_{t+i})}{\tau_{i+1}(\theta_{t+i+1})}. \tag{80}
$$

The $b_i(W)$, $b_i^*(W)$ *are from Definition 9.1.*

Proof. Immediate from Theorem 9.5(i), (iii) and Lemma 15.2. □

Theorem 15.4. *With reference to Definition 5.2,*

$$
a_0(W) = \theta_t + \frac{\varphi_1}{\theta_r^* - \theta_{r+1}^*}, \quad a_d(W) = \theta_{t+d} + \frac{\varphi_d}{\theta_{r+d}^* - \theta_{r+d-1}^*}, \tag{81}
$$

$$
a_0(W) = \theta_{t+d} + \frac{\phi_1}{\theta_r^* - \theta_{r+1}^*}, \quad a_d(W) = \theta_t + \frac{\phi_d}{\theta_{r+d}^* - \theta_{r+d-1}^*}. \tag{82}
$$

For $1 \leqslant i \leqslant d-1$,

$$a_i(W) = \theta_{t+i} + \frac{\varphi_i}{\theta^*_{r+i} - \theta^*_{r+i-1}} + \frac{\varphi_{i+1}}{\theta^*_{r+i} - \theta^*_{r+i+1}} \tag{83}$$

$$= \theta_{t+d-i} + \frac{\phi_i}{\theta^*_{r+i} - \theta^*_{r+i-1}} + \frac{\phi_{i+1}}{\theta^*_{r+i} - \theta^*_{r+i+1}}. \tag{84}$$

Proof. To obtain (83), we compute the coefficient of λ^i in u_{i+1} in two ways. One way is using Lemma 9.4 and Lemma 11.7. Using this approach, we find that the coefficient is equal to

$$-\sum_{l=0}^{i} \frac{a_l(W)}{p_{i+1}(\theta_t)}. \tag{85}$$

Another way is using (77). Using this approach, the coefficient is equal to

$$\frac{\tau^*_i\left(\theta^*_{r+i+1}\right)}{\varphi_1\varphi_2\cdots\varphi_i} - \sum_{l=0}^{i} \theta_{t+l} \frac{\tau^*_{i+1}\left(\theta^*_{r+i+1}\right)}{\varphi_1\varphi_2\cdots\varphi_{i+1}}. \tag{86}$$

Evaluating (85) using (79) and comparing the result with (86), we obtain (83). Similarly, we obtain the two equations in (81). We now prove (84). Observe that by Definitions 5.2 and 14.16, replacing E_{t+i} with E_{t+d-i} for $0 \leqslant i \leqslant d$ has the effect of switching $(a_i(W), \theta_{t+i}, \varphi_i)$ to $(a_i(W), \theta_{t+d-i}, \phi_i)$. Applying this switching to (83), we obtain (84). Similarly, we obtain the two equations in (82). $\square$

Theorem 15.5. *With reference to Definition 5.2,*

$$a^*_0(W) = \theta^*_r + \frac{\varphi_1}{\theta_t - \theta_{t+1}}, \quad a^*_d(W) = \theta^*_{r+d} + \frac{\varphi_d}{\theta_{t+d} - \theta_{t+d-1}}, \tag{87}$$

$$a^*_0(W) = \theta^*_{r+d} + \frac{\phi_d}{\theta_t - \theta_{t+1}}, \quad a^*_d(W) = \theta^*_r + \frac{\phi_1}{\theta_{t+d} - \theta_{t+d-1}}. \tag{88}$$

For $1 \leqslant i \leqslant d-1$,

$$a^*_i(W) = \theta^*_{r+i} + \frac{\varphi_i}{\theta_{t+i} - \theta_{t+i-1}} + \frac{\varphi_{i+1}}{\theta_{t+i} - \theta_{t+i+1}} \tag{89}$$

$$= \theta^*_{r+d-i} + \frac{\phi_{d-i+1}}{\theta_{t+i} - \theta_{t+i-1}} + \frac{\phi_{d-i}}{\theta_{t+i} - \theta_{t+i+1}}. \tag{90}$$

Proof. To obtain (87) and (89) argue similarly as in the proof of (83). We now prove (90). By Definitions 5.2 and 14.16, replacing E_{t+i} with E_{t+d-i} for $0 \leqslant i \leqslant d$ has the effect of switching $(a^*_i(W), \theta_{t+i}, \varphi_i)$ to $(a^*_{d-i}(W), \theta_{t+d-i}, \phi_i)$. Applying this switching to (89), we obtain

$$a^*_{d-i}(W) = \theta^*_{r+i} + \frac{\phi_i}{\theta_{t+d-i} - \theta_{t+d-i+1}} + \frac{\phi_{i+1}}{\theta_{t+d-i} - \theta_{t+d-i-1}}. \tag{91}$$

Changing i to $d-i$ in (91), we obtain (90). $\square$

Theorem 15.6. *For* $1 \leqslant i \leqslant d$, φ_i *is equal to each of the following*:

$$\left(\theta^*_{r+i} - \theta^*_{r+i-1}\right)\sum_{j=0}^{i-1}(\theta_{t+j} - a_j(W)), \quad \left(\theta^*_{r+i-1} - \theta^*_{r+i}\right)\sum_{j=i}^{d}(\theta_{t+j} - a_j(W)), \tag{92}$$

$$(\theta_{t+i} - \theta_{t+i-1})\sum_{j=0}^{i-1}\left(\theta^*_{r+j} - a^*_j(W)\right), \quad (\theta_{t+i-1} - \theta_{t+i})\sum_{j=i}^{d}\left(\theta^*_{r+j} - a^*_j(W)\right). \tag{93}$$

The $a_h(W)$, $a^*_h(W)$ *are from Definition 5.2.*

Proof. To obtain the expression on the left in (92), solve for φ_i recursively using (83). From this and Lemma 5.10(i), we obtain the expression on the right in (92). The remaining assertions can be similarly shown. $\square$

Theorem 15.7. *For $1 \leqslant i \leqslant d$, ϕ_i is equal to each of the following*:

$$\left(\theta^*_{r+i} - \theta^*_{r+i-1}\right)\sum_{j=0}^{i-1}(\theta_{t+d-j} - a_j(W)), \quad \left(\theta^*_{r+i-1} - \theta^*_{r+i}\right)\sum_{j=i}^{d}(\theta_{t+d-j} - a_j(W)), \tag{94}$$

$$(\theta_{t+d-i} - \theta_{t+d-i+1})\sum_{j=0}^{i-1}\left(\theta^*_{r+j} - a^*_{d-j}(W)\right), \quad (\theta_{t+d-i+1} - \theta_{t+d-i})\sum_{j=i}^{d}\left(\theta^*_{r+j} - a^*_{d-j}(W)\right). \tag{95}$$

The $a_h(W)$, $a^*_h(W)$ are from Definition 5.2.

Proof. Similar to the proof of Theorem 15.6. $\square$

Theorem 15.8. *For $0 \leqslant i \leqslant d$, the polynomial $p_i = p_i^W$ from Definition 6.1 is equal to both*

$$\sum_{h=0}^{i} \frac{\varphi_1\varphi_2\cdots\varphi_i\tau^*_h\left(\theta^*_{r+i}\right)}{\varphi_1\varphi_2\cdots\varphi_h\tau^*_i\left(\theta^*_{r+i}\right)}\tau_h, \quad \sum_{h=0}^{i} \frac{\phi_1\phi_2\cdots\phi_i\tau^*_h\left(\theta^*_{r+i}\right)}{\phi_1\phi_2\cdots\phi_h\tau^*_i\left(\theta^*_{r+i}\right)}\eta_h. \tag{96}$$

Proof. The expression on the left in (96) is equal to p_i by Definition 11.5, (77), and the equation on the left in (79). To show that p_i is equal to the expression on the right in (96), write u_i as a linear combination of $\{\eta_h\}_{h=0}^i$. Arguing as in the proof of (77), we find that

$$u_i = u_i(\theta_{t+d})\sum_{h=0}^{i} \frac{\tau^*_h\left(\theta^*_{r+i}\right)}{\phi_1\phi_2\cdots\phi_h}\eta_h. \tag{97}$$

To find $u_i(\theta_{t+d})$, we compute the coefficient of λ^i in u_i in two ways: one way is using (77) and another way is using (97). Comparing these results we obtain

$$u_i(\theta_{t+d}) = \frac{\phi_1\phi_2\cdots\phi_i}{\varphi_1\varphi_2\cdots\varphi_i}. \tag{98}$$

Evaluating p_i using Definition 11.5, (97), (98) and the equation on the left in (79), we find that p_i is equal to the expression on the right in (96). $\square$

Theorem 15.9. *For $0 \leqslant i \leqslant d$, the polynomial $p^*_i = p_i^{*W}$ from Definition 6.2 is equal to both*

$$\sum_{h=0}^{i} \frac{\varphi_1\varphi_2\cdots\varphi_i\tau_h(\theta_{t+i})}{\varphi_1\varphi_2\cdots\varphi_h\tau_i(\theta_{t+i})}\tau^*_h, \quad \sum_{h=0}^{i} \frac{\phi_d\phi_{d-1}\cdots\phi_{d-i+1}\tau_h(\theta_{t+i})}{\phi_d\phi_{d-1}\cdots\phi_{d-h+1}\tau_i(\theta_{t+i})}\eta^*_h. \tag{99}$$

Proof. The expression on the left in (99) is equal to p^*_i by Definition 11.5, (78), and the equation on the right in (79). We now prove that p^*_i is equal to the expression on the right in (99). Comparing the equation on the left in (94) and the equation on the right in (95), we find that interchanging A and A^* has the effect of switching ϕ_i to ϕ_{d-i+1} for $1 \leqslant i \leqslant d$. Applying this switching to the sum on the right in (96), we obtain the sum on the right in (99). $\square$

Lemma 15.10. *For $0 \leqslant i \leqslant d$,*

$$p_i(\theta_{t+d}) = \frac{\phi_1\phi_2\cdots\phi_i}{\tau^*_i\left(\theta^*_{r+i}\right)}, \quad p^*_i\left(\theta^*_{r+d}\right) = \frac{\phi_d\phi_{d-1}\cdots\phi_{d-i+1}}{\tau_i(\theta_{t+i})},$$

where $p_i = p_i^W$, $p^*_i = p_i^{*W}$ are from Definitions 6.1 and 6.2.

Proof. Immediate from the right side of lines (96) and (99). $\square$

Theorem 15.11. *For* $1 \leqslant i \leqslant d$,

$$c_i(W) = \phi_i \frac{\eta^*_{d-i}\left(\theta^*_{r+i}\right)}{\eta^*_{d-i+1}\left(\theta^*_{r+i-1}\right)}, \quad c^*_i(W) = \phi_{d-i+1} \frac{\eta_{d-i}(\theta_{t+i})}{\eta_{d-i+1}(\theta_{t+i-1})}. \tag{100}$$

The $c_i(W)$, $c^*_i(W)$ *are from Definition 9.1.*

Proof. We first verify the equation on the right in (100). By (29), replacing E_{t+i} with E_{t+d-i} for $0 \leqslant i \leqslant d$ switches $b^*_i(W)$ and $c^*_{d-i}(W)$. Applying this switching to the equation on the right in (80), we find that for $0 \leqslant i \leqslant d-1$,

$$c^*_{d-i}(W) = \phi_{i+1} \frac{\eta_i(\theta_{t+d-i})}{\eta_{i+1}(\theta_{t+d-i-1})}. \tag{101}$$

Changing i to $d-i$ in (101), we obtain the equation on the right in (100). We now verify the equation on the left in (100). Recall from the proof of Theorem 15.9 that interchanging A and A^* switches ϕ_i and ϕ_{d-i+1}. Applying this switching to the equation on the right in (100), we obtain the equation on the left in (100). $\square$

Theorem 15.12. *With reference to Definition 7.4,*

$$\nu(W) = \frac{\eta_d(\theta_t)\eta^*_d\left(\theta^*_r\right)}{\phi_1\phi_2\cdots\phi_d}. \tag{102}$$

Proof. Let $0 \neq v \in E^*_r W$. By Theorem 14.17, $(A^* - \theta^*_{r+i} I)\eta_i(A)v = \phi_i \eta_{i-1}(A)v$ for $1 \leqslant i \leqslant d$. Hence, $\eta^*_d(A^*)\eta_d(A)v = \phi_1\phi_2\cdots\phi_d v$. By (3) and (7), on W we have $\eta_d(A) = \eta_d(\theta_t)E_t$ and $\eta^*_d(A^*) = \eta^*_d(\theta^*_r)E^*_r$. Thus, $\eta^*_d(A^*)\eta_d(A)v = \eta^*_d(\theta^*_r)\eta_d(\theta_t)E^*_r E_t v$. From these comments and since $v \in E^*_r W$, we obtain $\phi_1\phi_2\cdots\phi_d v = \eta^*_d(\theta^*_r)\eta_d(\theta_t)E^*_r E_t E^*_r v$. Evaluate $E^*_r E_t E^*_r$ using Theorem 7.5(ii). The result follows. $\square$

Theorem 15.13. *With reference to Definitions 5.2 and 10.1,*

$$k_i(W) = \frac{\varphi_1\varphi_2\cdots\varphi_i}{\phi_1\phi_2\cdots\phi_i} \frac{\eta^*_d\left(\theta^*_r\right)}{\tau^*_i\left(\theta^*_{r+i}\right)\eta^*_{d-i}\left(\theta^*_{r+i}\right)} \qquad (0 \leqslant i \leqslant d), \tag{103}$$

$$k^*_i(W) = \frac{\varphi_1\varphi_2\cdots\varphi_i}{\phi_d\phi_{d-1}\cdots\phi_{d-i+1}} \frac{\eta_d(\theta_t)}{\tau_i(\theta_{t+i})\eta_{d-i}(\theta_{t+i})} \qquad (0 \leqslant i \leqslant d), \tag{104}$$

$$x_i(W) = \varphi_i\phi_i \frac{\tau^*_{i-1}\left(\theta^*_{r+i-1}\right)\eta^*_{d-i}\left(\theta^*_{r+i}\right)}{\tau^*_i\left(\theta^*_{r+i}\right)\eta^*_{d-i+1}\left(\theta^*_{r+i-1}\right)} \quad (1 \leqslant i \leqslant d), \tag{105}$$

$$x^*_i(W) = \varphi_i\phi_{d-i+1} \frac{\tau_{i-1}(\theta_{t+i-1})\eta_{d-i}(\theta_{t+i})}{\tau_i(\theta_{t+i})\eta_{d-i+1}(\theta_{t+i-1})} \quad (1 \leqslant i \leqslant d). \tag{106}$$

Proof. Evaluate the equations in (37), (38) and in Lemma 9.3(i), (iii), using Theorems 15.3 and 15.11. $\square$

For the rest of this section, we will find alternative formulae for the intersection and dual intersection numbers of W. The reason in doing this is that the formulae given in Theorems 15.3 and 15.11 involve huge products which may not be easy to compute. We will need the following lemma.

Lemma 15.14. *For* $0 \leqslant i \leqslant d$,

$$c_i(W)\tau_1^*\left(\theta^*_{r+i-1}\right) + a_i(W)\tau_1^*\left(\theta^*_{r+i}\right) + b_i(W)\tau_1^*(\theta^*_{r+i+1}) = \varphi_1 + \theta_{t+1}\tau_1^*\left(\theta^*_{r+i}\right), \tag{107}$$

$$c_i^*(W)\tau_1(\theta_{t+i-1}) + a_i^*(W)\tau_1(\theta_{t+i}) + b_i^*(W)\tau_1(\theta_{t+i+1}) = \varphi_1 + \theta^*_{r+1}\tau_1(\theta_{t+i}). \tag{108}$$

The $a_i(W)$, $b_i(W)$, $c_i(W)$ *(resp.,* $a_i^*(W)$, $b_i^*(W)$, $c_i^*(W)$*) are the intersection numbers (resp., dual intersection numbers) of* W.

Proof. By (49), $u_1^* = (\lambda - a_0^*(W))/b_0^*(W)$. Use this to evaluate (57) with $j = 1$. Eliminate $a_0^*(W)$ in the resulting equation using the expression on the left of (87). Simplify using Lemma 9.3(ii) to obtain (107). The proof of (108) is similar. □

Theorem 15.15. *The intersection numbers of* W *are as follows*:

$$b_0(W) = \frac{\varphi_1}{\theta^*_{r+1} - \theta^*_r}, \tag{109}$$

$$b_i(W) = \frac{(\theta_t - a_i(W))\left(\theta^*_{r+i} - \theta^*_{r+i-1}\right) + (\theta_t - \theta_{t+1})\left(\theta^*_r - \theta^*_{r+i}\right) + \varphi_1}{\theta^*_{r+i+1} - \theta^*_{r+i-1}} \quad (1 \leqslant i \leqslant d-1), \tag{110}$$

$$c_i(W) = \frac{(\theta_t - a_i(W))\left(\theta^*_{r+i} - \theta^*_{r+i+1}\right) + (\theta_t - \theta_{t+1})\left(\theta^*_r - \theta^*_{r+i}\right) + \varphi_1}{\theta^*_{r+i-1} - \theta^*_{r+i+1}} \quad (1 \leqslant i \leqslant d-1), \tag{111}$$

$$c_d(W) = \frac{\varphi_1 + (\theta_{t+1} - \theta_t)\left(\theta^*_{r+d} - \theta^*_r\right)}{\theta^*_{r+d-1} - \theta^*_{r+d}}. \tag{112}$$

To obtain $b_i^*(W)$ *and* $c_i^*(W)$, *replace* $(\theta_{t+j}, \theta^*_{r+j}, a_j(W))$ *with* $(\theta^*_{r+j}, \theta_{t+j}, a_j^*(W))$.

Proof. To obtain (109), eliminate $a_0(W)$ in the equation on the left of (81) using Lemma 9.3(ii). To obtain (110) and (111), solve the system of equations in Lemmas 9.3(ii) and (107). To obtain (112), set $i = d$ in (107) and eliminate $a_d(W)$ using Lemma 9.3(ii). The proof of the assertion regarding the dual intersection numbers of W is similar. □

By Theorems 15.4 and 15.15, the intersection numbers (resp., dual intersection numbers) of W can be expressed in terms of the parameter array of W. By (75) and (76), the parameter array of W is determined by the eigenvalue sequence of W, dual eigenvalue sequence of W, and $\varphi_1(W)$. Hence, we now solve for the intersection numbers (resp., dual intersection numbers) of W in terms of these parameters. But first we need the following lemmas.

Lemma 15.16. *Assume* $d \geqslant 2$. *Then the scalar* φ_2 *is equal to both*

$$\varphi_1\left(1 + \frac{\theta_{t+1} - \theta_{t+d-1}}{\theta_t - \theta_{t+d}}\right) + \left(\theta^*_{r+1} - \theta^*_r\right)(\theta_{t+d} + \theta_{t+d-1} - \theta_t - \theta_{t+1}) + \left(\theta^*_{r+2} - \theta^*_r\right)(\theta_{t+1} - \theta_{t+d}), \tag{113}$$

$$\varphi_1\left(1 + \frac{\theta^*_{r+1} - \theta^*_{r+d-1}}{\theta^*_r - \theta^*_{r+d}}\right) + (\theta_{t+1} - \theta_t)\left(\theta^*_{r+d} + \theta^*_{r+d-1} - \theta^*_r - \theta^*_{r+1}\right) + (\theta_{t+2} - \theta_t)\left(\theta^*_{r+1} - \theta^*_{r+d}\right). \tag{114}$$

Proof. To obtain (113), set $i = 2$ in (75) and evaluate ϕ_1 using (76). Comparing the formula for φ_i on the left in lines (92) and (93), we find that interchanging A and A^* has no effect on φ_i for $1 \leqslant i \leqslant d$. Applying this switching to (113), we obtain (114). □

Lemma 15.17. *Assume $d \geqslant 2$. Then for $0 \leqslant i \leqslant d$,*

$$c_i(W)\tau_2^*\left(\theta^*_{r+i-1}\right) + a_i(W)\tau_2^*\left(\theta^*_{r+i}\right) + b_i(W)\tau_2^*(\theta^*_{r+i+1}) = \varphi_2\tau_1^*\left(\theta^*_{r+i}\right) + \theta_{t+2}\tau_2^*\left(\theta^*_{r+i}\right), \tag{115}$$

$$c_i^*(W)\tau_2(\theta_{t+i-1}) + a_i^*(W)\tau_2(\theta_{t+i}) + b_i^*(W)\tau_2(\theta_{t+i+1}) = \varphi_2\tau_1(\theta_{t+i}) + \theta^*_{r+2}\tau_2(\theta_{t+i}). \tag{116}$$

The $a_i(W)$, $b_i(W)$, $c_i(W)$ (resp., $a_i^*(W)$, $b_i^*(W)$, $c_i^*(W)$) are the intersection numbers (resp., dual intersection numbers) of W.

Proof. Eliminating u_2^* in (57) with $j = 2$ using (78), we obtain

$$\begin{aligned}
&c_i(W) + a_i(W) + b_i(W) + \frac{\tau_1(\theta_{t+2})}{\varphi_1}\left(c_i(W)\tau_1^*\left(\theta^*_{r+i-1}\right) + a_i(W)\tau_1^*\left(\theta^*_{r+i}\right)\right.\\
&\quad\left.+ b_i(W)\tau_1^*\left(\theta^*_{r+i+1}\right)\right) + \frac{\tau_2(\theta_{t+2})}{\varphi_1\varphi_2}\left(c_i(W)\tau_2^*\left(\theta^*_{r+i-1}\right) + a_i(W)\tau_2^*\left(\theta^*_{r+i}\right)\right.\\
&\quad\left.+ b_i(W)\tau_2^*\left(\theta^*_{r+i+1}\right)\right)\\
&\quad= \theta_{t+2}\left(1 + \frac{\tau_1(\theta_{t+2})}{\varphi_1}\tau_1^*\left(\theta^*_{r+i}\right) + \frac{\tau_2(\theta_{t+2})}{\varphi_1\varphi_2}\tau_2^*\left(\theta^*_{r+i}\right)\right).
\end{aligned} \tag{117}$$

Simplify the first three terms of (117) using Lemma 9.3(ii). Evaluating the coefficient of $\tau_1(\theta_{t+2})/\varphi_1$ in (117) using (107), we routinely obtain (115). The proof of (116) is similar. □

Theorem 15.18. *The intersection numbers of W are as follows:*

$$b_0(W) = \frac{\varphi_1}{\theta^*_{r+1} - \theta^*_r}, \tag{118}$$

$$b_i(W) = \frac{\varphi_1 f_i^+ + g_i^+}{\left(\theta^*_{r+i+1} - \theta^*_{r+i}\right)\left(\theta^*_{r+i+1} - \theta^*_{r+i-1}\right)} \qquad (1 \leqslant i \leqslant d-1), \tag{119}$$

$$c_i(W) = \frac{\varphi_1 f_i^- + g_i^-}{\left(\theta^*_{r+i-1} - \theta^*_{r+i}\right)\left(\theta^*_{r+i-1} - \theta^*_{r+i+1}\right)} \qquad (1 \leqslant i \leqslant d-1), \tag{120}$$

$$c_d(W) = \frac{\varphi_1 + (\theta_{t+1} - \theta_t)\left(\theta^*_{r+d} - \theta^*_r\right)}{\theta^*_{r+d-1} - \theta^*_{r+d}}, \tag{121}$$

where

$$\begin{aligned}
f_i^{\pm} &= \theta^*_{r+1} - \theta^*_{r+i\mp1} - \frac{\left(\theta^*_{r+i} - \theta^*_r\right)\left(\theta^*_{r+1} - \theta^*_{r+d-1}\right)}{\left(\theta^*_{r+d} - \theta^*_r\right)},\\
g_i^{\pm} &= \left(\theta^*_{r+i} - \theta^*_r\right)\left((\theta_{t+2} - \theta_{t+1})\left(\theta^*_{r+i} - \theta^*_{r+d}\right) - (\theta_{t+1} - \theta_t)\left(\theta^*_{r+i\mp1} - \theta^*_{r+d-1}\right)\right),
\end{aligned}$$

provided $d \geqslant 2$. To obtain $b_i^*(W)$ and $c_i^*(W)$, replace $(\theta_{t+j}, \theta^*_{r+j})$ with $(\theta^*_{r+j}, \theta_{t+j})$.

Proof. Observe that (118), (121) are (109), (112). To obtain (119) and (120), eliminate $a_i(W)$ in (107) and (115) using Lemma 9.3(ii). Then for $1 \leqslant i \leqslant d-1$,

$$c_i(W)\left(\theta^*_{r+i-1}-\theta^*_{r+i}\right)+b_i(W)\left(\theta^*_{r+i+1}-\theta^*_{r+i}\right)=\varphi_1+(\theta_{t+1}-\theta_t)\left(\theta^*_{r+i}-\theta^*_r\right), \quad (122)$$

$$c_i(W)h_i\left(\theta^*_{r+i-1}\right)+b_i(W)h_i\left(\theta^*_{r+i+1}\right)$$
$$=\varphi_2\left(\theta^*_{r+i}-\theta^*_r\right)+(\theta_{t+2}-\theta_t)\left(\theta^*_{r+i}-\theta^*_r\right)\left(\theta^*_{r+i}-\theta^*_{r+1}\right), \quad (123)$$

where

$$h_i(\lambda)=(\lambda-\theta^*_{r+i})(\lambda+\theta^*_{r+i}-\theta^*_{r+1}-\theta^*_r).$$

Eliminate φ_2 in (123) using (114). Solving the system of equations (122), (123), we obtain (119) and (120). Argue similarly and evaluate φ_2 using (113) to obtain the formula for the dual intersection numbers of W. □

Lemma 15.19. *Given vertices y, z in X, let W (resp., W') be the trivial $T(y)$-module (resp., $T(z)$-module) of Γ. Then W and W' have the same parameter array.*

Proof. By Lemma 9.6, W and W' have the same intersection numbers and dual intersection numbers. Thus, W and W' both have eigenvalue sequence $\{\theta_i\}_{i=0}^D$ and dual eigenvalue sequence $\{\theta^*_i\}_{i=0}^D$. By (118), $\varphi_1(W)=\varphi_1(W')$. Using (75) and (76), we find that W and W' have the same first split sequence and second split sequence. □

Definition 15.20. By the *parameter array of Γ*, we mean the parameter array of the trivial $T(x)$-module. Observe that this parameter array is independent of the choice of x by Lemma 15.19.

16. Isomorphism classes of thin irreducible *T*-modules

In Corollary 9.7, we mentioned some set of scalars needed to determine the isomorphism class of a thin irreducible T-module. As we have seen in Theorem 15.18, there are many relations among these scalars.We now consider a much smaller set of scalars needed to determine the isomorphism class. Let us first consider some equations from (75) and (76).

Lemma 16.1. *Let W be as in Assumption 3.4. Let $\{\varphi_i\}_{i=1}^d$, $\{\phi_i\}_{i=1}^d$ denote the first split sequence and second split sequence of W, respectively. Then*

$$\phi_1=\varphi_1+\left(\theta^*_{r+1}-\theta^*_r\right)(\theta_{t+d}-\theta_t),$$
$$\phi_d=\varphi_1+\left(\theta^*_{r+d}-\theta^*_r\right)(\theta_{t+1}-\theta_t),$$
$$\varphi_d=\phi_1+\left(\theta^*_{r+d}-\theta^*_r\right)(\theta_{t+d-1}-\theta_{t+d}).$$

Proof. Immediate from (75) and (76). □

Lemma 16.2. *Suppose that W and W' are thin irreducible T-modules with the same endpoint, dual endpoint and diameter $d>0$. Then the following are equivalent*:

$$\begin{aligned}
\varphi_1(W)&=\varphi_1(W'), & \varphi_d(W)&=\varphi_d(W'),\\
\phi_1(W)&=\phi_1(W'), & \phi_d(W)&=\phi_d(W'),\\
a_0(W)&=a_0(W'), & a_d(W)&=a_d(W'),\\
a^*_0(W)&=a^*_0(W'), & a^*_d(W)&=a^*_d(W').
\end{aligned}$$

Proof. Combine Lemma 16.1, (81), (82), (87), (88). □

Theorem 16.3. *Suppose that W and W' are thin irreducible T-modules with common diameter d.*

(i) *Assume $d = 0$. Then W and W' are isomorphic as T-modules if and only if they have the same endpoint and dual endpoint.*
(ii) *Assume $d > 0$. Then W and W' are isomorphic as T-modules if and only if they have the same endpoint, dual endpoint and all of the quantities in Lemma 16.2.*

Proof. (i) Immediate from Lemma 9.7.

(ii) By Theorem 15.18, W and W' have the same intersection numbers (resp., dual intersection numbers) if and only if they have the same endpoint, dual endpoint, diameter and $\varphi_1(W) = \varphi_1(W')$. Combining this with Lemmas 9.7 and 16.2, we obtain the desired result. □

17. Two examples of Q-polynomial distance-regular graphs

In this section, we apply the results that we have obtained in Section 15 to several examples of Q-polynomial distance-regular graphs. We will continue talking about the T-module W in Assumption 3.4 but now we will impose extra conditions on Γ.

Definition 17.1. The graph Γ is said to have *q-Racah type* whenever its parameter array $(\{\theta_i\}_{i=0}^D, \{\theta_i^*\}_{i=0}^D, \{\varphi_i\}_{i=1}^D, \{\phi_i\}_{i=1}^D)$ satisfy the following:

For $0 \leqslant i \leqslant D$,

$$\theta_i = \theta_0 + hq^{-i}(1-q^i)(1-sq^{i+1}),$$
$$\theta_i^* = \theta_0^* + h^*q^{-i}(1-q^i)(1-s^*q^{i+1}).$$

For $1 \leqslant i \leqslant D$,

$$\varphi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-D-1})(1-r_1q^i)(1-r_2q^i),$$
$$\phi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-D-1})(r_1-s^*q^i)(r_2-s^*q^i)/s^*.$$

In the above, $q, h, h^*, r_1, r_2, s, s^*$ are complex scalars such that $r_1r_2 = ss^*q^{D+1}$, $hh^*ss^* \neq 0$, $q \notin \{-1, 0, 1\}$.

Lemma 17.2. *Let Γ be as in Definition 17.1. Then for $0 \leqslant i, j \leqslant D$,*

$$\theta_i - \theta_j = h(q^i - q^j)(sq - q^{-i-j}),$$
$$\theta_i^* - \theta_j^* = h^*(q^i - q^j)(s^*q - q^{-i-j}).$$

Proof. Routine calculation using Definition 17.1. □

Lemma 17.3. *With reference to Definition 17.1, none of $q^i, r_1q^i, r_2q^i, s^*q^i/r_1, s^*q^i/r_2$ is equal to 1 for $1 \leqslant i \leqslant D$. Moreover, neither of sq^i, s^*q^i is equal to 1 for $2 \leqslant i \leqslant 2D$.*

Proof. The first assertion follows from Definition 17.1 and the fact that for $1 \leqslant i \leqslant D$, $\theta_i \neq \theta_0, \theta_i^* \neq \theta_0^*$, $\varphi_i \neq 0$, $\phi_i \neq 0$. The second assertion is immediate from Lemma 17.2 and the fact that the eigenvalues (resp., dual eigenvalues) of Γ are mutually distinct. □

Lemma 17.4. *Let Γ be as in Definition 17.1. Let $\{\varphi_i(W)\}_{i=1}^d$ and $\{\phi_i(W)\}_{i=1}^d$ be the first split sequence and second split sequence of W, respectively. Then there exists $\tau(W) \in \mathbb{C}$ such that for $1 \leqslant i \leqslant d$,*

$$\varphi_i(W) = hh^*(1-q^i)(1-q^{d-i+1})\left(\tau(W) - ss^*q^{r+t+i+1} - q^{-r-t-i-d}\right), \tag{124}$$

$$\phi_i(W) = hh^*(1-q^i)(1-q^{d-i+1})\left(\tau(W) - s^*q^{r-t-d+i} - sq^{t-r-i+1}\right). \tag{125}$$

Proof. Since h, h^* are both nonzero and q, q^d are both not equal to 1, there exists $\tau(W)$ such that (124) holds for $i = 1$. Plugging $\varphi_1(W)$ in (76) and using Lemma 17.2, we routinely obtain that (125) holds for $1 \leqslant i \leqslant d$. Evaluating (75) using (125) with $i = 1$ and repeating the same argument above, we find that (124) holds for $1 \leqslant i \leqslant d$. □

We make a comment about our notation used in Lemma 17.4. In the proof of [11, Theorem 35.15], there are scalars τ, h, h^*. Our present h, h^* are the same as those in [11, Theorem 35.15]. However, our $\tau(W)$ is equal to τ/hh^*.

Theorem 17.5. *Let Γ be as in Definition 17.1. Let $r_1(W), r_2(W)$ be the roots of*

$$\lambda^2 - \tau(W)q^{r+t+d}\lambda + ss^*q^{2r+2t+d+1} = 0,$$

where $\tau(W)$ is from Lemma 17.4. Then for $1 \leqslant i \leqslant d$,

$$\varphi_i(W) = hh^*q^{1-2i-t-r}(1-q^i)(1-q^{i-d-1})(1-r_1(W)q^i)(1-r_2(W)q^i), \tag{126}$$

$$\phi_i(W) = hh^*q^{1-2i-t-r}(1-q^i)(1-q^{i-d-1})(r_1(W)-s^*q^{i+2r})(r_2(W)-s^*q^{i+2r})/s^*q^{2r}. \tag{127}$$

Proof. Note that

$$r_1(W)r_2(W) = ss^*q^{2r+2t+d+1}, \quad r_1(W)+r_2(W) = \tau(W)q^{r+t+d}. \tag{128}$$

Eliminating $\tau(W), ss^*$ in (124) using (128), we routinely obtain (126). Arguing similarly, we obtain (127). □

The next theorem will involve basic hypergeometric series. For the definition, see [6, p. 4].

Theorem 17.6. *Let Γ be as in Definition 17.1. Then*

$$u_i(\theta_{t+j}) = {}_4\phi_3\left(\left.\begin{matrix} q^{-i}, s^*q^{2r+i+1}, q^{-j}, sq^{2t+j+1} \\ r_1(W)q, r_2(W)q, q^{-d} \end{matrix}\right| q, q\right) \quad (0 \leqslant i, j \leqslant d),$$

where $u_i = u_i^W$, $r_1(W)$, $r_2(W)$ are from Definitions 11.5 and 17.5.

Proof. Routine calculation using (77) and Lemmas 17.2 and 17.5. □

The polynomials u_i are q-Racah polynomials. For the definition of q-Racah polynomials, see [1].

Theorem 17.7. *Let Γ be as in Definition 17.1. Then the intersection numbers of W are as follows:*

$$b_0(W) = \frac{hq^{-t}(1-q^{-d})(1-r_1(W)q)(1-r_2(W)q)}{(1-s^*q^{2r+2})},$$

$$b_i(W) = \frac{hq^{-t}(1-q^{i-d})(1-s^*q^{2r+i+1})(1-r_1(W)q^{i+1})(1-r_2(W)q^{i+1})}{(1-s^*q^{2r+2i+1})(1-s^*q^{2r+2i+2})} \quad (1 \leqslant i \leqslant d-1),$$

$$c_i(W) = \frac{hq^{-t}(1-q^i)(1-s^*q^{2r+i+d+1})(r_1(W)-s^*q^{2r+i})(r_2(W)-s^*q^{2r+i})}{s^*q^{2r+d}(1-s^*q^{2r+2i})(1-s^*q^{2r+2i+1})} \quad (1 \leqslant i \leqslant d-1),$$

$$c_d(W) = \frac{hq^{-t}(1-q^d)(r_1(W)-s^*q^{2r+d})(r_2(W)-s^*q^{2r+d})}{s^*q^{2r+d}(1-s^*q^{2r+2d})},$$

$$a_i(W) = \theta_t - b_i(W) - c_i(W) \quad (0 \leqslant i \leqslant d),$$

where $r_1(W), r_2(W)$ are from Theorem 17.5. To obtain the dual intersection numbers of W, replace (h, s^*, r, t) with (h^*, s, t, r).

Proof. Evaluate the equations on the left in (80) and (100) using Lemma 17.2 and Theorem 17.5. □

Corollary 17.8. *Let Γ be as in Definition 17.1. Then the intersection numbers of Γ are as follows*:

$$\begin{aligned}
b_0 &= \frac{h(1-q^{-D})(1-r_1q)(1-r_2q)}{(1-s^*q^2)},\\
b_i &= \frac{h(1-q^{i-D})(1-s^*q^{i+1})(1-r_1q^{i+1})(1-r_2q^{i+1})}{(1-s^*q^{2i+1})(1-s^*q^{2i+2})} \quad (1 \leqslant i \leqslant D-1),\\
c_i &= \frac{h(1-q^{i})(1-s^*q^{i+D+1})(r_1-s^*q^{i})(r_2-s^*q^{i})}{s^*q^D(1-s^*q^{2i})(1-s^*q^{2i+1})} \quad (1 \leqslant i \leqslant D-1),\\
c_D &= \frac{h(1-q^{D})(r_1-s^*q^{D})(r_2-s^*q^{D})}{s^*q^D(1-s^*q^{2D})},\\
a_i &= \theta_0 - b_i - c_i \quad (0 \leqslant i \leqslant D),
\end{aligned}$$

where r_1, r_2 are from Definition 17.1. To obtain the dual intersection numbers of Γ, replace (h, s^*) with (h^*, s).

Proof. Apply Theorem 17.7 with W equal to the trivial T-module and use Lemma 9.6. □

We now turn our attention to graphs with classical parameters.

Definition 17.9. Let $b, \alpha, \sigma \in \mathbb{C}$ with $b \notin \{-1, 0, 1\}$. The graph Γ is said to have *classical parameters* (D, b, α, σ) whenever

$$\begin{aligned}
c_i &= \frac{b^i-1}{b-1}\left(1+\alpha\frac{b^{i-1}-1}{b-1}\right) \quad (1 \leqslant i \leqslant D),\\
b_i &= \frac{b^D-b^i}{b-1}\left(\sigma-\alpha\frac{b^{i}-1}{b-1}\right) \quad (0 \leqslant i \leqslant D-1).
\end{aligned}$$

Theorem 17.10 [3, Corollary 8.4.2]**.** *Let Γ be as in Definition 17.9. The following hold:*

(i) *There exists an ordering $\{\theta_i\}_{i=0}^D$ of the eigenvalues such that for $0 \leqslant i \leqslant D$,*

$$\theta_i = \eta + \mu b^i + h b^{-i},$$

where

$$\begin{aligned}
\eta &= \frac{(\sigma-1)(1-b)-\alpha(b^D+1)}{(b-1)^2},\\
\mu &= \frac{\alpha-b+1}{(b-1)^2},\\
h &= \frac{b^D(\sigma b-\sigma+\alpha)}{(b-1)^2}.
\end{aligned}$$

(ii) *Γ is Q-polynomial with respect to $\{\theta_i\}_{i=0}^D$.*

Let E be the primitive idempotent of Γ corresponding to θ_1. Let $\{\theta_i^*\}_{i=0}^D$ be its corresponding dual eigenvalue sequence. Our next goal is to express these values in terms of α, b, σ, D.

Theorem 17.11 [3, Corollary 8.4.2]**.** *Let $\{\theta_i^*\}_{i=0}^D$ be the dual eigenvalues corresponding to E. Then for $0 \leqslant i \leqslant D$,*

$$\theta_i^* = \eta^* + h^* b^{-i}, \tag{129}$$

where

$$\eta^* = \theta_0^* \left(1 + \frac{b}{b-1} \frac{(\sigma - \alpha)(b^{D-1} - 1) - b + 1 - \sigma(b^D - 1)}{\sigma(b^D - 1)}\right),$$
$$h^* = \theta_0^* - \eta^*. \tag{130}$$

Observe that by (129), $h^* \neq 0$ since $\theta_i^* \neq \theta_0^*$ for $1 \leqslant i \leqslant D$. Later in this section, we will express θ_0^* in terms of α, b, σ, D.

Lemma 17.12. *Let Γ be as in Definition 17.9. Then for $0 \leqslant i, j \leqslant D$,*

$$\theta_i - \theta_j = (b^i - b^j)(\mu - hb^{-i-j}),$$
$$\theta_i^* - \theta_j^* = h^* b^{-i-j}(b^j - b^i),$$

where μ, h, h^* are the from Theorems 17.10 and 17.11.

Proof. Routine calculation using Theorems 17.10 and 17.11. $\square$

Lemma 17.13. *Let Γ be as in Definition 17.9. Then there exists $\tau(W) \in \mathbb{C}$ such that the first split sequence and second split sequence of W are given by*

$$\varphi_i(W) = (1 - b^i)(1 - b^{d-i+1})(\tau(W) - hh^* b^{-r-t-i-d}) \quad (1 \leqslant i \leqslant d), \tag{131}$$
$$\phi_i(W) = (1 - b^i)(1 - b^{d-i+1})(\tau(W) - h^* \mu b^{-r+t-i}) \quad (1 \leqslant i \leqslant d), \tag{132}$$

where μ, h, h^* are from Theorems 17.10 and 17.11.

Proof. Similar to the proof of Lemma 17.4. $\square$

Applying Lemma 17.13 with W equal to the trivial T-module and using Definition 15.20, we obtain the following corollary.

Corollary 17.14. *Let Γ be as in Definition 17.9. Then the first split sequence and second split sequence of Γ are as follows:*

$$\varphi_i = (1 - b^i)(1 - b^{D-i+1})(\tau - hh^* b^{-i-D}) \quad (1 \leqslant i \leqslant D),$$
$$\phi_i = (1 - b^i)(1 - b^{D-i+1})(\tau - h^* \mu b^{-i}) \quad (1 \leqslant i \leqslant D),$$

where μ, h, h^* are from Theorems 17.10 and 17.11 and τ is the $\tau(W)$ associated with the trivial T-module.

Observe that the parameter h given in Theorem 17.10 may or may not be zero. Consider a thin irreducible T-module W. Note that if $h = 0$, then $\tau(W) \neq 0$. This follows from (131) and the fact that $\varphi_i(W) \neq 0$ for $1 \leqslant i \leqslant d$.

Theorem 17.15. *Let Γ be as in Definition 17.9. For $0 \leqslant i, j \leqslant d$,*

$$u_i(\theta_{t+j}) = \begin{cases} {}_3\phi_2\left(\begin{matrix} b^{-i}, b^{-j}, \frac{\mu b^{2t+j}}{h} \\ b^{-d}, \frac{\tau(W)b^{r+t+d+1}}{hh^*} \end{matrix} \middle| b, b\right) & \text{if } h \neq 0 \\ {}_2\phi_1\left(\begin{matrix} b^{-i}, b^{-j} \\ b^{-d} \end{matrix} \middle| b, \frac{\mu h^* b^{-r+t+j-d}}{\tau(W)}\right) & \text{if } h = 0 \end{cases},$$

where $u_i = u_i^W$ is from Definition 11.5 and $\mu, h, h^*, \tau(W)$ are from Theorems 17.10 and 17.11 and Lemma 17.13.

Proof. Routine calculation using (77) and Lemmas 17.12 and 17.13. □

Theorem 17.16. *Let Γ be as in Definition 17.9. Then the intersection numbers of W are as follows:*

$$b_i(W) = b^{r+2i+1}(1 - b^{d-i})(\tau(W) - hh^*b^{-r-t-i-d-1})/h^* \quad (0 \leqslant i \leqslant d-1), \tag{133}$$

$$c_i(W) = b^{r+i}(b^i - 1)(\tau(W) - h^*\mu b^{-r+t-i})/h^* \quad (1 \leqslant i \leqslant d), \tag{134}$$

$$a_i(W) = \theta_t - b_i(W) - c_i(W) \quad (0 \leqslant i \leqslant d),$$

where $\mu, h, h^*, \tau(W)$ are from Theorems 17.10 and 17.11 and Lemma 17.13.

Proof. Evaluate the equations on the left in (80) and (100) using Lemmas 17.12 and 17.13. □

Theorem 17.17. *Let Γ be as in Definition 17.9. Then the dual intersection numbers of W are as follows:*

$$b_0^*(W) = \frac{(b^d - 1)(\tau(W) - hh^*b^{-r-t-d-1})}{\mu b^t - hb^{-t-1}},$$

$$b_i^*(W) = \frac{b^{-i}(b^{d-i} - 1)(\tau(W) - hh^*b^{-r-t-i-d-1})(\mu b^t - hb^{-t-i})}{(\mu b^t - hb^{-t-2i-1})(\mu b^t - hb^{-t-2i})} \qquad (1 \leqslant i \leqslant d-1), \tag{135}$$

$$c_i^*(W) = \frac{b^{d-2i+1}(1 - b^i)(\tau(W) - h^*\mu b^{-r+t-d+i-1})(\mu b^t - hb^{-t-i-d})}{(\mu b^t - hb^{-t-2i})(\mu b^t - hb^{-t-2i+1})} \quad (1 \leqslant i \leqslant d-1), \tag{136}$$

$$c_d^*(W) = \frac{b^{-d+1}(1 - b^d)(\tau(W) - h^*\mu b^{-r+t-1})}{\mu b^t - hb^{-t-2d+1}},$$

$$a_i^*(W) = \theta_r^* - b_i^*(W) - c_i^*(W) \quad (0 \leqslant i \leqslant d),$$

whete $\mu, h, h^*, \tau(W)$ are from Theorems 17.10 and 17.11 and Lemma 17.13.

Proof. Note that for $1 \leqslant i \leqslant d-1$, $\mu b^t - hb^{-t-2i-1} \neq 0$ since this is a factor of $\theta_{t+i} - \theta_{t+i+1}$ by Lemma 17.12 and the eigenvalues of Γ are mutually distinct. Similarly, $\mu b^t - hb^{-t-2i}$ and $\mu b^t - hb^{-t-2i+1}$ are nonzero since these are factors of $\theta_{t+i-1} - \theta_{t+i+1}$ and $\theta_{t+i} - \theta_{t+i-1}$, respectively. Arguing as in the proof of Theorem 17.16, we obtain the desired result. □

In Definition 17.9, we gave a formula for the intersection numbers of Γ in terms of α, b, σ, D. We now give an alternate formula in terms of μ, h, h^*.

Theorem 17.18. *Let Γ be as in Definition 17.9. Then the intersection numbers of Γ are as follows*:

$$\begin{aligned} b_i &= b^{2i+1}(1-b^{D-i})(\tau - hh^*b^{-i-D-1})/h^* \quad (0 \leqslant i \leqslant D-1), \\ c_i &= b^i(b^i-1)(\tau - h^*\mu b^{-i})/h^* \quad (1 \leqslant i \leqslant D), \\ a_i &= \theta_0 - b_i - c_i \quad (0 \leqslant i \leqslant D), \end{aligned} \tag{137}$$

where μ, h, h^*, τ are from Theorems 17.10 and 17.11 and Corollary 17.14.

Proof. Immediate from Lemmas 9.6 and 17.16. □

We now give a formula for the dual intersection numbers of Γ.

Theorem 17.19. *Let Γ be as in Definition 17.9. Then the dual intersection numbers of Γ are as follows*:

$$\begin{aligned} b_0^* &= \frac{(b^D-1)(\tau - hh^*b^{-D-1})}{\mu - hb^{-1}}, \\ b_i^* &= \frac{b^{-i}(b^{D-i}-1)(\tau - hh^*b^{-i-D-1})(\mu - hb^{-i})}{(\mu - hb^{-2i-1})(\mu - hb^{-2i})} \quad (1 \leqslant i \leqslant D-1), \\ c_i^* &= \frac{b^{D-2i+1}(1-b^i)(\tau - h^*\mu b^{-D+i-1})(\mu - hb^{-i-D})}{(\mu - hb^{-2i})(\mu - hb^{-2i+1})} \quad (1 \leqslant i \leqslant D-1), \\ c_D^* &= \frac{b^{-D+1}(1-b^D)(\tau - h^*\mu b^{-1})}{(\mu - hb^{-2D+1})}, \\ a_i^* &= \theta_0^* - b_i^* - c_i^* \quad (0 \leqslant i \leqslant D), \end{aligned} \tag{138}$$

where μ, h, h^*, τ are from Theorems 17.10 and 17.11 and Corollary 17.14.

Proof. Immediate from Lemma 9.6 and Theorem 17.17 □

Recall that in Lemma 17.11, we gave a formula for the dual eigenvalues of Γ in terms of $\alpha, b, \sigma, \theta_0^*$. We are now ready to solve for θ_0^*.

Lemma 17.20. *Let Γ be as in Definition 17.9. Let h^* and τ be as in Theorem 17.11 and Corollary 17.14, respectively. Then*

$$h^* = -\frac{b^{D-1}(\mu b^2 - h)(\mu b - h)}{(\mu b^{D+1} - h)(b^{D-1} + \mu(b-1)(b^{D-1}-1))}, \tag{139}$$

$$\tau = \frac{h^*(1 + \mu b - \mu)}{b(b-1)}, \tag{140}$$

$$\theta_0^* = \frac{h^*\sigma(b^D-1)(1-b)}{b((\sigma-\alpha)(b^{D-1}-1) - b + 1 - \sigma(b^D-1))}. \tag{141}$$

Proof. To obtain (139) and (140), solve the system of equations in (137) and (138) with $i = 1$ and use the fact that $c_1 = 1 = c_1^*$. Line (141) is immediate from (130). □

In Theorem 17.16, we gave a formula for the intersection numbers of W. We now give an alternate formula which is reminiscent of Definition 17.9

Theorem 17.21. *Let Γ be as in Definition 17.9. Then*

$$c_i(W) = \frac{b^i - 1}{b - 1}\left(c_1(W) + \alpha(W)\frac{b^{i-1} - 1}{b - 1}\right) \quad (1 \leqslant i \leqslant d), \tag{142}$$

$$b_i(W) = \frac{b^d - b^i}{b - 1}\left(\sigma(W) - \alpha(W)\frac{b^i - 1}{b - 1}\right) \quad (0 \leqslant i \leqslant d - 1), \tag{143}$$

where

$$\alpha(W) = \tau(W)b^{r+1}(b-1)^2/h^*,$$

$$\sigma(W) = \frac{hb^{-t}(b-1)^2 - \alpha(W)b^d}{b^d(b-1)}.$$

Proof. Comparing the right side of (142) with that of (134), we find that (142) holds. Comparing the right side of (143) with that of (133), we obtain (143). □

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References

[1] R. Askey, J.A. Wilson, A set of orthogonal polynomials that generalize the Racah coefficients or $6-j$ symbols, SIAM J. Math. Anal. 10 (1979) 1008–1016.
[2] E. Bannai, T. Ito, Algebraic Combinatorics I: Association Schemes, Benjamin/Cummings, London, 1984.
[3] A.E. Brouwer, A.M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer-Verlag, Berlin, 1989.
[4] J.S. Caughman IV, The Terwilliger algebra of bipartite P- and Q-polynomial schemes, Discrete Math. 196 (1999) 65–95.
[5] B. Curtin, The Terwilliger algebra of a 2-homogeneous bipartite distance-regular graph, J. Combin. Theory 81 (B) (2001) 125–141.
[6] G. Gasper, M. Rahman, Basic hypergeometric series, Cambridge University Press, Cambridge, 1990.
[7] J.T. Go, The Terwilliger algebra of the hypercube, Eur. J. Combin. 23 (2002) 399–429.
[8] P. Terwilliger, The subconstituent algebra of an association scheme (part I), J. Algebraic Combin. 1 (4) (1992) 363–388.
[9] P. Terwilliger, The subconstituent algebra of an association scheme (part II), J. Algebraic Combin. 2 (1) (1993) 73–103.
[10] P. Terwilliger, Two linear transformations each tridiagonal with respect to an eigenbasis of the other, Linear Algebra Appl. 330 (2001) 149–203.
[11] P. Terwilliger, Two linear transformations each tridiagonal with respect to an eigenbasis of the other; an algebraic approach to the Askey scheme of orthogonal polynomials. Available from: arxiv:math/0408390v3.