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Permutations Restricted by Two Distinct Patterns of Length Three

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Define $S_n(R;T)$ to be the set of permutations on n letters which avoid all patterns in the set R and contain each pattern in the multiset T exactly once. In this paper we enumerate $S_n(\alpha;\beta)$ and $S_n(\varnothing;\{\alpha,\beta\})$ for all $\alpha \neq \beta \in S_3$. © 2001 Elsevier Science

1. INTRODUCTION

Let $\pi \in S_n$ be a permutation of $[n] = \{1, 2, \ldots, n\}$ written as a word. Let $\alpha \in S_k$, $k \leq n$. We say that π *contains the pattern* α if there exist indices $i_1, i_2, \ldots, i_k$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ is equivalent to α, where we define equivalence as follows. Define $\bar{\pi}_{i_j} = |\{m : \pi_{i_m} \leq \pi_{i_j}, m = 1, 2, \ldots, k\}|$. If $\alpha = \bar{\pi}_{i_1}\bar{\pi}_{i_2}\cdots\bar{\pi}_{i_k}$ then we say that α and $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ are equivalent. For example, if $\tau = 124635$ then τ contains the pattern 213 by noting that $\tau_3\tau_5\tau_6 = 435$ is equivalent to 213. We say that π *avoids the pattern* α if π does not contain the pattern α. In our above example, τ avoids the pattern 321.

Let $\alpha \neq \beta$ be patterns of length three. In this article we enumerate all permutations which contain α exactly once and avoid β as well as those permutations which contain each of α and β exactly once.

2. SOME HISTORY

The investigation of permutations which avoid a pattern of length three started well over a hundred years ago as exhibited in [C] and references therein. Knuth [Kn] investigated permutations which avoid any single pattern of length 3 and showed that, regardless of the pattern, such

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permutations are enumerated by the Catalan numbers. Bijective results are given in [Ri], [Krt], [SS], and [W1]. To describe the enumeration results more succinctly we introduce the following notation. Let $S_n(R)$ be the set of permutations on $[n]$ which avoid all patterns in the set R, where we omit the set notation if $|R| = 1$ and let $s_n(R) = |S_n(R)|$. Knuth's result can then be stated as $s_n(\alpha) = \frac{1}{n+1}\binom{2n}{n}$ for all $\alpha \in S_3$.

Following Knuth's result, two natural progressions were made: the investigation of $S_n(R)$ for $R \subseteq S_3$ and the investigation of $S_n(\beta)$ for $\beta \in S_4$. With respect to the former investigation, Simion and Schmidt [SS] gave a complete study of $s_n(R)$ for all $R \subseteq S_3$. With respect to the latter investigation, in two beautiful papers, Gessel [Ge] found $s_n(1234)$ and Bóna [B1] found $s_n(1342)$. Further results on $S_n(\alpha)$ for $\alpha \in S_4$ are given by West in [W1] and [W2] and by Stankova in [S]. The exact enumeration of 1324-avoiding permutations is still an open question, with the only result being a lower bound given by Bóna in [B2].

Several logical extensions followed: the investigation of $S_n(R)$ for $R \subseteq S_4$, the investigation of $S_n(S \cup T)$ for $S \subseteq S_3$ and $T \subseteq S_4$, and the investigation of $S_n(R)$ for $R \subseteq S_j$, $j > 4$. Guibert, in [Gu], showed that for certain $R \subseteq S_4$ with two elements, the corresponding $s_n(R)$ are given by Schröder numbers. In [B3] and [Kr], Bóna and Kremer, respectively, gave further extensions for $R \subseteq S_4$ with two elements. Mansour [M] completely enumerated $S_n(R \cup \{\alpha\})$ for $R \subseteq S_3$ and $\alpha \in S_4$. Results for permutations avoiding patterns of length greater than four can be found in [BLPP1], [BLPP2], [CW], and [Kr].

A natural generalization of pattern-avoiding permutations is pattern-containing permutations. To aid in the discussion of pattern-containing permutations we introduce the following notation. Let $S_n(R;T)$ be the set of permutations on $[n]$ which avoid all patterns in the set R and contain each pattern in the multiset T exactly once, where we again omit the set notation for singleton sets and let $s_n(R;T) = |S_n(R;T)|$.

Recently, there has been much research focused on $S_n(R;T)$ for various sets R and multisets T. Below, we give some results in this direction. First, in [N], Noonan proved that $s_n(\varnothing; 123) = \frac{3}{n}\binom{2n}{n+3}$, a remarkably elegant formula. Bóna, in [B4], then showed that $s_n(\varnothing; 132) = \binom{2n-3}{n-3}$, an even simpler formula, proving a conjecture presented in [NZ]. These two results give $s_n(\varnothing; \alpha)$ for all $\alpha \in S_3$ by applying the following two bijections (given in [SS]).

Reversal: Define $r : S_n \to S_n$ by $r(\pi_1\pi_2\cdots\pi_n) = \pi_n\pi_{n-1}\cdots\pi_1$.

Complementation: Define $c : S_n \to S_n$ by $c(\pi_1\pi_2\cdots\pi_n) = (n-\pi_1+1)(n-\pi_2+1)\cdots(n-\pi_n+1)$.

We will also have need of a third bijection (given in [SS]) which is defined as follows.

Inverse: Define $i: S_n \to S_n$ as the group theoretic inverse.

It is easy to see that if π contains exactly $s \geq 0$ occurrences of the pattern α, then $r(\pi)$ (resp. $c(\pi)$, $i(\pi)$) contains exactly s occurrences of the pattern $r(\alpha)$ (resp. $c(\pi)$, $i(\pi)$). By applying r, c, and $r \circ c$ we see that $s_n(\emptyset; 123) = s_n(\emptyset; 321)$ and $s_n(\emptyset; 132) = s_n(\emptyset; 231) = s_n(\emptyset; 312) = s_n(\emptyset; 213)$.

In [B4], Bóna also gave the generating function for $\{s_n(\emptyset; \{132, 132\})\}_n$. In [R], the formulas for $s_n(132; 123)$, $s_n(123; 132)$, and $s_n(\emptyset; \{123, 132\})$ are given. These results were extended in [RWZ] to give the generating function for $\{s_n(132; \{123^r\})\}_{r,n\geq 0}$ in the form of a continued fraction. Mansour and Vainshtein [MV1] generalized this result to give the generating function for $\{s_n(132; \{(123\cdots k)^r\})\}_{r,n}$ for a given k and showed the relation of such permutations to Chebyshev polynomials of the second kind. In [CW] other similar permutations were first shown to be related to the Chebyshev polynomials of the second kind. Independently, Jani and Rieper [JR] also extended the result in [RWZ] to find the generating function given in [MV1] using the theory of ordered trees. Shortly thereafter, Krattenthaler, in [Krt], used Dyck path bijections to reprove elegantly the results in [MV1] and [JR], extend results given in [CW], give a precise asymptotic formula for $s_n(132, \{(123\cdots k)^r\})$, and show that $s_n(132, \{(123\cdots k)^r\}) \asymp s_n(123, \{((k-1)(k-2)\cdots 1k)^r\})$.

3. PRELIMINARIES

In this section we give some definitions and state a known result (without proof) upon which we will need to draw.

In order to discuss our analysis we have need of the following two definitions. The first definition has become a standard definition, while the second definition is new.

DEFINITION (Wilf class). Let S_1 and S_2 be two sets. If $s_n(S_1) = s_n(S_2)$ then we say that S_1 and S_2 are in the same *Wilf class* or are *Wilf equivalent*.

EXAMPLE. There is only one Wilf class for permutations avoiding a single pattern of length 3 since $s_n(\alpha) = \frac{1}{n+1}\binom{2n}{n}$ for any $\alpha \in S_3$.

DEFINITION (almost-Wilf class[1]). Let S_1 and S_2 be two sets and let T_1 and T_2 be two multisets. If $s_n(S_1; T_1) = s_n(S_2; T_2)$ then we say that $(S_1; T_1)$ and $(S_2; T_2)$ are in the same *almost-Wilf class* or are *almost-Wilf equivalent*.

[1]As an aside, Herb Wilf has told the author that he is not fond of the monicker Wilf class; however, in honor of Herb (and due to the lack of a better name), we extend what has become the standardized definition of pattern-avoiding permutation classes.

THEOREM 3.1 (Simion and Schmidt, [SS]).

1. *For* $\{\alpha, \beta\} \in \{\{123, 132\}, \{123, 213\}, \{132, 213\}, \{132, 231\}, \{132, 312\}, \{213, 231\}, \{213, 312\}, \{231, 312\}, \{231, 321\}, \{312, 321\}\}$ *we have* $s_n(\{\alpha, \beta\}) = 2^{n-1}$ *for* $n \geq 2$ *and* $s_1(\{\alpha, \beta\}) = 1$.

2. *For* $\{\alpha, \beta\} \in \{\{123, 231\}, \{123, 312\}, \{132, 312\}, \{213, 321\}\}$ *we have* $s_n(\{\alpha, \beta\}) = \binom{n}{2} + 1$.

3. $s_n(\{123, 321\}) = 0$ *for* $n \geq 5$.

4. ON $S_N(\alpha; \beta)$

As seen in Section 2 we know $s_n(\alpha; \beta)$ for $(\alpha, \beta) \in \{(123, 132), (132, 123)\}$. Using the reversal and complementation bijections presented in Section 2 we see that the following is true.

THEOREM 4.1. *For* $(\alpha, \beta) \in \{(123, 132), (123, 213), (132, 123), (213, 123), (231, 321), (312, 321), (321, 231), (321, 312)\}$ *we have* $s_n(\alpha; \beta) = (n-2)2^{n-3}$ *for* $n \geq 3$.

To complete the enumeration $s_n(\alpha; \beta)$ for all $\alpha \neq \beta \in S_3$ we must consider the following classes, which can be obtained through application of the reversal, complementation, and inverse bijections.

(1) $\{(123; 321), (321; 123)\}$

(2) $\{(123, 231), (123, 312), (321, 132), (321, 213)\}$

(3) $\{(132; 213), (213; 132), (231; 312), (312; 231)\}$

(4) $\{(132; 231), (132; 312), (213; 231), (213; 312), (231; 132), (231; 213), (312; 132), (312; 213)\}$

(5) $\{(132, 321), (213, 321), (231, 123), (312, 123)\}$

Trivially, we have $s_n(123; 321) = 0$ for $n \geq 6$. The enumeration concerning the remaining classes follows from results, which will be noted below, given by Mansour and Vanshtein in [MV2] and [MV3].

THEOREM 4.2. *For* $(\alpha, \beta) \in \{(123, 231), (123, 312), (321, 132), (321, 213)\}$ *we have* $s_n(\alpha; \beta) = 2n - 5$ *for* $n \geq 3$.

Proof. This is a particular case of Theorem 3.3 in [MV3] (with $m = 2$ and $k = 3$). ∎

THEOREM 4.3. *For* $(\alpha, \beta) \in \{(132, 213), (213, 132), (231, 312), (312, 231)\}$ *we have* $s_n(\alpha; \beta) = n2^{n-5}$ *for* $n \geq 4$ *and* $s_3(\alpha; \beta) = 1$.

Proof. This follows from Example 3.2 in [MV2] with $p = 1, m = 2$, and $k = 3$. ∎

THEOREM 4.4. *For* $(\alpha, \beta) \in \{(132, 231), (132, 312), (213, 231), (213, 312), (231, 132), (231, 213), (312, 132), (312, 213)\}$ *we have* $s_n(\alpha; \beta) = 2^{n-3}$ *for* $n \geq 3$.

Proof. This is a particular case of Theorem 3.4 in [MV2] (with $m = 1$ and $k = 3$). ∎

THEOREM 4.5. *For* $(\alpha, \beta) \in \{(132, 321), (213, 321), (231, 123), (312, 123)\}$ *we have* $s_n(\alpha; \beta) = 2n - 5$ *for* $n \geq 3$.

Proof. This follows immediately from Theorem 3.2 in [MV2]. ∎

Remark. Notice that *a priori* there were six classes we had to consider (by Theorems 4.1 through 4.5 and the trivial case). (This is one less than the seven classes to consider before [R] showed that (123;132) and (132;123) are almost-Wilf equivalent.) However, the results above show that there are in fact only five almost-Wilf classes associated with $S_n(\alpha; \beta)$, $\alpha \neq \beta \in S_3$. Some explanation of this is given in the following section.

4.1. *Generating* $S_n(123; 312)$ *and* $S_n(312; 123)$

In this section we investigate why $s_n(123; 312) = s_n(312; 123)$ (which are both equal to $2n - 5$). We will show that the two sets considered here are generated by almost exactly the same rule and let the reader infer a bijection from this result. Define $\phi : S_{m-1} \to S_m$ by $\phi(\pi_1\pi_2 \cdots \pi_{m-1}) = (\pi_1 + 1)(\pi_2 + 1) \cdots (\pi_{m-1} + 1)1$.

It is clear for any $\sigma \in S_{n-1}(123; 312)$ and any $\tau \in S_{n-1}(312; 123)$ that $\phi(\sigma) \in S_n(123; 312)$ and $\phi(\tau) \in S_n(312; 123)$. Since $S_3(123; 312) = \{312\}$ and $S_3(312; 123) = \{123\}$ we can use the rules below to generate $S_n(123; 312)$ and $S_n(312; 123)$.

Generating Rule for $S_n(123; 312)$: By Theorem 4.2, it is trivial to check that $S_n(123; 312) = \{\phi(\pi) : \pi \in S_{n-1}(123; 312)\} \cup \{31n(n-1)(n-2) \cdots 542, (n-2)(n-3) \cdots 32n1(n-1)\}$.

Generating Rule for $S_n(312; 123)$: By Theorem 4.5, it is trivial to check that $S_n(312; 123) = \{\phi(\pi) : \pi \in S_{n-1}(312; 123)\} \cup \{1(n-1)n(n-2)(n-3) \cdots 32, (n-2)(n-1)(n-3)(n-4) \cdots 21n\}$.

5. ON $S_N(\emptyset; \{\alpha, \beta\})$

We first note that trivially $s_n(\emptyset; \{123, 321\}) = 0$ for $n \geq 6$. Next, using the bijections r and c, we have four classes to consider:

(1) $\overline{\{123, 231\}} = \{\{123, 231\}, \{123, 312\}, \{132, 321\}, \{213, 321\}\}$

(2) $\overline{\{123, 132\}} = \{\{123, 132\}, \{123, 213\}, \{231, 321\}, \{312, 321\}\}$

(3) $\overline{\{132, 213\}} = \{\{132, 213\}, \{231, 312\}\}$

(4) $\overline{\{132, 231\}} = \{\{132, 231\}, \{132, 312\}, \{213, 231\}, \{213, 312\}\}$.

Class (2) was enumerated in [R] giving the following theorem.

THEOREM 5.1. *For* $\{\alpha, \beta\} \in \{\{123, 132\}, \{123, 213\}, \{231, 321\}, \{312, 321\}\}$ *we have* $s_n(\emptyset, \{\alpha, \beta\}) = (n-3)(n-4)2^{n-5}$ *for* $n \geq 5$.

Except for the proof of Theorem 5.4, in the proofs below we will isolate either the element 1 or the element n in each permutation, π. Denote by $\pi(1)$ the elements (in order) to the left of the isolated element and by $\pi(2)$ the elements (in order) to the right of the isolated element. Hence, we have $\pi = \pi(1)1\pi(2)$ or $\pi = \pi(1)n\pi(2)$. We start with class (1).

THEOREM 5.2. *For* $\{\alpha, \beta\} \in \{\{123, 231\}, \{123, 312\}, \{132, 321\}, \{213, 321\}\}$ *we have* $s_n(\emptyset, \{\alpha, \beta\}) = 2n - 5$ *for* $n \geq 5$ *and* $s_4(\emptyset, \{\alpha, \beta\}) = 2$.

Proof. We will use $\{123, 312\}$ for our proof. Let $f_n = s_n(\emptyset; \{123, 312\})$, let $\pi \in S_n(\emptyset; \{123, 312\})$, and let $\pi_i = n$.

We have three cases to consider: (i) the (312) pattern occurs with n as the '3' and the $(12) \in \pi(2)$, (ii) the pattern $(312) \in \pi(1)$, and (iii) the '2' in the (312) pattern is in $\pi(2)$ while $(31) \in \pi(1)$.

We start with case (i): the (312) pattern occurs with n as the '3' and the $(12) \in \pi(2)$. Let x be the '1' and y be the '2' in the (312) pattern.

Write $\pi = \pi(1)n\,A\,x\,B\,y\,C$, where A, B, and C represent the portions of π between two distinguished elements (either n and x, x and y, or y and the end of π).

We will first show that A is empty. Assume otherwise and let $a \in A$. Then either nay is another (312) occurrence (if $a < y$) or axy is another (312) occurrence (if $a > y$). Hence, A is empty. Next, we will show that B must be empty. Assume otherwise and let $b \in B$. Then either nby is another (312) occurrence (if $b < y$) or nxb is another (312) occurrence (if $b > y$). Hence, B must also be empty. Thus we may write $\pi = \pi(1)\,n\,x\,y\,C$.

Next, we notice that for any $c \in C$ we must have $c < x$ and for any $p \in \pi(1)$ we must have $p < y$ to avoid another (312) occurrence. Furthermore, if C were to contain a (12) pattern then we would have another occurrence of (312) with n acting as the '3'. Hence, the elements of C must be in decreasing order and thus our (123) pattern must start in $\pi(1)$. Similarly, the elements of $\pi(1)$ must be in decreasing order or we would have at least two occurrences of (123) with both n and y serving as the '3' in the (123) pattern. Hence, there exists $r \in \pi(1)$ with $r < x$ which produces rxy as our (123) pattern. Furthermore, all other elements in $\pi(1)$ must be larger than x or else we would have another occurrence of (123). Hence, we must have $r = \pi_{i-1}$ since the elements in $\pi(1)$ are decreasing. However, if $i \neq 2$ then $\pi_1 rx$ would be another (312) pattern. Thus, $i = 2$. The last piece of information we need is that since all elements in C are less than x, we must have $x = n-2$. Thus we see that our permutations in this case are of the form $\pi = r\,n\,(n-2)\,(n-1)\,C$ with the elements of C in decreasing order. Since we have $n-3$ choices for r, we have $n-3$ permutations in this case.

Next, we look at case (ii): the pattern $(312) \in \pi(1)$.

Let zxy be the (312) pattern and write $\pi = A\,z\,B\,x\,C\,y\,D\,n\,\pi(2)$. Notice that in this case we already have our (123) pattern, namely, xyn.

We first show that A, B, and C are empty. Assume otherwise and let $a \in A$, $b \in B$, and $c \in C$. For any $a \in A$ we see that either ayn would give another (123) occurrence (if $a < y$) or that axy would give another (312) occurrence (if $a > y$). For $b \in B$ we see that either byn would give another (123) occurrence (if $b < y$) or that bxy would give another (312) occurrence (if $b > y$). For $c \in C$, either xcn would be another (123) occurrence (if $c > y$) or zcy would be another (312) occurrence (if $c < y$). Hence, A, B, and C must all be empty so we may write $\pi = z\,x\,y\,D\,n\,\pi(2)$.

Next, we notice that for any element in D or $\pi(2)$, that element must be less than x, for otherwise we would have either another occurrence of (312) with z and x or another (123) occurrence with x and y. This restriction gives us $z = n-1$, $x = n-3$, and $y = n-2$. Furthermore, the elements in D must be decreasing (to avoid another (123) with n), and the elements in $\pi(2)$ must be decreasing (to avoid another (312) with n). Even further, for all $d \in D$ and all $p \in \pi(2)$ we must have $d > p$ or else we would have another (312) occurrence with zdp. Hence, the elements in both D and $\pi(2)$ are determined by the position of n. Since we have $n-3$ choices for the position of n, we have $n-3$ permutations in this case.

Last, we look at case (iii): the '2' in the (312) pattern is in $\pi(2)$ while $(31) \in \pi(1)$.

Let zxy be the (312) pattern and write $\pi = A\,z\,B\,x\,C\,n\,D\,y\,E$.

We first show that B and C are empty. Assume otherwise and let $b \in B$ and $c \in C$. For $b \in B$, either bxy is another occurrence of (312) (if $b > y$) or zby is another occurrence of (312) (if $b < y$). For $c \in C$, either we get two occurrences of (123) with zcn and xcn (if $c > z$), we get another (312) occurrence with zxc (if $x < c < z$), or we get another (312) occurrence with zcy (if $c < x$). Hence, we may write $\pi = A\,z\,x\,n\,D\,y\,E$.

Next, notice that the elements in D must be decreasing and the elements in E must be decreasing for if $d_1 < d_2 \in D$ and $e_1 < e_2 \in E$ then nd_1d_2 and ne_1e_2 are (312) patterns which would give too many (312) occurrences. Furthermore, for $d \in D$ and $e \in E$ we must have $d > z$ and $e < x$. Otherwise, if $d < z$ then we would have another (312) occurrence with zxd (if $d > x$) or zdy (if $d < x$), and if $e > x$ we would have another (312) occurrence with zxe (if $e < z$) or nye (if $e > z$). Next, we see that for all $a \in A$ we must have $x < a < y$, otherwise if $a > y$ we would obtain another (312) occurrence with x and y, and if $a < x$ we would have two occurrences of (123) with axn and axy. We also note that A must contain exactly one element since for any $a \in A$, azn produces a (123) pattern and if A is empty we cannot obtain a (123) occurrence. Since A is not empty we now see that D must be empty to avoid another (123) occurrence with a and z. We may now write $\pi = a\,z\,x\,n\,y\,E$, where $x < a < y$.

Since all elements in E must be smaller than x we see that $x = n-4$, $y = n-2$, $z = n-1$, and $a = n-3$. Finally, since the elements in E must be decreasing we see that we only have a single permutation in this case (provided $n \geq 5$).

Summing over all cases we have $s_n(\emptyset, \{123, 312\}) = 2n - 5$ for $n \geq 5$. ∎

Remark. Notice that we have the interesting result that $s_n(123; 312) = s_n(\emptyset; \{123, 312\})$ and hence (123;312) and $(\emptyset; \{123, 312\})$ are almost-Wilf equivalent (for $n \geq 5$). This is the first nontrivial case of a "mixed restriction" equivalence.

We now move on to class (3) and prove the following theorem.

THEOREM 5.3. *For* $\{\alpha, \beta\} \in \{\{132, 213\}, \{231, 312\}\}$ *we have* $s_n(\emptyset, \{\alpha, \beta\}) = (n^2 + 21n - 28)2^{n-9}$ *for* $n \geq 7$, $s_6(\emptyset, \{\alpha, \beta\}) = 17$, $s_5(\emptyset, \{\alpha, \beta\}) = 6$, *and* $s_4(\emptyset, \{\alpha, \beta\}) = 3$.

Proof. We will use $\{231, 312\}$ for our proof. Let $f_n = s_n(\emptyset; \{231, 312\})$, let $\pi \in S_n(\emptyset; \{231, 312\})$, and let $\pi_i = 1$.

We have three cases to consider: (i) the pattern $(312) \in \pi(1)$, (ii) the pattern $(312) \in \pi(2)$, and (iii) the (312) pattern straddles 1; i.e., the '3' is in $\pi(1)$, the '2' is in $\pi(2)$, and 1 serves as the '1' in the pattern.

We start with case (i): the pattern $(312) \in \pi(1)$.

Let zxy be our (312) pattern and write $\pi = A\,z\,B\,x\,C\,y\,D\,1\,\pi(2)$. Note that we already have our (231) pattern with $xy1$.

We first argue that A, B, C, and D must all be empty. Assume otherwise and let $a \in A$, $b \in B$, $c \in C$, and $d \in D$. We start with $c \in C$. Clearly we must have $c > z$ to avoid another (312) occurrence. However, this produces $zc1$ which is another (231) occurrence. Hence, C must be empty. Next, we move to $b \in B$. We see here that either zby is another occurrence of (312) (if $b < y$) or bxy is another occurrence of (312) (if $b > y$). Hence, B must also be empty. Now, we look at $a \in A$. Here, either axy is another (312) occurrence (if $a > y$) or $ay1$ is another (231) occurrence (if $a < y$). Last, for $d \in D$, either $xd1$ would be another occurrence of (231) (if $d > x$) or xyd would be another occurrence of (231) (if $d < x$). Hence, we may now write $\pi = z\,x\,y\,1\,\pi(2)$. Since we already have both of the required patterns we see that $\pi(2) \in S_{n-4}(\{312, 231\})$. For $p \in \pi(2)$, to avoid another occurrence of (312) we must have $p > z$, thus determining the values of x, y, and z. Hence, by Theorem 3.1 we have 2^{n-5} permutations in this case for $n \geq 5$, and 1 permutation for $n = 4$.

Next we look at case (ii): the pattern $(312) \in \pi(2)$.

Let zxy be our (312) pattern and write $\pi = \pi(1)\,1\,A\,z\,B\,x\,C\,y\,D$.

We first show that B must be empty. Assume otherwise and let $b \in B$. Then either zby is another (312) (if $b < y$) or bxy is another (312) (if $b > y$). We next note that for any $c \in C$ we must have $c > z$ to avoid another (312) occurrence. Hence, zcy is a (231) pattern for any $c \in C$. Thus, $|C| \leq 1$.

We first consider the subcase $|C| = 1$. Let $c \in C$ so that we have both of the required patterns in our permutation. Write $\pi = \pi(1)\,1\,A\,z\,x\,c\,y\,D$. Notice that for any $p \in \pi(1)$ and $a \in A$ we must have $p < a$ to avoid another (312) occurrence with $p1a$. We then see that for any $a \in A$ we must have $a < x$ to avoid another (231) occurrence with azx (if $a < z$) or another (312) occurrence with axy (if $a > z$). Last, we note that for any $d \in D$ we require $d > c$ to avoid (231) with zcd (if $d < z$) or another (312) with cyd (if $z < d < c$). Now since our elements in A and D are either less than x or greater than c, we see that $y = x + 1$, $z = x + 2$, and $c = x + 3$.

We now notice that $\pi(1)\,1\,A$ read as a permutation must avoid both (231) and (312). Likewise, D must avoid both (231) and (312). Since the value of x determines its position, by Theorem 3.1 we have $\sum_{x=2}^{n-4} 2^{x-2}2^{n-x-4} + 2^{n-5} = (n-3)2^{n-6}$ permutations for $n \geq 6$, one permutation for $n = 5$, and none for $n \leq 4$ in this subcase.

Next, consider the subcase $|C| = 0$. Write $\pi = \pi(1)\,1\,A\,z\,x\,y\,D$. We have four subsubcases to consider:

(a) There exists a unique $d \in D$ with $d < x$. This gives xyd as our (231) pattern.

(b) There exists a unique $a \in A$ with $x < a < y$. This gives azx as our (231) pattern.

(c) All elements in $\pi(1)$ and A are smaller than x and our (231) pattern is within $\pi(1)\,1\,A$ while D avoids both patterns.

(d) Our (231) pattern is contained within D while $\pi(1)\,1\,A$ avoids both patterns.

In all subsubcases below let $z = \pi_j$ for some $j > i$.

We start with subsubcase (a). We must have $d = \pi_{j+3}$ in order to avoid another occurrence of (231). Write $\pi = \pi(1)\,1\,A\,z\,x\,y\,d\,\widehat{D}$. We note that for all $\hat{d} \in \widehat{D}$ and all $p \in \pi(1)\,1\,A$ we must have $\hat{d} > z$ and $p < x$ to avoid another (312) or (231) occurrence. Thus, we have $y = x + 1$ and $z = x + 2$. Hence, the value of d determines the value of j (the position of z). Last, we obviously need $\pi(1)\,1\,A$ and $\widehat{D}$ to be $\{231, 312\}$-avoiding. By Theorem 3.1, we now see that we have $\sum_{j=2}^{n-4} 2^{j-2}2^{n-j-4} + 2^{n-5}$ permutations in this subsubcase. Hence, we have $(n-3)2^{n-6}$ permutations for $n \geq 6$, one permutation for $n = 5$, and none for $n \leq 4$ in this subsubcase.

On to subsubcase (b). We must have $a = \pi_{j-1}$ to avoid another occurrence of (312). Write $\pi = \pi(1)\,1\,\widehat{A}\,a\,z\,x\,y\,D$. For all $\hat{a} \in \widehat{A}$ and for all $d \in D$ we must have $\hat{a} < x$ and $d > z$ in order to avoid another occurrence of either pattern. Thus, we have $a = x+1$, $y = x+2$, and $z = x+3$. As in subsubcase (a), we have $(n-3)2^{n-6}$ permutations for $n \geq 6$, one permutation for $n = 5$, and none for $n \leq 4$ in this subsubcase.

Next, consider subsubcase (c). We must have $\pi(1)\,1\,A \in S_{j-1}(312;231)$ and $D \in S_{n-j-2}(\{231, 312\})$. From Theorems 3.1 and 4.3, for each $j \geq 5$ we have $(j-1)2^{j-6}2^{n-j-3} = (j-1)2^{n-9}$ permutations, for $j = 4$ we have 2^{n-7} permutations, and for $j \leq 3$ we have none. Summing over all valid j we have $(n-4)(n+1)2^{n-10}$ permutations for $n \geq 7$, one permutation for $n = 6$, and none for $n \leq 5$ in this subsubcase.

Last, we have subsubcase (d). A result similar to that of subsubcase (c) holds. Noting that $\pi(1)\,1\,A \in S_{j-1}(\{231, 312\})$ and $D \in S_{n-j-2}(312;231)$, from Theorems 3.1 and 4.3, for each $j \leq n-6$ we have $2^{j-2}(n-j-2) \times 2^{n-j-7} = (n-j-2)2^{n-9}$ permutations. For $j = n-5$ we have 2^{n-7} permutations, and for $j \geq n-4$ we have none. Summing over all valid j we have $(n^2-7n+8)2^{n-10}$ permutations for $n \geq 7$ and none for $n \leq 6$ in this subsubcase.

Summing over all subsubcases, we see that we have $(n^2+11n-46)2^{n-9}$ permutations for $n \geq 6$, two permutations for $n = 5$, and none for $n \leq 4$ in the subcase $|C| = 0$.

Our last case to consider is (iii): the (312) pattern straddles 1; i.e., the '3' is in $\pi(1)$, the '2' is in $\pi(2)$, and 1 serves as the '1' in the pattern.

Let $z1y$ be our (312) pattern and write $\pi = A\,z\,B\,1\,C\,y\,D$.

We first show that B must be empty. Assume otherwise and let $b \in B$. Then we either have another occurrence of (312) with $b1y$ (if $b > y$) or another occurrence of (312) with zby (if $b < y$).

Next, we show that $|A| + |C| \leq 1$. Let $a \in A$ and $c \in C$. We first note that we must have $c > z$ in order to avoid another (312) occurrence with $z1c$. We then note that we must have $a < y$ in order to avoid another (312) occurrence with $a1y$. Hence, for every $a \in A$, $az1$ gives a (231) occurrence, and for every $c \in C$, zcy gives a (231) occurrence. Thus, $|A| + |C| \leq 1$.

If $|A| = 1$, let $a \in A$ and write $\pi = a\,z\,1\,y\,D$, with $D \in S_{n-4}(\{231, 312\})$. Furthermore, all elements in D must be larger than z so that we avoid another occurrence of (312) with z and 1. By Theorem 3.1, we have 2^{n-5} permutations here for $n \geq 5$ and one permutation for $n = 4$.

If $|C| = 1$ we also have 2^{n-5} permutations for $n \geq 5$ and one permutation for $n = 4$ via an argument very similar to that found in the preceding paragraph.

If $|A| + |C| = 0$ we write $\pi = z\,1\,y\,D$, where $D \in S_{n-4}(312;231)$ and again all elements in D are larger than z. By Theorem 4.3 we have

$(n-3)2^{n-8}$ permutations for $n \geq 7$, one permutation for $n = 6$, and none for $n \leq 5$ here.

Hence, case (iii) yields $(n+13)2^{n-8}$ permutations for $n \geq 7$, five permutations for $n = 4.5$, two permutations for $n = 4.5$, and none for $n \geq 3$.

Summing the number of permutations from all three cases proves the theorem. ∎

For our final class (class (4)) we have the following theorem, whose proof is more interesting than those above.

THEOREM 5.4. *For* $\{\alpha, \beta\} \in \{\{132, 231\}, \{132, 312\}, \{213, 231\}, \{213, 312\}\}$ *we have* $s_n(\emptyset, \{\alpha, \beta\}) = 2^{n-3}$ *for* $n \geq 4$.

Proof. We will use $\{132, 312\}$ for our proof. Let $f_n = s_n(\emptyset; \{132, 312\})$.

Let xzy be our (132) pattern and write $\pi = A\,x\,B\,z\,C\,y\,D$. First, we show that B must be empty. Assume otherwise and let $b \in B$. Then either bzy or xby is another occurrence of (132) (depending on whether $b < y$ or $b > y$).

Now let $a \in A$. We must have $y < a < z$ for otherwise we would have another (132) occurrence with azy or more than one (312) occurrence with axz and axy. Also, for $c \in C$ we must have $c < y$ or else we would have another occurrence of (132) with xcy.

We now turn our attention to D. For any $d \in D$ we must have $d < x$ or $d > z$ in order to avoid another (132) occurrence with x and z. Furthermore, those elements in D which are larger than z must be in increasing order so that we avoid another (132) occurrence with x, and those elements in D which are smaller than x must be in decreasing order so that we avoid more than one occurrence of (312) with x, y, and z.

Turning back to A and C we now argue that $|A| + |C| = 1$. To see this, note that for any $a \in A$ and any $c \in C$ both axy and zcy are (312) patterns. Since we may only have one such pattern we see that $|A| + |C| \leq 1$. Now assume that both A and C are empty. With the restrictions on D in the previous paragraph we see that the pattern (312) is avoided with this assumption. Hence, $|A| + |C| \geq 1$.

Before putting this all together we note that the above restrictions show that we have $\pi = a\,x\,z\,y\,D$ or $\pi = x\,z\,c\,y\,D$ with all elements in D either smaller than x or larger than z. Hence, the elements preceding D must be four consecutive integers which contain both the patterns (132) and (312) exactly once.

Thus, we have $f_n = f_4 \sum_{i=1}^{n-3} \binom{n-4}{i-1} = 2^{n-3}$ permutations in this case (for $n \geq 4$). This holds since there are f_4 ways to arrange the first four consecutive elements; we may choose $i = 1, 2, \ldots, n-3$ for the value of the minimal element preceding D and since we are choosing $i-1$ spaces from the $n-4$ spaces after y in which to place the decreasing elements of D. ∎

Remark. We again see another interesting "mixed restriction" result with $s_n(132;231) = s_n(\emptyset;\{132,231\})$; i.e., $(132;231)$ and $(\emptyset;\{132,231\})$ are almost-Wilf equivalent.

5.1. *Generating* $S_n(312;123)$ *and* $S_n(\emptyset;\{123,312\})$: *On the Almost-Wilf Equivalence of* $(312;123)$ *and* $(\emptyset;\{123,312\})$

In this short section we show that the two sets considered are generated by almost the same rule and let the reader infer a bijection from these rules. In the following, let $n \geq 5$.

Recall (from Section 4.1) that we have defined $\phi : S_{m-1} \to S_m$ by $\phi(\pi_1\pi_2\cdots\pi_{m-1}) = (\pi_1+1)(\pi_2+1)\cdots(\pi_{m-1}+1)1$. We have also seen the following rule for generating $S_n(312;123)$.

Generating Rule for $S_n(312;123)$: By Theorem 4.5, it is trivial to check that $S_n(312;123) = \{\phi(\pi) : \pi \in S_{n-1}(312;123)\} \cup \{1(n-1)n(n-2)(n-3)\cdots 32, (n-2)(n-1)(n-3)(n-4)\cdots 21n\}$.

Finally, we can generate $S_n(\emptyset;\{123,312\})$ by the following similar rule.

Generating Rule for $S_n(\emptyset;\{123,312\})$: By Theorem 5.2, it is trivial to check that $S_n(\emptyset;\{123,312\}) = \{\phi(\pi) : \pi \in S_{n-1}(\emptyset;\{123,312\})\} \cup \{1n(n-2)(n-1)(n-3)(n-4)\cdots 32, (n-1)(n-3)(n-2)\cdots 21n\}$.

5.2. *Generating* $S_n(132;312)$ *and* $S_n(\emptyset;\{132,312\})$: *On the Almost-Wilf Equivalence of* $(132;312)$ *and* $(\emptyset;\{132,312\})$

In this short section we show that the two sets considered are generated by exactly the same rule and let the reader infer a bijection from these rules. In the following, let $n \geq 4$.

From above we have $\phi : S_{m-1} \to S_m$ by $\phi(\pi_1\pi_2\cdots\pi_{m-1}) = (\pi_1+1)(\pi_2+1)\cdots(\pi_{m-1}+1)1$. We also define $\Phi : S_{m-1} \to S_m$ by $\Phi(\pi_1\pi_2\cdots\pi_{m-1}) = \pi_1\pi_2\cdots\pi_{m-1}m$.

It is easy to check that the following generation rule generates both $S_n(132;312)$ and $S_n(\emptyset;\{132,312\})$. The difference in the sets comes from the initial sets: $S_4(132;312) = \{3124, 4231\}$ and $S_4(\emptyset;\{132,312\}) = \{2413, 3142\}$.

Generating Rule for both $S_n(132;312)$ *and* $S_n(\emptyset;\{132,312\})$: To obtain $S_n(\bullet;\bullet)$ from $S_{n-1}(\bullet;\bullet)$ take $S_n(\bullet;\bullet) = \{\phi(\pi), \Phi(\pi) : \pi \in S_{n-1}(\bullet;\bullet)\}$.

6. SUMMARY

Table I summarizes this paper's results and presents some remaining questions.

TABLE I

Almost-Wilf Class, $\mathscr{W}$	$s_n(T)$, $T \in \mathscr{W}$
$A = \overline{(123;321)}$	0 for $n \geq 6$
$B = \overline{(123;132)}$	$(n-2)2^{n-3}$ for $n \geq 3$
$C = \overline{(123;231)}$	$2n-5$ for $n \geq 3$
$D = \overline{(132;213)}$	$n2^{n-5}$ for $n \geq 4$
$E = \overline{(132;231)}$	2^{n-3} for $n \geq 3$
$F = \overline{(\emptyset;\{123,321\})}$	0 for $n \geq 6$
$G = \overline{(\emptyset;\{123,231\})}$	$2n-5$ for $n \geq 5$
$H = \overline{(\emptyset;\{123,132\})}$	$\binom{n-3}{2}2^{n-4}$ for $n \geq 5$
$I = \overline{(\emptyset;\{132,213\})}$	$(n^2+21n-28)2^{n-9}$ for $n \geq 7$
$J = \overline{(\emptyset;\{132,231\})}$	2^{n-3} for $n \geq 4$

Note. $A = \{(123;321),(321;123)\}$. $B = \{(123;132),(123;213),(132;123),(213;123),(231;321),(312;321),(321;231),(321;312)\}$. $C = \{(123;231),(123;312),(132;321),(213;321),(231;123),(312;123),(321;132),(321;213)\}$. $D = \{(132;213),(213;132),(231;312),(312;231)\}$. $E = \{(132;231),(132;312),(213;231),(213;312),(231;132),(231;213),(312;132),(312;213)\}$. $F = \{(\emptyset;\{123,321\}),(\emptyset;\{321;123\})\}$. $G = \{(\emptyset;\{123;231\}),(\emptyset;\{123,312\}),(\emptyset;\{132,321\}),(\emptyset;\{213,321\})\}$. $H = \{(\emptyset;\{123,132\}),(\emptyset;\{123,213\}),(\emptyset;\{231,321\}),(\emptyset;\{312,321\})\}$. $I = \{(\emptyset;\{132,213\}),(\emptyset;\{231,312\})\}$. $J = \{(\emptyset;\{132,231\}),(\emptyset;\{132,312\}),(\emptyset;\{213,231\}),(\emptyset;\{213,312\})\}$.

REMARK: Alek Vainshtein has informed me, via email, that he and Toufik Mansour have recently shown that $s_n(\emptyset;\{(132)^2\}) = ((n-2)^2(n+21)-4)/(2n(n-1))\binom{2n-6}{n-4}$. To finish the study of $s_n(\emptyset;\{\alpha,\beta\})$ for all $\alpha,\beta \in S_3$, all that remains is to determine a formula for $s_n(\emptyset;\{(123)^2\})$. I thank Alek for calling my attention to some relevant references.

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