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The Star Height of Reset-Free Events and Strictly Locally Testable Events

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An algorithm is presented for determining the star height of reset-free events and strictly locally testable events.

1. INTRODUCTION

Eggan (1963) posed the problem of determining the star height of regular events, and presented regular events of arbitrary star height, which are strictly locally testable. McNaughton (1967) established the pathwise homonorphism theorem, and presented an algorithm for determining the star height of pure-group events. Cohen and Brzozowski (1970), and Cohen (1970, 1971) investigated many properties of star height, some of which provide algorithms for determining the star height of certain reset-free events.

In this paper we obtain an algorithm for determining the star height of reset-free events and strictly locally testable events. The class of reset-free events properly contains the class of pure-group events, and it is known that there exist strictly locally testable events of arbitrary star height over the two letter alphabet (Hashiguchi and Honda, 1976). It turns out that we can reduce the problem of determining the star height of any event in our class to the problem for a finite set of related star events and their root events. As a corollary we present a star event whose star height is equal to that of its root event.

2. PRELIMINARIES

We assume that the reader is familiar with regular events, regular expressions, and finite automata. In this section we present notation and some definitions.

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Let Σ be a finite nonempty alphabet; Σ^*, the set of all words over Σ; λ, the null word; Σ^+, the set of all nonnull words over Σ; and ϕ, the empty event. For any $k \geqslant 0$, $\Sigma(k)$ is the set of words over Σ of length less than or equal to k. For $w \in \Sigma^*$, $\ell(w)$ is the length of w, and $\# Q$ the cardinality of the set Q. In this paper "regular expressions" use only the operators union ($\cup$), concatenation ($\cdot$), and star ($*$). Let $| E |$ be the regular event represented by a regular expression E.

DEFINITION 2.1. The apparent star height $h_\alpha(E)$ of a regular expression E is defined inductively as follows:

(1) For $a \in \Sigma$, λ, and ϕ, $h_\alpha(a) = h_\alpha(\lambda) = h_\alpha(\phi) = 0$.

(2) $h_\alpha(E_1 \cup E_2) = h_\alpha(E_1 E_2) = \max\{h_\alpha(E_1), h_\alpha(E_2)\}$, and $h_\alpha(E^*) = h_\alpha(E) + 1$.

DEFINITION 2.2. The star height $h(R)$ of a regular event R is defined by $h(R) = \min\{h_\alpha(E) \mid E \text{ is a regular expression representing } R\}$.

Let $\mathscr{A} = \langle \Sigma, Q, M, S, F \rangle$ be a (finite) automaton over an input alphabet Σ, where Q is the set of states, M is the transition function from $Q \times (\Sigma \cup \{\lambda\})$ to 2^Q, and $S, F \subseteq Q$ are the sets of initial states and final states, respectively. $\mathscr{A}$ is deterministic if M is a (partial) function from $Q \times \Sigma$ to Q. M is extended to $Q \times 2^{\Sigma^*} \to 2^Q$ in the usual way. The event accepted by $\mathscr{A}$ is denoted by $R(\mathscr{A})$, and $R(\mathscr{A}) = \{w \in \Sigma^* \mid M(S, w) \cap F \neq \phi\}$. When S and F are irrelevant to the context, $\mathscr{A}$ is denoted by the triple $\langle \Sigma, Q, M \rangle$. $\leftrightarrow(\mathscr{A})$, or $\leftrightarrow$ when no ambiguity arises, is the relation of strong connectedness over Q. $\urcorner\!\leftrightarrow(\mathscr{A})$ (or $\urcorner\!\leftrightarrow$) is the negation of $\leftrightarrow(\mathscr{A})$ ($\leftrightarrow$). Thus for any $q, q' \in Q$, $q \leftrightarrow q'$ iff $q' \in M(q, \Sigma^*)$, and $q \in M(q', \Sigma^*)$. $\mathscr{A}$ is strongly connected (s.c.) if for any $q, q' \in Q$, $q \leftrightarrow q'$. A subautomaton of $\mathscr{A}$ is an automaton $\mathscr{A}_1 = \langle \Sigma, Q_1, M_1 \rangle$ such that $Q_1 \subseteq Q$, and for all $(q, a) \in Q_1 \times (\Sigma \cup \{\lambda\})$, $M_1(q, a) \subseteq M(q, a) \cap Q_1$. A section of $\mathscr{A}$ is a maximal s.c. subautomaton of $\mathscr{A}$. A section $\mathscr{S} = \langle \Sigma, Q_1, M_1 \rangle$ of $\mathscr{A}$ is trivial if $\# Q_1 = 1$, and $M_1(Q_1, \Sigma) = \phi$. For $Q_1 \subseteq Q$, $\mathscr{A} - [Q_1]$ is the maximal subautomaton of $\mathscr{A}$ whose set of states is $Q - Q_1$. $Q(\mathscr{A})$ is the set of states of $\mathscr{A}$.

DEFINITION 2.3. The (cycle) rank $r(\mathscr{A})$ of an automaton $\mathscr{A}$ is defined inductively as follows:

(1) If all sections of $\mathscr{A}$ are trivial, then $r(\mathscr{A}) = 0$.

(2) If $\mathscr{A}$ has a nontrivial section, then $r(\mathscr{A}) = \max\{\min\{r(\mathscr{S} - [q]) \mid q \in Q(\mathscr{S})\} \mid \mathscr{S} \text{ is a nontrivial section of } \mathscr{A}\} + 1$, where if $Q(\mathscr{S}) = \{q\}$, then $r((\mathscr{S}) - [q]) = 0$.

Any $q \in Q$ is a rank center of $\mathscr{A}$ if $r(\mathscr{A}) = r(\mathscr{A} - [q]) + 1$.

LEMMA 2.1. *For any automaton* $\mathscr{A}$, $r(\mathscr{A}) = \max\{r(\mathscr{S}) \mid \mathscr{S}$ *is a section of* $\mathscr{A}\}$.

Eggan (1963) established the following theorem.

Eggan's theorem. For any regular event R, $h(R) = \min\{r(\mathscr{A}) \mid \mathscr{A}$ is an automaton accepting $R\}$.

By Eggan's theorem, we assume the following (1), (2), and (3) in the rest of the paper.

(1) For any automaton $\mathscr{A} = \langle \Sigma, Q, M, S, F \rangle$ and $q \in Q$, $q \in M(S, \Sigma^*)$, and $M(q, \Sigma^*) \cap F \neq \phi$.

(2) q_d is a special symbol such that for any automaton $\mathscr{A}$, $q_d \notin Q(\mathscr{A})$.

(3) For any automaton $\mathscr{A} = \langle \Sigma, Q, M \rangle$, M_d is the function from $(Q \cup \{q_d\}) \times (\Sigma \cup \{\lambda\})$ to $2^{Q \cup \{q_d\}}$ such that for all $a \in \Sigma \cup \{\lambda\}$, and $q \in Q \cup \{q_d\}$, if $M(q, a) \neq \phi$, then $M_d(q, a) = M(q, a)$; otherwise $M_d(q, a) = \{q_d\}$. M_d is extended to $2^{Q \cup \{q_d\}} \times 2^{\Sigma^*} \to 2^{Q\{\cup q_d\}}$ in the usual way.

For any automaton $\mathscr{A} = \langle \Sigma, Q, M \rangle$, and $t, t' \subseteq Q$, we define $\mathscr{A}(t, t') = \{w \in \Sigma^* \mid$ for each $q \in t$, $M(q, w) \cap t' \neq \phi\}$, and $\mathscr{A}(t, t', \bigcup) = \{w \in \Sigma^* \mid$ for some $q \in t$, $M(q, w) \cap t' \neq \phi\}$. For a regular event $R \subseteq \Sigma^*$, $\mathscr{A}[R]$ is the reduced automaton accepting R, and $R^+ = R^* - \{\lambda\}$. For $R_1, R_2, R_3 \subseteq \Sigma^*$, $R_1 \backslash R_2$, R_1/R_2, and $R_1 \backslash R_2/R_3$ are the events, $\{y \in \Sigma^* \mid xy \in R_2$ for some $x \in R_1\}$, $\{x \in \Sigma^* \mid xy \in R_1$ for some $y \in R_2\}$, and $\{y \in \Sigma^* \mid xyz \in R_2$ for some $x \in R_1$, and $z \in R_3\}$, respectively. A regular event $R \subseteq \Sigma^*$ is a star event if $R = R^*$. For a star event $R \subseteq \Sigma^*$, we define the root (event) of R, $\operatorname{root}(R) = R - (R^+)^2$. Note that $R = (\operatorname{root}(R))^*$.

THEOREM 2.1. (Cohen and Brzozowski, 1970). *For any regular event $R \subseteq \Sigma^*$, and $R_1, R_2 \subseteq \Sigma^*$, $h(R_1 \backslash R/R_2) \leqslant h(R)$.*

COROLLARY 2.1. *For any deterministic automaton $\mathscr{A} = \langle \Sigma, Q, M \rangle$ and $q, q' \in Q$, if $q \leftrightarrow q'$, then $h(\mathscr{A}(q, q)) = h(\mathscr{A}(q', q))$.*

Proof. Assume $q' = M(q, w)$, and $q = M(q', w')$ for $w, w' \in \Sigma^*$. Then $\mathscr{A}(q, q) = w' \backslash \mathscr{A}(q', q)$, and $\mathscr{A}(q', q) = w \backslash \mathscr{A}(q, q)$. By the theorem the result follows.

For a deterministic automaton $\mathscr{A} = \langle \Sigma, Q, M \rangle$, we define $\Sigma_p(\mathscr{A}) = \{a \in \Sigma \mid$ for all $q \in Q$, $M(q, a) \neq \phi$, and $M(q, a) \notin M(Q - \{q\}, a)\}$, and $\Sigma_f(\mathscr{A}) = \{a \in \Sigma \mid$ for all $q \in Q$, $M(q, a) = \phi$, or $M(q, a) \notin M(Q - \{q\}, a)\}$. Clearly $\Sigma_p(\mathscr{A}) \subseteq \Sigma_f(\mathscr{A})$. A regular event $R \subseteq \Sigma^*$ is (1) a pure-group event, and (2) a reset-free event if (1) $\Sigma = \Sigma_p(\mathscr{A}[R])$, and (2) $\Sigma = \Sigma_f(\mathscr{A}[R])$, respectively.

Let $k \geqslant 1$ be an integer. For $w \in \Sigma^*$ of length $\geqslant k$, $L_k(w)$, $R_k(w)$, and $I_k(w)$ are the initial segment of w of length k, the terminal segment of w of length k, and the set of interior segment of w of length k, respectively. $k(w)$ is the set, $I_k(w) \cup \{L_k(w), R_k(w)\}$. If $\ell(w) < k$, $k(w) = \phi$. For $R \subseteq \Sigma^*$, define $L_k(R) = \{L_k(w) \mid w \in R \cap \Sigma^k\Sigma^*\}$, $R_k(R) = \{R_k(w) \mid w \in R \cap \Sigma^k\Sigma^*\}$, and $I_k(R) = \{x \in I_k(w) \mid w \in R \cap \Sigma^k\Sigma^*\}$. $R \subseteq \Sigma^*$ is strictly k-testable if for all $w \in \Sigma^*$ of length $\geqslant k$, $w \in R$ iff $L_k(w) \in L_k(R)$, $I_k(w) \subseteq I_k(R)$, and $R_k(w) \in R_k(R)$. $R \subseteq \Sigma^*$ is strictly locally testable if it is strictly k-testable for some $k \geqslant 1$.

In the rest of this section, let $R \subseteq \Sigma^*$ be regular, and $\mathscr{A} = \langle \Sigma, Q, M, \{s\}, F \rangle$ the reduced automaton accepting R.

DEFINITION 2.4. A problem on $\mathscr{A}$ is a triple (t, R_0, t') such that $t, t' \subseteq Q$, $R_0 \subseteq \mathscr{A}(t, t')$, and R_0 is regular. $T(\mathscr{A})$ is the set of problems on $\mathscr{A}$. For each $(t, R_0, t') \in T(\mathscr{A})$, define the solution $r_m(t, R_0, t')$ by

$$r_m(t, R_0, t') = \min\{r(\mathscr{A}_0) \mid \mathscr{A}_0 \text{ is an automaton and } R_0 \subseteq R(\mathscr{A}) \subseteq \mathscr{A}(t, t')\}$$

An automaton $\mathscr{A}_0$ is (1) a candidate, and (2) a proper candidate for $(t, R_0, t') \in T(\mathscr{A})$ if (1) $R_0 \subseteq R(\mathscr{A}_0) \subseteq \mathscr{A}(t, t')$, and (2) $R_0 \subseteq R(\mathscr{A}_0) \subseteq \mathscr{A}(t, t')$, and $r(\mathscr{A}_0) = r_m(t, R_0, t')$, respectively.

The following lemma connects the problem of determining $h(R)$ to the problem of determining $r_m(t, R_0, t')$ for $(t, R_0, t') \in T(\mathscr{A})$.

LEMMA 2.2. $h(R) = r_m(s, R, F)$.

LEMMA 2.3. *For all* (t, R_0, t'), $(t_1, R_1, t_1') \in T(\mathscr{A})$,

(1) $r_m(t, R_0, t') = 0$ *iff* R_0 *is finite*;

(2) *if* $t = t_1$, *and* $t' = t_1'$, *then* $(t, R_0 \cup R_1, t') \in T(\mathscr{A})$, *and* $r_m(t, R_0 \cup R_1, t') = \max\{r_m(t, R_0, t'), r_m(t, R_1, t')\}$;

(3) *if* $t' \subseteq t_1$, *then* $(t, R_0R_1, t_1') \in T(\mathscr{A})$, *and* $r_m(t, R_0R_1, t_1') \leqslant \max\{r_m(t, R_0, t'), r_m(t_1, R_1, t_1')\}$;

(4) *if* $t = t'$, *then* $(t, R_0^*, t) \in T(\mathscr{A})$, *and* $r_m(t, R_0^*, t) \leqslant r_m(t, R_0, t) + 1$.

The following definition will be used to check whether or not an arbitrary automaton $\mathscr{A}_0$ is a candidate for some $(t, R_0, t') \in T(\mathscr{A})$.

DEFINITION 2.5. For any automaton $\mathscr{A}_0 = \langle \Sigma, Q_0, M_0, S_0, F_0 \rangle$ and $t \subseteq Q$, $\Gamma[\mathscr{A}, \mathscr{A}_0, t]$ is the mapping from Q_0 to the power set of $Q \cup \{q_d\}$ such that for all $q \in Q_0$, $\Gamma[\mathscr{A}, \mathscr{A}_0, t](q) = M_d(t, \mathscr{A}_0(S_0, q, \bigcup))$.

LEMMA 2.4. *For any automaton* $\mathscr{A}_0$, q, $q' \in Q(\mathscr{A}_0)$, *and* t, $t' \subseteq Q$, $\Gamma[\mathscr{A}, \mathscr{A}_0, t](q') \supseteq M_d(\Gamma[\mathscr{A}, \mathscr{A}_0, t](q), \mathscr{A}_0(q, q'))$.

Definition 2.5 and Lemma 2.4 are explained as follows: $\Gamma[\mathscr{A}, \mathscr{A}_0, t](q)$ is the set of states in $Q \cup \{q_d\}$ to which $\mathscr{A}$ moves from some state in $t \subseteq Q$ by M_d reading some word in $\mathscr{A}_0(S_0, q, \bigcup)$. Here, $\mathscr{A}_0(S_0, q, \bigcup)$ is the set of words by which $\mathscr{A}_0$ moves from some $s_0 \in S_0$ to q. Lemma 2.4 asserts that the set of states in $Q \cup \{q_d\}$ to which $\mathscr{A}$ moves from some state in $t \subseteq Q$ by M_d reading some words in $\mathscr{A}_0(S_0, q', \bigcup)$ includes $M_d(\Gamma[\mathscr{A}, \mathscr{A}_0, t](q), \mathscr{A}_0(q, q'))$ for any $q \in Q$.

3. The Star Height of Reset-Free Events

In this section we obtain an algorithm for determining the star height of reset-free events. Throughout this section, let $R \subseteq \Sigma^*$ be reset-free, and $\mathscr{A} = \langle \Sigma, Q, M, \{s\}, F \rangle$ the reduced automaton accepting R. T is the set of problems on $\mathscr{A}$. K is the syntactic semigroup of R, and α the homomorphism mapping Σ^* onto K such that for all $v, w \in \Sigma^*$, $\alpha(v) = \alpha(w)$ iff for all $q \in Q$, $M(q, v) = M(q, w)$. Let $M_{(r)}$ be the "reverse" transition function from $\Sigma^* \times 2^Q$ to 2^Q such that for all $w \in \Sigma^*$ and $t \subseteq Q$, $M_{(r)}(w, t) = \{q \in Q \mid M(q, w) \in t\}$. For all $w \in \Sigma^*$, we define $Q(w) = \{q \in Q \mid M(q, w) \neq \phi\}$, and $M[w]$ is the function from $Q(w)$ to $M(Q, w)$ such that for all $q \in Q(w)$, $M[w](q) = M(q, w)$.

Lemma 3.1. *For all $w \in \Sigma^*$, if $Q(w) \neq \phi$, then $M[w]$ is bijective.*

Proof. The proof is by induction on $\ell(w)$. If $\ell(w) \leqslant 1$, the lemma is obvious. If $w = w'a$, $w' \in \Sigma^+$, $a \in \Sigma$, and $Q(w) \neq \phi$, then $Q(w')$, $Q(a) \neq \phi$, and for any $q, q' \in Q(w)$ with $q \neq q'$, $M[w](q) = M(q, w'a) = M(M(q, w'), a) = M[a](M[w'](q)) \neq M[a](M[w'](q')) = M[w](q', w)$, where the inequality follows by the inductive hypothesis. This implies that $M[w]$ is bijective.

Corollary 3.1. *For all $v, w \in \Sigma^*$, and $t, t', t'' \subseteq Q$,*

(1) *if $M_d(t, v) = M_d(t', v)$, and $q_d \notin M_d(t, v)$, then $t = t'$;*

(2) *if $M_d(t, v) \subseteq t'$, and $M_d(t', w) \subseteq t$, then $\# t = \# t'$, $M_d(t, v) = t'$, and $M_d(t', w) = t$;*

(3) *if $M_d(t', w) = t''$, and $M_d(t, vw) = t''$, then $M_d(t, v) = t'$.*

Definition 3.1. $w \in \Sigma^*$ is identity-like (w.r.t. ($\mathscr{A}$)) if for all $q \in Q(w)$, $M(q, w) = q$. $[K_I]$ is the set of identity-like words in Σ^*(w.r.t. ($\mathscr{A}$)).

Lemma 3.2. *There exists an integer $e \geqslant 1$ such that for all $w \in \Sigma^*$, w^e is identity-like.*

Proof. We shall define a mapping β from $Q \times \Sigma^*$ to the set, $\{i \mid i$ is an integer, and $1 \leqslant i \leqslant \# Q\}$. Let $(q, w) \in Q \times \Sigma^*$. Let $i \geqslant 0$ and $j \geqslant 1$ be the smallest integers such that $M(q, w^i) = M(q, w^{i+j})$. If $M(q, w^i) = \phi$, put $\beta(q, w) = i$. Otherwise put $\beta(q, w) = j$. Note that in the latter case, $M(q, w^i) = M(q, w^{j+i}) = M(M(q, w^j), w^i)$, and $q = M(q, w^j)$ by Corollary 3.1. Consider the set $B = \{\beta(q, w) \mid (q, w) \in Q \times \Sigma^*\}$. Clearly $\# B \leqslant \# Q$. Let e be the least common multiple of all integers in B. Then the lemma follows.

In the rest of this section, e denotes the integer defined in the preceding lemma.

The following lemma, which resembles Theorem 4 in McNaughton (1967), presents certain deterministic properties of transitions in nondeterministic automata.

LEMMA 3.3. *Let* $t, t' \subseteq Q$, *and* $\mathscr{A}_0 = \langle \Sigma, Q_0, M_0, S_0, F_0 \rangle$ *be an automaton. Then* $R(\mathscr{A}_0) \subseteq \mathscr{A}(t, t')$ *iff for all* $q, q' \in Q_0$, *the following* (1), (2), *and* (3) *hold, where* $\Gamma = \Gamma[\mathscr{A}, \mathscr{A}_0, t]$:

(1) $q_d \notin \Gamma(q)$;

(2) *if* $q \in F_0$, *then* $\Gamma(q) \subseteq t'$;

(3) *for any* $v, w \in \Sigma^*$, *if* $v \in \mathscr{A}_0(q, q')$, *and* $w \in \mathscr{A}_0(q', q)$, *then* $\Gamma(q') = M_d(\Gamma(q), v) = M_{(r)}(w, \Gamma(q))$, *and* $\#\Gamma(q) = \#\Gamma(q')$.

Proof. It is easy to see that $R(\mathscr{A}_0) \subseteq \mathscr{A}(t, t')$ iff (1) and (2) hold. Assume that $R(\mathscr{A}_0) \subseteq \mathscr{A}(t, t')$, $v \in \mathscr{A}_0(q, q')$, and $w \in \mathscr{A}_0(q', q)$ for $q, q' \in Q_0$, and $v, w \in \Sigma^*$. Then (3) follows from (1), $M_d(\Gamma(q), v) \subseteq \Gamma(q')$, $M_d(\Gamma(q'), w) \subseteq \Gamma(q)$, and Corollary 3.1.

DEFINITION 3.2. Let $\mathscr{A}_0 = \langle \Sigma, Q_0, M_0, \{s_0\}, F_0 \rangle$ be a deterministic automaton.

(1) For each $w \in R(\mathscr{A}_0)$, the section-wise transition of w in $\mathscr{A}_0$ is a sequence, s.w. $(w, \mathscr{A}_0) = (q_{11}, x_1, q_{12}, a_1, \ldots, a_{m-1}, q_{m1}, x_m, q_{m2})$, such that (i) $m \geqslant 1$, $q_{11} = s_0$, $q_{m2} \in F$, and $w = x_1 a_1 x_2 \cdots a_{m-1} x_m$, and (ii) for $i = 1, \ldots, m$, and $j = 1, \ldots, m-1$, $q_{i1}, q_{i2} \in Q_0$, $q_{i1} \leftrightarrow q_{i2}$, $q_{i2} = M_0(q_{i1}, x_i)$, $q_{j2} \urcorner\!\leftrightarrow q_{j+1,1}$, $a_j \in \Sigma$, and $q_{j+1,1} = M_0(q_{j2}, a_j)$. s.w.$(w, Q_0, \mathscr{A}_0)$ is the sequence, $(q_{11}, q_{12}, q_{21}, q_{22}, \ldots, q_{m1}, q_{m2})$.

(2) A complete subevent of $\mathscr{A}_0$ is an event, $R_0 = H_1 a_1 H_2 \cdots a_{m-1} H_m$, such that for some $w \in R(\mathscr{A}_0)$, s.w.$(w, \mathscr{A}_0) = (q_{11}, x_1, q_{12}, a_1, \ldots, q_{m-1}, q_{m1}, x_m, q_{m2})$, and $H_i = \mathscr{A}_0(q_{i1}, q_{i2}) \cap [K_l]\, \alpha^{-1}\alpha(x_i)$ for $i = 1, \ldots, m$. s.w.$(R_0, \mathscr{A}_0)$ is the sequence, $(q_{11}, \alpha(x_1), q_{12}, a_1, \ldots, a_{m-1}, q_{m1}, \alpha(x_m), q_{m2})$. $C_f(\mathscr{A}_0)$ is the set of complete subevents of $\mathscr{A}_0$.

LEMMA 3.4. *For any reduced automaton* $\mathscr{A}_0$, $C_f(\mathscr{A}_0)$ *is finite, and* $R(\mathscr{A}_0) = \{w \in R_0 \mid R_0 \in C_f(\mathscr{A}_0)\}$.

DEFINITION 3.3. Let $\mathscr{A}_0 = \langle \Sigma, Q_0, M_0, \{s_0\}, F_0 \rangle$ be a deterministic automaton, $R_0 \in C_f(\mathscr{A}_0)$, and s.w.$(R_0, \mathscr{A}_0) = (q_{11}, \alpha(x_1), q_{12}, a_1, \ldots, a_{m-1}, q_{m1}, \alpha(x_m), q_{m2})$. Let $\mathscr{A}_1 = \langle \Sigma, Q_1, M_1, S_1, F_1 \rangle$ be an automaton. A sequence, $(q_1, \ldots, q_m) \in (Q_1)^m$, is a complete sequence of $\mathscr{A}_1$ w.r.t.$(R_0, \mathscr{A}_0)$ if (1) there exist $z_0 \in [K_l]$, $z_m \in [K_l]\, \alpha^{-1}\alpha(x_m)[K_l]$, and $z_i \in [K_l] \cdot \alpha^{-1}\alpha(x_i) \cdot [K_l] \cdot a_i \cdot [K_l]$ for $i = 1, \ldots, m-1$ such that $q_1 \in M_1(S_1, z_0)$, $M_1(q_m, z_m) \cap F_1 \neq \phi$, and $q_{i+1} \in M_1(q_i, z_i)$ for $i = 1, \ldots, m-1$, and (2) for each $i = 1, \ldots, m$, and $w \in \mathscr{A}_0(q_{i1}, q_{i1})$, $xwy \in \mathscr{A}_1(q_i, q_i) \cap [K_l]$ for some $x \in [K_l]$, and $y \in \Sigma^*$.

The following lemma resembles the lemma to Theorem 6 in McNaughton (1967).

LEMMA 3.5. *Let* $t,\ t' \subseteq Q$, $\mathscr{A}_0 = \langle \Sigma, Q_0, M_0, \{\mathfrak{s}_0\}, F_0 \rangle$ *be a deterministic automaton*, $R(\mathscr{A}_0) \subseteq \mathscr{A}(t, t')$, $R_0 \in C_f(\mathscr{A}_0)$, and $\mathscr{A}_1 = \langle \Sigma, Q_1, M_1, S_1, F_1 \rangle$ *be a candidate for* $(t, R(\mathscr{A}_0), t') \in T$. *Then there exists a complete sequence of* $\mathscr{A}_1$ w.r.t$(R_0, \mathscr{A}_0)$.

Proof. We shall construct a complete sequence of $\mathscr{A}_1$ w.r.t. $(R_0, \mathscr{A}_0)$. Let s.w. $(R_0, \mathscr{A}_0) = (q_{11}, \alpha(x_1), q_{12}, a_1, ..., a_{m-1}, q_{m1}, \alpha(x_m), q_{m2})$, and $x_j \in \mathscr{A}_0(q_{j1}, q_{j2})$ for $j = 1,..., m$. For each $i \geqslant 0$, and $j = 1,..., m$, define $B_{ij} = \{v \in \mathscr{A}_0(q_{j1}, q_{j1}) \mid \ell(v) \leqslant i\}$. B_{ij} is finite. Put $z_{ij} = (v_1)^e(v_2)^e \cdots (v_k)^e$, where $B_{ij} = \{v_1, v_2, ..., v_k\}$. We note that for each $v \in B_{ij}$, $z_{ij} = xvy$ for some $x \in [K_l]$, and $y \in \Sigma^*$. Now put $z_i = (z_{i1})^n x_1 a_1 \cdots a_{m-1}(z_{im})^n x_m$, where $n = \#Q_1$. Then s.w. $(z_i, \mathscr{A}_0) = (q_{11}, (z_{i1})^n x_1, q_{12}, a_1, ..., a_{m-1}, q_{m1}, (z_{im})^n x_m, q_{m2})$, and $z_i \in R_0 \subseteq R(\mathscr{A}_1)$. Consider the transitions induced by z_i in $\mathscr{A}_1$. Since $n = \#Q_1$, there exists a sequence, $(q'_{i1}, q'_{i2}, ..., q'_{im}) \in (Q_1)^m$, such that $\mathscr{A}_1(S_1, q'_{i1}, \bigcup) \cap [K_l] \neq \phi$, $\mathscr{A}_1(q'_{im}, F_1) \cap ([K_l]x_m) \neq \phi$, $\mathscr{A}_1(q'_{ij}, q'_{ij+1}) \cap ([K_l]\ x_j a_j [K_l]) \neq \phi$ for $j = 1,..., m-1$, and $\mathscr{A}_1(q'_{ik}, q'_{ik}) \cap (z_{ik})^+ \neq \phi$ for $k = 1,..., m$. Put $\mathfrak{s}(i) = (q'_{i1}, q'_{i2}, ..., q'_{im})$. Consider the infinite sequence, $\mathfrak{s}(0)$, $\mathfrak{s}(1)$, $\mathfrak{s}(2)$,... . Since m and n are finite, there exists $\mathfrak{s}(i)$ which appears infinitely many times in the sequence, $\mathfrak{s}(0)$, $\mathfrak{s}(1)$, $\mathfrak{s}(2)$,... . It is easy to see that $\mathfrak{s}(i)$ is a complete sequence of $\mathscr{A}_1$ w.r.t. $(R_0, \mathscr{A}_0)$, which completes the proof.

DEFINITION 3.4. An automaton $\mathscr{A}_0 = \langle \Sigma, Q_0, M_0, S_0, F_0 \rangle$ is a subset automaton of $\mathscr{A}$ if (1) $Q_0 \subseteq 2^Q$, (2) $\mathscr{A}_0$ is deterministic, (3) $\#S_0 = \#F_0 = 1$, (4) for all $q, q' \in Q_0$ and $a \in \Sigma$, $M_0(q, a) = q'$ only if $M_d(q, a) \subseteq q'$, and (5) for all $w, w' \in R(\mathscr{A}_0)$, s.w. $(w, Q_0, \mathscr{A}_0) =$ s.w. $(w', Q_0, \mathscr{A}_0)$. It is a complete subset automaton of $\mathscr{A}$ if it is a subset automaton of $\mathscr{A}$, and for $S_0 = \{\mathfrak{s}_0\}$, and $F_0 = \{f_0\}$, $\mathfrak{s} \in \mathfrak{s}_0$, and $f_0 \subseteq F$. Let s.a. $(\mathscr{A})$, and c.s.a. $(\mathscr{A})$ be the sets of subset automata and complete subset automata of $\mathscr{A}$, respectively. For any $C \subseteq$ c.s.a. $(\mathscr{A})$, define $R(C) = \{w \in R(\mathscr{A}_0) \mid \mathscr{A}_0 \in C\}$, and $r(C) = \max\{r(\mathscr{A}_0) \mid \mathscr{A}_0 \in C\}$.

LEMMA 3.6. *For any* $\mathscr{A}_0 \in$ *c.s.a.* $(\mathscr{A})$, $R(\mathscr{A}_0) \subseteq R$.

We are now ready to state an algorithm for determining the star height of reset-free events.

ALGORITHM 3.1. *For a reset-free event* $R \subseteq \Sigma^*$, *and the reduced automaton* $\mathscr{A}$ *accepting* R,

$$h(R) = \min\{r(C) \mid C \subseteq \textit{c.s.a.}\ (\mathscr{A}),\ \textit{and}\ R(C) = R\}.$$

We can determine the right side of the equation by constructing all finitely many complete subset automata of $\mathscr{A}$, determining their rank, and obtaining the events accepted by them. In the rest of this section, we shall prove the correctness of the algorithm. Clearly $h(R) \leqslant \min\{r(C) \mid C \subseteq$ c.s.a. $(\mathscr{A})$, and $R(C) = R\}$. We shall prove that the converse of the inequality also holds.

LEMMA 3.7. *For any* $\mathscr{A}_0 = \langle \Sigma, Q_0, M_0, \{t\}, \{t'\}\rangle \in$ *s.a.* $(\mathscr{A})$, *if* $\#t = \#t'$, *then* $r(\mathscr{A}_0) = r_m(t, R(\mathscr{A}_0), t')$.

Proof. Clearly $r_m(t, R(\mathscr{A}_0), t') \leqslant r(\mathscr{A}_0)$. We shall prove that $r_m(t, R(\mathscr{A}_0), t') \geqslant r(\mathscr{A}_0)$ by induction on $r_m(t, R(\mathscr{A}_0), t')$. If $r_m(t, R(\mathscr{A}_0), t') = 0$, then the inequality is obvious. Assume $r_m(t, R(\mathscr{A}_0), t') > 0$. Let $R_0 \in C_f(\mathscr{A}_0)$, and s.w. $(R_0, \mathscr{A}_0) = (q_{11}, \alpha(x_1), q_{12}, a_1, \ldots, a_{m-1}, q_{m1}, \alpha(x_m), q_{m2})$. Let $\mathscr{A}_1 = \langle \Sigma, Q_1, M_1, S_1, F_1\rangle$ be a proper candidate for $(t, R(\mathscr{A}_0), t')$. By Lemma 3.5, there exists a complete sequence, $(q_1, q_2, \ldots, q_m) \in (Q_1)^m$ of $\mathscr{A}_1$ w.r.t. $(R_0, \mathscr{A}_0)$. Let $\mathscr{S}_{0i} = \langle \Sigma, Q_{0i}, M_{0i}\rangle$, and $\mathscr{S}_{1i} = \langle \Sigma, Q_{1i}, M_{1i}\rangle$ be the sections of $\mathscr{A}_0$ and $\mathscr{A}_1$ such that $q_{i1} \in Q_{0i}$, and $q_i \in Q_{1i}$ for $i = 1, \ldots, m$. Now it will suffice to show that $r(\mathscr{S}_{0i}) \leqslant r(\mathscr{S}_{1i})$ for $i = 1, \ldots, m$. Let $q_i' \in Q_{1i}$ be a rank center of $\mathscr{S}_{1i}$. Let $\Gamma = \Gamma[\mathscr{A}, \mathscr{A}_1, t]$. By Corollary 3.1 and Lemma 3.5, one can see that $\Gamma(q_i) = q_{i1} \subseteq Q$. Put $t_0 = \Gamma(q_i')$. Let s.a. $(\mathscr{S}_{0i}, t_0)$ be the set of automata which belong to s.a. $(\mathscr{A})$, and are subautomata of $\mathscr{S}_{0i} - [t_0]$. Clearly $r(\mathscr{S}_{0i}) \leqslant \max\{r(\mathscr{A}_2) \mid \mathscr{A}_2 \in \text{s.a.}\ (\mathscr{S}_{0i}, t_0)\} + 1$. To complete the proof, we shall show that for each $\mathscr{A}_2 \in$ s.a. $(\mathscr{S}_{0i}, t_0)$, $r(\mathscr{A}_2) \leqslant r(\mathscr{S}_{1i} - [q_i'])$. (Note that $r(\mathscr{S}_{1i}) = r(\mathscr{S}_{1i} - q_i']) + 1$). Let $\mathscr{A}_2 = \langle \Sigma, Q_2, M_2, \{t_1\}, \{t_2\}\rangle \in$ s.a. $(\mathscr{S}_{0i}, t_0)$. Let $\mathscr{S}_{1i} - [q_i'] = \langle \Sigma, Q_3, M_3\rangle$. Define $S_3 = \{q'' \in Q_3 \mid \Gamma(q'') = t_1\}$, and $F_3 = \{q'' \in Q_3 \mid \Gamma(q'') = t_2\}$. Let $\mathscr{A}_3 = \langle \Sigma, Q_3, M_3, S_3, F_3\rangle$. By Corollary 3.1, and Lemma 3.3, one can see that $R(\mathscr{A}_2) \subseteq R(\mathscr{A}_3) \subseteq \mathscr{A}(t_1, t_2)$. Thus $r_m(t_1, R(\mathscr{A}_2), t_2) \leqslant r(\mathscr{A}_3) \leqslant r(\mathscr{S}_{1i} - [q_i'])$. By the inductive hypothesis, $r(\mathscr{A}_2) \leqslant r_m(t_1, R(\mathscr{A}_2), t_2)$. Hence $r(\mathscr{A}_2) \leqslant r(\mathscr{S}_{1i} - [q_i'])$, completing the proof of the lemma.

To complete the proof of correctness of Algorithm 3.1, we shall show that $h(R) \geqslant \min\{r(C) \mid C \subseteq \text{c.s.a.}\ (\mathscr{A}), \text{and}\ R(C) = R\}$. Let $\mathscr{A}_1 = \langle \Sigma, Q_1, M_1, S_1, F_1\rangle$ be a proper candidate for $(\mathscr{s}, R, F)$. Thus $r(\mathscr{A}_1) = h(R)$, and $R(\mathscr{A}_1) = R$. It will suffice to show that for each $R_0 \in C_f(\mathscr{A})$, there exists $\mathscr{A}_0 \in$ c.s.a. $(\mathscr{A})$ such that $R_0 \subseteq R(\mathscr{A}_0)$, and $r(\mathscr{A}_0) \leqslant r(\mathscr{A}_1)$. Let $R_0 \in C_f(\mathscr{A})$, and s.w. $(R_0, \mathscr{A}) = (q_{11}, \alpha(x_1), q_{12}, a_1, \ldots, a_{m-1}, q_{m1}, \alpha(x_m), q_{m2})$ with $x_i \in \mathscr{A}(q_{i1}, q_{i2})$ for $i = 1, \ldots, m$. In the following we shall construct some $\mathscr{A}_0 \in$ c.s.a. $(\mathscr{A})$ such that $R_0 \subseteq R(\mathscr{A}_0)$, and $r(\mathscr{A}_0) \leqslant r(\mathscr{A}_1)$. By Lemma 3.5, there exists a complete sequence, $(q_1, \ldots, q_m) \in (Q_1)^m$, of $\mathscr{A}_1$ w.r.t. $(R_0, \mathscr{A})$. For each $i = 1, \ldots, m$, let $\mathscr{S}_i = \langle \Sigma, Q_{i0}, M_{i0}\rangle$, and $\mathscr{S}_{1i} = \langle \Sigma, Q_{1i}, M_{1i}\rangle$ be the sections of $\mathscr{A}$ and $\mathscr{A}_1$ such that $q_{i1} \in Q_{i0}$, and $q_i \in Q_{1i}$. Let Γ be the function $\Gamma[\mathscr{A}, \mathscr{A}_1, \mathscr{s}]$. Put $t_i = \Gamma(q_i)$ for $i = 1, \ldots, m$. By definition of Γ, it is easy to see that $q_{i1} \in t_i$. Let $\mathscr{S}_{0i}$ be the automaton, $\langle \Sigma, Q_{0i}, M_{0i}, \{t_i\}, \{t_i'\}\rangle$, such that $Q_{0i} = \{t \subseteq Q \mid t = M(t_i, v)$ for some $q \in Q_{i0}$, and $v \in \mathscr{A}(q_{i1}, q)\}$, $t_i' = M(t_i, x_i)$, and for each $q, q' \in Q_{0i}$ and $a \in \Sigma$, $M_{0i}(q, a) = q'$ iff $M(q, a) = q'$. It is easy to see that $\mathscr{S}_{0i} \in$ s.a. $(\mathscr{A})$, and $\mathscr{A}(q_i, q_{i2}) \cap [K_I]\, \alpha^{-1}\alpha(x_i) \subseteq R(\mathscr{S}_{0i}) \subseteq \mathscr{A}(t_i, t_i')$ by Corollary 3.1, and Lemma 3.3. Moreover we have $r(\mathscr{S}_{0i}) \leqslant r(\mathscr{A}_1)$ as explained below. Define $S_i = \{q \in Q_{1i} \mid \Gamma(q) = t_i\}$, and $F_i = \{q \in Q_{1i} \mid \Gamma(q) = t_i'\}$. Let $\mathscr{A}_{1i} = \langle \Sigma, Q_{1i}, M_{1i}, S_i, F_i\rangle$. By Corollary 3.1, and Lemma 3.3, $R(\mathscr{A}_{1i}) \subseteq \mathscr{A}(t_i, t_i')$. One can see that $\mathscr{A}_{1i}$ is a candidate for $(t_i, R(\mathscr{S}_{0i}), t_i')$ as follows. Consider any $w \in R(\mathscr{S}_{0i})$. Since $\mathscr{S}_{1i}$ is strongly connected, $M(t_i', v) = t_i$ for some $v \in \Sigma^*$. Then $M(q_{i1}, (wv)^e) = q_{i1}$. By Corollary 3.1, and Lemma 3.3, one can see that $R(\mathscr{S}_{0i}) \subseteq R(\mathscr{A}_{1i})$. Thus $r_m(t_i, R(\mathscr{S}_{0i}), t_i') \leqslant r(\mathscr{A}_{1i}) \leqslant r(\mathscr{A}_1)$. By Lemma 3.7, $r(\mathscr{S}_{0i}) \leqslant r(\mathscr{A}_1)$.

Now consider any Q_{0i} and Q_{0j} for $1 \leqslant i < j \leqslant m$. Let $t \in Q_{0i}$ and $t' \in Q_{0j}$. It is easy to see that for some $q_{i0} \in Q_{i0}$, and $v \in \Sigma^*$, $q_{i0} \in t$, $M(t, v) \subseteq t'$, and $q_{i0} \neg\!\leftrightarrow M(q_{i0}, v)$. It $t = t'$, then $M(t, v^e) = t'$, and $M(q_{i0}, v^e) = q_{i0}$, which is a contradiction to $q_{i0} \neg\!\leftrightarrow M(q_{i0}, v)$. Thus $Q_{0i} \cap Q_{0j} = \phi$. Now it is easy to see that one can construct $\mathscr{A}_0 \in$ c.s.a. $(\mathscr{A})$ such that $r(\mathscr{A}_0) \leqslant h(R)$, and $R_0 \subseteq R(\mathscr{A}_0)$ by a series connection of $\mathscr{S}_{01}$, $\mathscr{A}[a_1]$, $\mathscr{S}_{02}$, $\mathscr{A}[a_2]$,..., $\mathscr{A}[a_{m-1}]$, and $\mathscr{S}_{0m}$ in the obvious manner. This completes the proof of Algorithm 3.1.

Lemma 3.7 provides the following corollary.

COROLLARY 3.2. (McNaughton (1967), Cohen (1970)). *If* $\#F = 1$, *then* $h(R) = r(\mathscr{A})$.

Remark 3.1. One can alternatively prove $r(\mathscr{S}_{1i}) \geqslant r(\mathscr{S}_{0i})$ for $i = 1,..., m$ in the above proof using the pathwise homomorphism theorem in McNaughton (1967).

Remark 3.2. It is known that there exists a regular event R such that $h(R) < \min\{r(C) \mid C \in \text{c.s.a.}\ (\mathscr{A}[R]) \text{ and } R(C) = R\}$. (See Example 6.3 in Cohen and Brzozowski (1970)).

Remark 3.3. Cohen (1970) presented the hollowing theorem. Let $R \subseteq \Sigma^*$ be a regular event with the finite intersection property, i.e., for all $x, y \in \Sigma^*$, $x\backslash R = y\backslash R$, or $x\backslash R \cap y\backslash R$ is finite. Then $h(R) = r(\mathscr{A}[R])$.

One can alternatively prove that $h(R) = r(\mathscr{A}[R])$ as follows. We may assume that $h(R) > 0$. Let $\mathscr{A}[R] = \mathscr{A} = \langle \Sigma, Q, M, \{s\}, F\rangle$. Put $k = \max\{\ell(w) \mid w \in x\backslash R \cap y\backslash R$ for some $x, y \in \Sigma^*$ with $x\backslash R \neq y\backslash R\} + 1$. For each $w \in R_k(R)$, there exists a unique state $q(w) \in Q$ such that $M(q(w), w) \in F$, and let $\mathscr{A}[w]$ be the automaton, $\langle \Sigma, Q(w), M(w), \{s\}, \{q(w)\}\rangle$, where $Q(w) = \{q \in Q \mid \mathscr{A}(q, F) \cap \Sigma^* w \neq \phi\}$, and $M(w)$ is the restriction of M to $Q(w) \times \Sigma \to Q(w)$. One can see that (1) $\mathscr{A}[w]$ is reset-free, (2) $\mathscr{A}[w] = \mathscr{A}[R/w]$, (3) $r(\mathscr{A}) = \max\{r(\mathscr{A}[w]) \mid w \in R_k(R)\}$, and (4) $R - \Sigma(k-1) = (R/w_1) \cdot w_1 \cup \cdots \cup (R/w_m) \cdot w_m$, where $R_k(R) = \{w_1, ..., w_m\}$. Then $h(R) = \max\{h(R/w) \mid w \in R_k(R)\}$ by (4) and Theorem 2.1, and $r(\mathscr{A}) = \max\{h(R/w) \mid w \in R_k(R)\}$ by (1), (2), (3) and Corollary 3.2. Hence $h(R) = r(\mathscr{A})$.

4. THE STAR HEIGHT OF STRICTLY LOCALLY TESTABLE EVENTS

In this section we obtain an algorithm for determining the star height of strictly locally testable events. We first present an alternative characterization of strictly locally testable events.

DEFINITION 4.1. Let $k \geqslant 0$ be an integer. A regular event $R \subseteq \Sigma^*$ is k-reset if for all $x, y \in \Sigma^*$ and $w \in \Sigma^k$, $xw\backslash R$, $yw\backslash R \neq \phi$ implies $xw\backslash R = yw\backslash R$. An automaton $\mathscr{A} = \langle \Sigma, Q, M, S, F \rangle$ is k-reset if it is deterministic, and for any $w \in \Sigma^k$, $\#M(Q, w) \leqslant 1$. A regular event (an automaton) is reset if it is k-reset for some $k \geqslant 0$.

PROPOSITION 4.1. *For any $k \geqslant 0$, and any regular event $R \subseteq \Sigma^*$, the following are equivalent*:

(1) *R is k-reset.*

(2) *$\mathscr{A}[R]$ is k-reset.*

(3) *R is accepted by some k-reset automaton.*

PROPOSITION 4.2. *$R \subseteq \Sigma^*$ is reset iff it is strictly locally testable.*

The proposition follows from the following two lemmata.

LEMMA 4.1. *If $R \subseteq \Sigma^*$ is k-reset for some $k \geqslant 0$, then it is strictly $(k + 1)$-testable.*

Proof. Let $R \subseteq \Sigma^*$ be k-reset, and $\mathscr{A} = \langle \Sigma, Q, M, \{s\}, F \rangle = \mathscr{A}[R]$. Let $w \in \Sigma^{k+1}\Sigma^*$. If $w \in R$, then $L_{k+1}(w) \in L_{k+1}(R)$, $R_{k+1}(w) \in R_{k+1}(R)$, and $I_{k+1}(w) \subseteq I_{k+1}(R)$ by definition. Conversely let $L_{k+1}(w) \in L_{k+1}(R)$, $R_{k+1}(w) \in R_{k+1}(R)$, and $I_{k+1}(w) \subseteq I_{k+1}(R)$. Assume that $\ell(w) > k + 2$. (In case $\ell(w) \leqslant k + 2$, the argument is similar). Let $w = w_1 a_1 a_2 \cdots a_m$, $w_1 \in \Sigma^{k+1}$, $m \geqslant 2$, and $a_i \in \Sigma$ for $i = 1,..., m$. Put $q_0 = M_d(s, w_1)$, and $q_i = M_d(s, w_1 a_1 \cdots a_i)$ for $i = 1,..., m$. Since $L_{k+1}(w) \in L_{k+1}(R)$, $q_0 \neq q_d$. Let $w_1 = bw_1'$, $b \in \Sigma$, and $w_1' \in \Sigma^k$. Then $w_1' a_1 \in I_{k+1}(R)$. Since $\mathscr{A}$ is k-reset, $M(Q, w_1' a_1) = M(M(Q, w_1'), a_1) = \{M(q_0, a_1)\} = \{q_1\}$ and $q_1 \neq q_d$. By the same arguments, $q_2, ..., q_m \neq q_d$. Since $R_{k+1}(w) \in R_{k+1}(R)$, and $\mathscr{A}$ is k-reset, it follows that $M(Q, R_{k+1}(w)) = \{q_m\} \subseteq F$. Hence $w \in R$, completing the proof.

LEMMA 4.2. *If $R \subseteq \Sigma^*$ is strictly k-testable for some $k \geqslant 1$, then it is $(k + 1)$-reset.*

Proof. Assume that $R \subseteq \Sigma^*$ is strictly k-testable. Let $x, y \in \Sigma^*$, $w \in \Sigma^{k+1}$, and $xw\backslash R$, $yw\backslash R \neq \phi$. Let $z \in xw\backslash R$. Then $xwz \in R$. Since $yw\backslash R \neq \phi$, $ywz_0 \in R$ for some $z_0 \in \Sigma^*$. Then $L_k(ywz) = L_k(ywz_0) \in L_k(R)$, $R_k(ywz) = R_k(xwz) \in R_k(R)$. Moreover $I_k(ywz) = I_k(yw) \cup I_k(wz) \subseteq I_k(R)$. Thus $ywz \in R$, and $xw\backslash R \subseteq yw\backslash R$. Similarly $yw\backslash R \subseteq xw\backslash R$. Hence $xw\backslash R = yw\backslash R$, completing the proof.

Remark 4.1. For any $k \geqslant 1$, $(01^{k-1} \cup 1^{k-1}0)$ is k-reset, but not strictly k-testable since $01^{k-1}0 \notin (01^{k-1} \cup 1^{k-1}0)$. Conversely $(0^k \cup 0^{k+1})$ is strictly k-testable, but not k-reset.

For any $w = a_1a_2 \cdots a_m \in \Sigma^+$, $a_i \in \Sigma$ for $i = 1,..., m$, w^T is the reverse of w, $w^T = a_m a_{m-1} \cdots a_1$. $\lambda^T = \lambda$. For any $R \subseteq \Sigma^*$, $R^T = \{w^T \mid w \in R\}$.

PROPOSITION 4.3. *If $R \subseteq \Sigma^*$ is k-reset for some $k \geqslant 0$, then R^T is k-reset.*

Proof. Let $\mathscr{A}[R] = \mathscr{A} = \langle \Sigma, Q, M, \{\sigma\}, F\rangle$, and R be k-reset. Let x, $y \in \Sigma^*$, $w \in \Sigma^k$, and R/wx, $R/wy \neq \phi$. Let $z \in R/wx$. Then $zwx \in R$. Since $R/wy \neq \phi$, $z_0wy \in R$ for some $z_0 \in \Sigma^*$. Since R is k-reset, $\{M(\sigma, zwy)\} = M(Q, wy) = \{M(\sigma, z_0wy)\} \subseteq F$. Thus $z \in R/wy$, and $R/wx \subseteq R/wy$. Similarly $R/wy \subseteq R/wx$, completing the proof.

The following lemma reduces the problem of determining the star height of strictly k-testable events to the problem of $(k-1)$-reset events.

LEMMA 4.3. *If $R \subseteq \Sigma^*$ is strictly k-testable for some $k \geqslant 1$, then* (1) *$\Sigma\backslash R/\Sigma$ is $(k-1)$-reset, and* (2) *$h(R) = h(\Sigma\backslash R/\Sigma)$.*

Proof. Let $R \subseteq \Sigma^*$ be strictly k-testable, and $R_0 = \Sigma\backslash R/\Sigma$.

(1) Let x, $y \in \Sigma^*$, $w \in \Sigma^{k-1}$, and $xw\backslash R_0$, $yw\backslash R_0 \neq \phi$. Let $z \in xw\backslash R_0$. Then for some a, $b \in \Sigma$, $axwzb \in R$. Since $yw\backslash R_0 \neq \phi$, $a'ywz'b' \in R$ for some a', $b' \in \Sigma$, and $z' \in \Sigma^*$. Consider $a'ywzb$. It is easy to see that $L_k(a'ywzb) \in L_k(R)$, $R_k(a'ywzb) \in R_k(R)$, and $I_k(a'ywzb) = k(yw) \cup k(wz) \subseteq I_k(R)$. Then $z \in yw\backslash R_0$, and $xw\backslash R_0 \subseteq yw\backslash R_0$. Similarly $yw\backslash R_0 \subseteq xw\backslash R_0$. Thus $xw\backslash R_0 = yw\backslash R_0$.

(2) By Theorem 2.1 what must be proved is that $h(R) \leqslant h(\Sigma\backslash R/\Sigma)$. Clearly $R = (R \cap \Sigma(2k+1)) \cup (R - \Sigma(2k+1))$, and $h(R) = h(R - \Sigma(2k+1))$ since $R \cap \Sigma(2k+1)$ is finite. We shall prove that $h(R - \Sigma(2k+1)) \leqslant h(\Sigma\backslash R/\Sigma)$. Let $\Sigma \times \Sigma = \{(a_1, b_1),..., (a_m, b_m)\}$, and $\Sigma^{k+1} \times \Sigma^{k+1} = \{(x_1, y_1),..., (x_n, y_n)\}$. Then $\Sigma\backslash R/\Sigma = a_1\backslash R/b_1 \cup \cdots \cup a_m\backslash R/b_m$, and $R - \Sigma(2k+1) = x_1(x_1\backslash R/y_1)y_1 \cup \cdots \cup x_n(x_n\backslash R/y_n)y_n$. Now it will suffice to show that for any x, $y \in \Sigma^{k+1}$, $h(x\backslash R/y) \leqslant h(\Sigma\backslash R/\Sigma)$. Let $x, y \in \Sigma^{k+1}$, $x = ax'$, $y = y'b$, and $a, b \in \Sigma$. Then $x\backslash R/y = \{z \in \Sigma^* \mid x'zy' \in a\backslash R/b\} = \{z \in \Sigma^* \mid x'zy' \in \Sigma^+\backslash R/\Sigma^+\}$, and $a\backslash R/b \subseteq \Sigma\backslash R/\Sigma \subseteq \Sigma^+\backslash R/\Sigma^+$. Then

$$x\backslash R/y = \{z \in \Sigma^* \mid x'zy' \in \Sigma\backslash R/\Sigma\} = x'\backslash(\Sigma\backslash R/\Sigma)/y'.$$

By Theorem 2.1, the result follows.

In the rest of this section we shall present an algorithm for determining the star height of reset events. Let $R \subseteq \Sigma^*$ be k-reset, and $\mathscr{A} = \langle \Sigma, Q, M, \{\sigma\}, F\rangle = \mathscr{A}[R]$. If $k = 0$, then $R = \phi$, or $R = \Sigma_0^*$ for some $\Sigma_0 \subseteq \Sigma$, and it is easy to determine $h(R)$. We assume that $k \geqslant 1$. For each $q \in Q$, define $\Sigma(k, q) = \{w \in \Sigma^k \mid M(Q, w) = \{q\}\}$. For any automaton $\mathscr{A}_0 = \langle \Sigma, Q_0, M_0, S_0, F_0\rangle$, define $R[\mathscr{A}_0] = R(\langle \Sigma, Q_0, M_0, Q_0, Q_0\rangle)$. We note that $R[\mathscr{A}_0] = \Sigma^*\backslash R(\mathscr{A}_0)/\Sigma^*$.

LEMMA 4.4. *For any section $\mathscr{S}$ of $\mathscr{A}$, q, $q' \in Q(\mathscr{S})$, and any subautomaton $\mathscr{A}_0$ of $\mathscr{A}$, if $\mathscr{S}$ is a subautomaton of $\mathscr{A}_0$, then $h(R(\mathscr{A}_0)) \geqslant h(\mathscr{A}(q, q'))$.*

Proof. If $h(\mathscr{A}(q, q')) = 0$, the assertion is obvious. Assume that $h(\mathscr{A}(q, q')) \geqslant 1$, and $\mathscr{S}$ is a subautomaton of $\mathscr{A}_0$. Let $\mathscr{A}_0 = \langle \Sigma, Q_0, M_0, S_0, F_0 \rangle$. Then there exist $x, y \in \Sigma^k\Sigma^*$, $q_0 \in S_0$, and $q_1 \in F_0$ such that $q = M_0(q_0, x)$, and $q_1 = M_0(q', y)$. Let $\Sigma(k, q') = \{v_1, v_2, \ldots, v_n\}$, and for each $v \in \Sigma(k, q')$, define $R_0(v) = (x\backslash R(\mathscr{A}_0)/vy) \cdot v$. By Theorem 2.1, it will suffice to show that

$$\mathscr{A}(q, q') - \Sigma(k-1) = R_0(v_1) \cup \cdots \cup R_0(v_n).$$

One can see that

$$\begin{aligned} \mathscr{A}(q, q') - \Sigma(k-1) &= \{z \in \mathscr{A}(q, q') \mid z = z_0 v \text{ for some } v \in \Sigma(k, q')\} \\ &= \{z \mid z = z_0 v \text{ and } x z_0 v y \in R(\mathscr{A}_0) \text{ for some } v \in \Sigma(k, q')\} \\ &= R_0(v_1) \cup \cdots \cup R_0(v_n). \end{aligned}$$

This completes the proof.

LEMMA 4.5. (1) *For any $q, q' \in Q$, if $q \leftrightarrow q'$, then $h(\mathscr{A}(q, q)) = h(\mathscr{A}(q, q')) = h(\mathscr{A}(q', q)) = h(\mathscr{A}(q', q'))$.*

(2) $h(R) = h(R[\mathscr{A}]) = \max\{h(\mathscr{A}(q, q')) \mid q, q' \in Q\} = \max\{h(\mathscr{A}(q, q')) \mid q, q' \in Q$ *and* $q \leftrightarrow q'\} = \max\{h(\mathscr{A}(q, q)) \mid q \in Q\}$.

Proof. (1) is immediate from Lemma 4.4. (2) Put $B = \max\{h(\mathscr{A}(q, q')) \mid q, q' \in Q\}$, $S = \max\{h(\mathscr{A}(q, q')) \mid q, q' \in Q$ and $q \leftrightarrow q'\}$, and $L = \max\{h(\mathscr{A}(q, q)) \mid q \in Q\}$. Clearly $L \leqslant S \leqslant B$. From (1), $L = S$. We shall prove that (a) $B \leqslant S$, (b) $h(R)$, $h(R[\mathscr{A}]) \leqslant B$, and (c) $L \leqslant h(R)$, $h(R[\mathscr{A}])$. For any $q, q' \in Q$, we can obtain a regular expression representing $\mathscr{A}(q, q')$ from the set of regular expressions, $\{w \in \Sigma^* \mid \ell(w) \leqslant \#Q\} \cup \{E \mid |E| = \mathscr{A}(q_0, q_1)$ for some q_0, $q_1 \in Q$ with $q_0 \leftrightarrow q_1\}$, by concatenation and union. Thus $B \leqslant S$. Similarly $h(R)$, $h(R[\mathscr{A}]) \leqslant B$. By Lemma 4.4, $L \leqslant h(R)$, $h(R[\mathscr{A}])$. This completes the proof.

COROLLARY 4.1. (1) $h(R) = \max\{h(R[\mathscr{S}]) \mid \mathscr{S}$ *is a section of* $\mathscr{A}\}$. (2) *If* $\mathscr{A}$ *is s.c., then* $h(R) = h(\mathscr{A}(q, q))$ *for any* $q \in Q$.

In Section 2, we define the set of problems $T(\mathscr{A})$ on the reduced automaton $\mathscr{A}$ in connection with the problem of determining $h(R(\mathscr{A}))$. When $\mathscr{A}$ is reset, we define the set of problems on $R[\mathscr{A}]$ as in the following definition.

DEFINITION 4.2. $T(R)$ is the set of problems on R, $\{R_0 \mid R_0$ is regular and $R_0 \subseteq \Sigma^*\backslash R/\Sigma^*\}$. For each $R_0 \in T(R)$, define the solution $r_m(R_0)$ by

$$r_m(R_0) = \min\{r(\mathscr{A}_0) \mid \mathscr{A}_0 \text{ is an automaton, and } R_0 \subseteq R[\mathscr{A}_0] \subseteq \Sigma^*\backslash R/\Sigma^*\}.$$

We say an automaton $\mathscr{A}_0$ is (1) a candidate, and (2) a proper candidate for $R_0 \in T(R)$ if (1) $R_0 \subseteq R[\mathscr{A}_0] \subseteq \Sigma^*\backslash R/\Sigma^*$, and (2) $R_0 \subseteq R[\mathscr{A}_0] \subseteq \Sigma^*\backslash R/\Sigma^*$, and $r(\mathscr{A}_0) = r_m(R_0)$, respectively.

PROPOSITION 4.4. $h(R) = r_m(R)$.

Proof. Let $\mathscr{A}_0$ be an automaton such that $R(\mathscr{A}_0) = R$, and $r(\mathscr{A}_0) = h(R)$. Then $\mathscr{A}_0$ is a candidate for R_0, and $r_m(R) \leqslant r(\mathscr{A}_0) = h(R)$. Conversely let $\mathscr{A}_0$ be a proper candidate for R_0. Then $h(R) = h(\Sigma^*\backslash R/\Sigma^*) = h(\Sigma^*\backslash R(\mathscr{A}_0)/\Sigma^*) \leqslant h(R(\mathscr{A}_0)) \leqslant r(\mathscr{A}_0) = r_m(\mathscr{A}_0)$ by Lemma 4.5, and Theorem 2.1.

PROPOSITION 4.5. *For any* $R_0 \in T(R)$, $r_m(R_0) = 0$ *iff* R_0 *is finite.*

LEMMA 4.6. *For any* R_0, R_1, $R_2 \in T(R)$,

(1) *if* $R_0 \subseteq \Sigma^*\backslash R_1/\Sigma^*$, *then* $r_m(R_0) \leqslant r_m(R_1)$;

(2) $r_m(R_0 \cup R_1) = \max\{r_m(R_0), r_m(R_1)\}$;

(3) *if* $R_0 \subseteq R_1 \cdot R_2$, *then* $r_m(R_0) \leqslant \max\{r_m(R_1), r_m(R_2)\}$;

(4) *if* $R_0 = R_1R_2$, *then* $r_m(R_0) = \max\{r_m(R_1), r_m(R_2)\}$;

(5) *for any* $i, j \geqslant 0$, *and* R_0', $R_1' \subseteq \Sigma^*$, $r_m(R_0) = r_m(\Sigma^i\backslash R_0/\Sigma^j)$, $r_m(R_0) \geqslant r_m(R_0'\backslash R_0/R_1')$, *and if* R_0' *is finite, then* $r_m(R_0) = r_m(R_0 \cup R_0')$.

Proof. (1) and (2) are clear. (3) Let $\mathscr{A}_1 = \langle \Sigma, Q_1, M_1\rangle$ and $\mathscr{A}_2 = \langle \Sigma, Q_2, M_2\rangle$ be proper candidates for R_1 and R_2, respectively. Consider any $w \in R_0$. Let $w = w_1w_2$, $w_1 \in R_1$ and $w_2 \in R_2$. If $\ell(w_i) < 2k$, put $H_i = \{w_i\}$ for $i = 1, 2$. If $\ell(w_i) \geqslant 2k$, put $H_i = v \cdot R(\langle \sum, Q_i, M_i, S_i', F_i'\rangle) \cdot v'$, where $w_i = vxv'$, $v, v' \in \Sigma^k$, $x \in \Sigma^*$, $S_i' = \{q \in Q_i \mid q \in M_i(Q_i, v)\}$, and $F_i' = \{q \in Q_i \mid M_i(q, v') \neq \phi\}$ for $i = 1, 2$. Now put $R_0(w) = H_1 \cdot H_2$. It is easy to see that $w \in R_0(w)$, $R_0(w) \subseteq \Sigma^*\backslash R/\Sigma^*$, and $r_m(R_0(w)) \leqslant \max\{r_m(R_1), r_m(R_2)\}$. Consider the set, $B = \{R_0(w) \mid w \in R_0\}$. Since Σ^{2k} is finite, B is finite, and $R_0 = R_{01} \cup R_{02} \cup \cdots \cup R_{0m}$, where $B = \{R_{01}, R_{02}, \ldots, R_{0m}\}$. Thus $r_m(R_0) \leqslant \max\{r_m(R_1), r_m(R_2)\}$. (4) We note that R_1, $R_2 \subseteq \sum^*\backslash R_0/\Sigma^*$. By (1), $r_m(R_0) \geqslant \max\{r_m(R_1), r_m(R_2)\}$. The converse inequality follows from (3). (5) The last two statements and $r_m(R_0) \geqslant r_m(\Sigma^i\backslash R_0/\Sigma^j)$ are clear. We shall prove that $r_m(R_0) \leqslant r_m(\Sigma^i\backslash R_0/\Sigma^j)$. Let $\mathscr{A}_1$ be a proper candidate for $\Sigma^i\backslash R_0/\Sigma^j$. Let $\{(v, w, x, y) \mid v \in L_i(R_0), vw \in L_{i+k}(R_0), xy \in R_{j+k}(R_0)$, and $y \in R_j(R_0)\} = \{(v_1, w_1, x_1, y_1), \ldots, (v_n, w_n, x_n, y_n)\}$. Then $R_0 - \Sigma(i + j + 2k - 1) \subseteq v_1w_1(w_1\backslash R[\mathscr{A}_1]/x_1)\, x_1y_1 \cup \cdots \cup v_nw_n(w_n\backslash R[\mathscr{A}_1]/x_n)\, x_ny_n \subseteq \Sigma^*\backslash R/\Sigma^*$. Thus $r_m(R_0) \leqslant r_m(R[\mathscr{A}_1]) = r_m(\Sigma^i\backslash R_0/\Sigma^j)$, completing the proof.

PROPOSITION 4.6. *For any* $R_0 \in T(R)$, $r_m(R_0) = \max\{r_m(R[\mathscr{S}]) \mid \mathscr{S}$ *is a section of* $\mathscr{A}[R_0]\}$.

Proposition 4.6 follows from Lemma 4.6 immediately.

In the sequel, we obtain an algorithm for determining $r_m(R_0)$ for all $R_0 \in T(R)$. Let $R_0 \in T(R)$. By Proposition 4.6, we may assume $\mathscr{A}[R_0]$ is s.c. Let $\mathscr{A}_0 = \mathscr{A}[R_0] = \langle \Sigma, Q_0, M_0\rangle$. For $t \subseteq Q$, define $\Sigma^{2k}(t) = \{vw \mid v, w \in \Sigma^k, M(Q, v) \cap t \neq \phi$, and $q_d \notin M_d(t, w)\}$, $L(t, t) = \text{root}(\mathscr{A}(t, t))$, and $R(\mathscr{A}_0, t) = R[\mathscr{A}_0] \cap$

$L(t, t)$. We say that $t \subseteq Q$ is a possible rank center of $\mathscr{A}_0$ if $R[\mathscr{A}_0] \subseteq \Sigma^*\backslash R(\mathscr{A}_0, t)^*/\Sigma^*$. Let p.c. $(\mathscr{A}_0)$ be the set of possible rank centers of $\mathscr{A}_0$.

LEMMA 4.7. *For any* $t \subseteq Q$, $x, z \in \Sigma^k$, *and* $y, y' \in \Sigma^*$, *if* $xyy'z \in L(t, t)$, *then* $xy \notin \mathscr{A}(Q, t, \bigcup)$ *or* $y'z \notin \mathscr{A}(t, Q)$.

Proof. Assume that $xyy'z \in L(t, t)$, $xy \in \mathscr{A}(Q, t, \bigcup)$, and $y'z \in \mathscr{A}(t, Q)$. Then $M_d(t, xy) \subseteq t$, and $M_d(t, y't) = M_d(t, xyy'z) \subseteq t$, which is a contradiction to $xyy'z \in L(t, t)$.

LEMMA 4.8. *For any* $t \subseteq Q$, *if* $t \notin$ *p.c.* $(\mathscr{A}_0)$, *then* $r_m(R_0) = r_m(R[\mathscr{A}_0] - \Sigma^*\Sigma^{2k}(t)\Sigma^*)$.

Proof. Assume $t \notin$ p.c. $(\mathscr{A}_0)$. One can see that $r_m(R_0) = r_m(R[\mathscr{A}_0]) \geqslant r_m(R[\mathscr{A}_0] - \Sigma^*\Sigma^{2k}(t)\Sigma^*)$ by Lemma 4.6.(1). Then it will suffice to show that $R[\mathscr{A}_0] - \Sigma(2k) \subseteq R[\mathscr{A}_0] - \Sigma^*\Sigma^{2k}(t)\Sigma^*$. Assume that for some $w \in \Sigma^*\Sigma^{2k}$, $w \in R[\mathscr{A}_0] \cap \Sigma^*\Sigma^{2k}(t)\Sigma^*$. Let $w = x_0y_0y_1x_1$, x_0, $x_1 \in \Sigma^*$, y_0, $y_1 \in \Sigma^k$, and $y_0y_1 \in \Sigma^{2k}(t)$. Consider any $z \in R[\mathscr{A}_0]$. Since $\mathscr{A}_0$ is s.c., $x_0y_0y_1x_1vzv'x_0y_0 \in R[\mathscr{A}_0]$ for some $v, v' \in \Sigma^*$. Then $M_d(t, y_1x_1vzv'x_0y_0) = M_d(t, y_0) \subseteq t$. Let

$$y_1x_1vzv'x_0y_0 = v_1 \cdots v_m,$$

and $v_i \in L(t, t)$ for $i = 1,\ldots, m$. Then $z \in \Sigma^*\backslash R(\mathscr{A}_0, t)^*/\Sigma^*$. This is a contradiction to $t \notin$ p.c. $(\mathscr{A}_0)$. Hence $R[\mathscr{A}_0] - \Sigma(2k) \subseteq R[\mathscr{A}_0] - \Sigma^*\Sigma^{2k}(t)\Sigma^*$, completing the proof.

PROPOSITION 4.7. *Let* $R_0 \in T(R)$, $r_m(R_0) > 0$, *and* $\mathscr{A}[R_0] = \mathscr{A}_0$ *be s.c. Then* $r_m(R_0) = \min\{r_m(R(\mathscr{A}_0, t)) \mid t \in p.c. (\mathscr{A}_0)\} + 1$.

Proof. It is easy to see that for each $t \in$ p.c. $(\mathscr{A}_0)$, $r_m(R_0) \leqslant r_m(R(\mathscr{A}_0, t)) + 1$. Conversely let $\mathscr{A}_1 = \langle \Sigma, Q_1, M_1 \rangle$ be a proper candidate for R_0. Let $q_0 \in Q_1$ be a rank center of $\mathscr{A}_1$. Put $t = \{q \in Q \mid$ for some $v \in \Sigma^k$, $q_0 \in M_1(Q_1, v)$ and $\{q\} = M(Q, v)\}$. By Lemma 4.7, one can see that $\Sigma^k\backslash R(\mathscr{A}_0, t)/\Sigma^k \subseteq R[\mathscr{A}_1 - [q_0]]$. Thus $r_m(R_0) = r(\mathscr{A}_1) \geqslant r(\mathscr{A}_1 - [q_0]) + 1 \geqslant r_m(R(\mathscr{A}_0, t)) + 1$. Now it is left to show that $t \in$ p.c. $(\mathscr{A}_0)$. Assume that $t \notin$ p.c. $(\mathscr{A}_0)$. One ca see that $\Sigma^k\backslash(R[\mathscr{A}_0] - \Sigma^*\Sigma^{2k}(t)\Sigma^*)/\Sigma^k \subseteq R[\mathscr{A}_1 - [q_0]]$. Then $r(\mathscr{A}_1) = r(\mathscr{A}_1 - [q_0]) + 1 \geqslant r_m(R[\mathscr{A}_0] - \Sigma^*\Sigma^{2k}(t)\Sigma^*) + 1$ by Lemma 4.6.(1), which is a contradiction to Lemma 4.8. This completes the proof.

Proposition 4.7 can be restated as follows.

COROLLARY 4.2. *Let* $R_0 \in T(R)$, $r_m(R_0) > 0$, *and* $\mathscr{A}[R_0] = \mathscr{A}_0$ *be s.c. Then* $r_m(R_0) = \min\{r_m(R[\mathscr{A}_0] - \Sigma^*\Sigma^{2k}(t)\Sigma^*) \mid t \subseteq Q\} + 1$.

Proof. By the above proof, we can see that $r_m(R_0) \geqslant r_m(R[\mathscr{A}_0] - \Sigma^*\Sigma^{2k}(t)\Sigma^*) + 1$ for some $t \subseteq Q$. By Lemma 4.8 and the proposition, it will suffice to show that for any $t \in$ p.c. $(\mathscr{A}_0)$, $R[\mathscr{A}_0] - \Sigma^*\Sigma^{2k}(t)\Sigma^* \supseteq R(\mathscr{A}_0, t)$. By

Lemma 4.7, we can see that for each $x \in R(\mathscr{A}_0, t)$, $x \notin \Sigma^*\Sigma^{2k}(t)\Sigma^*$. This completes the proof.

COROLLARY 4.3. *Let* $R_0 \in T(R)$, *and* $\mathscr{A}[R_0] = \mathscr{A}_0$ *be s.c. If there exist* Σ_1, $\Sigma_2 \subseteq \Sigma$, *and* R_1, $R_2 \subseteq R[\mathscr{A}_0]$ *such that* $\Sigma_1 \cap \Sigma_2 = \phi$, $R_1 \subseteq \Sigma^*$, $R_2 \subseteq \Sigma_2$, *and* $\Sigma_1\Sigma_2 \cap \Sigma^*\backslash R/\Sigma^* = \phi$, *then* $r_m(R_0) \geqslant \min\{r_m(R_1), r_m(R_2)\} + 1$.

Proof. By Corollary 4.2, it will suffice to show that for any $t \subseteq Q$, $r_m(R[\mathscr{A}_0] - \Sigma^*\Sigma^{2k}(t)\Sigma^*) \geqslant \min\{r_m(R_1), r_m(R_2)\}$. Let $t \subseteq Q$. If $(\Sigma^*\backslash R_1/\Sigma^*) \cap \Sigma^{2k}(t)$, $(\Sigma^*\backslash R_2/\Sigma^*) \cap \Sigma^{2k}(t) \neq \phi$, then $\Sigma_1\Sigma_2 \cap \Sigma^*\backslash R/\Sigma^* \neq \phi$, a contradiction. Thus $(\Sigma^*\backslash R_1/\Sigma^*) \cap \Sigma^{2k}(t) = \phi$ or $(\Sigma^*\backslash R_2/\Sigma^*) \cap \Sigma^{2k}(t) = \phi$. Then $R_1 \subseteq R[\mathscr{A}_0] - \Sigma^*\Sigma^{2k}(t)\Sigma^*$, or $R_2 \subseteq R[\mathscr{A}_0] - \Sigma^*\Sigma^{2k}(t)\Sigma^*$, and the result follows.

Propositions 4.4 ~ 4.7 provide the following algorithm for determining the star height of reset events.

ALGORITHM 4.1. *Let* $R \subseteq \Sigma^*$ *be k-reset*, $k \geqslant 1$, *and* $\mathscr{A}$ *the reduced automaton accepting R as above. Determine* $r(\mathscr{A})$ *by definition, and determine* $r_m(R)$ *by the procedure below with* $r_0(R) = r(\mathscr{A})$. *Put* $h(R) = r_m(R)$.

Procedure. Let $R_0 \in T(R)$ *with* $r_0(R_0)$ *given.*

Step 1. *If* R_0 *is finite, or* $r_0(R_0) = 0$, *then put* $r_m(R_0) = 0$. *Otherwise proceed to Step* 2.

Step 2. *Let* $\{\mathscr{S} \mid \mathscr{S}$ *is a nontrivial section of* $\mathscr{A}[R_0]\} = \{\mathscr{S}_1, \ldots, \mathscr{S}_m\}$. *For each* $i = 1, \ldots, m$, *and* $t \in p.c.(\mathscr{S}_i)$, *apply Procedure to* $R(\mathscr{S}_i, t)$ *with* $r_0(R(\mathscr{S}_i, t)) = r_0(R_0) - 1$ *to determine* $r_m(R(\mathscr{S}_i, t))$.

Step 3. *Determine* $r_m(R_0)$ *by the following two equations*:

(1) $r_m(R_0) = \max\{r_m(R(\mathscr{S}_i)) \mid i = 1, \ldots, m\}$;

(2) *for each* $i = 1, \ldots, m$, $r_m(R(\mathscr{S}_i)) = \min\{r_m(R(\mathscr{S}_i, t)) \mid t \in p.c.(\mathscr{S}_i)\} + 1$.

Remark 4.2. Eggan (1963) presented the following class of regular events of arbitrary star height inductively. Let $a_i \in \Sigma$, $i = 1, 2, 3, \ldots$.

(1) $E_1 = a_1$, $E_2 = (a_1^* a_2^* a_3)$, $F_1 = a_2$, and $F_2 = (a_4^* a_5^* a_6)$.

(2) $E_k = (E_{k-1}^* F_{k-1}^* a_{2^k-1})$, and F_k is obtained from E_k by adding $2^k - 1$ to the subscripts of all a_i.

Then $h(| E_k^* |) = k$.

One can alternatively prove that $h(| E_k^* |) = k$ as follows. One can see that $| E_k^* |$ is 1-reset by noting for each $a \in \Sigma$, a appears at most once in the expression E_k. By Proposition 4.4, it will suffice to show that for each $| E_i^* |$, $i \leqslant k$, $r_m(| E_i^* |) = i$. Clearly $r_m(| E_i^* |) \leqslant i$. Conversely note that for all $w \in | E_i^* |$, $| E_i^* | = w\backslash | E_i^* |$. Thus $\mathscr{A}[| E_i^* |]$ is s.c. One can prove that $r_m(| E_i^* |) = i$ by induction on i, using Lemma 4.6 and Corollary 4.3.

In the rest of the paper we shall present an alternative algorithm for determining the star height of 1-reset events.

DEFINITION 4.3. Let $\mathscr{A}_0 = \langle \Sigma, Q_0, M_0 \rangle$ be an automaton.

(1) $E(\mathscr{A}_0)$ is the set of edges in $\mathscr{A}_0$, that is, $E(\mathscr{A}_0) = \{(q, a, q') \mid q, q' \in Q_0, a \in \Sigma \cup \{\lambda\}, q' \in M_0(q, a)$, and if $a = \lambda$, then $q \neq q'\}$;

(2) Any $E_0 \subseteq E(\mathscr{A}_0)$ is an "admissible" set of edges in $\mathscr{A}_0$ if for some $Q_1 \subseteq Q_0, E_0 = \{(q, a, q') \mid q \in Q_1, a \in \Sigma \cup \{\lambda\}, q' \in M_0(q, a)$, and $q_d \notin M_{0_d}(Q_1, a)\}$;

(3) a.e. $(\mathscr{A}_0)$ is the set of admissible sets of edges in $\mathscr{A}_0$;

(4) For any $E_0 \in E(\mathscr{A}_0)$, $\mathscr{A}_0 - E_0$ is a subautomaton of $\mathscr{A}_0$, $\langle \Sigma, Q_0, M_1 \rangle$, such that for all $q, q' \in Q_0$, and $a \in \Sigma \cup \{\lambda\}$, $q' \in M_1(q, a)$ iff $q' \in M_0(q, a)$, and $(q, a, q') \notin E_0$.

DEFINITION 4.4. Let $\mathscr{A}_0 = \langle \Sigma, Q_0, M_0 \rangle$ be an automaton.

(1) For any subautomaton $\mathscr{A}_1 = \langle \Sigma, Q_1, M_1 \rangle$ of $\mathscr{A}_0$, the edge rank $e(\mathscr{A}_1, \mathscr{A}_0)$ of $\mathscr{A}_1$ w.r.t. $\mathscr{A}_0$ is defined inductively as follows:

(1.1) If all sections of $\mathscr{A}_1$ are trivial, then $e(\mathscr{A}_1, \mathscr{A}_0) = 0$;

(1.2) If $\mathscr{A}_1$ has a nontrivial section, then $e(\mathscr{A}_1, \mathscr{A}_0) = \max\{\min e(\mathscr{S} - E_0 \cap E(\mathscr{S}), \mathscr{A}_0) \mid E_0 \in \text{a.e.}(\mathscr{A}_0)\} \mid \mathscr{S}$ is a section of $\mathscr{A}_1\} + 1$.

(2) The edge rank $e(\mathscr{A}_0)$ of $\mathscr{A}_0$ is defined by

$$e(\mathscr{A}_0) = e(\mathscr{A}_0, \mathscr{A}_0).$$

LEMMA 4.9. *For any automaton $\mathscr{A}_0$, and a subautomaton $\mathscr{A}_1$ of $\mathscr{A}_0$, $e(\mathscr{A}_1, \mathscr{A}_0) = \max\{e(\mathscr{S}, \mathscr{A}_0) \mid \mathscr{S}$ is a section of $\mathscr{A}_1\}$, and $e(\mathscr{A}_0) \leqslant r(\mathscr{A}_0)$.*

ALGORITHM 4.2. *Let $R \subseteq \Sigma^*$ be 1-reset. Then $h(R) = e(\mathscr{A}[R])$.*

Proof. Let $\mathscr{A}[R] = \langle \Sigma, Q, M, \{s\}, F \rangle$. We define the set of problems on $\mathscr{A}$, $T_0(\mathscr{A})$, as follows: $T_0(\mathscr{A}) = \{\mathscr{A}_0 \mid \mathscr{A}_0$ is a subautomaton of $\mathscr{A}$, and $Q(\mathscr{A}_0) = Q\}$. For each $\mathscr{A}_0 \in T_0(\mathscr{A})$, $r_m(R[\mathscr{A}_0])$ is defined as above. By Proposition 4.4, it will suffice to show that for each $\mathscr{A}_0 \in T_0(\mathscr{A})$, $r_m(R[\mathscr{A}_0]) = e(\mathscr{A}_0, \mathscr{A})$. Let $\mathscr{A}_0 = \langle \Sigma, Q, M_0 \rangle$. The proof is by induction on $\#E(\mathscr{A}_0)$. If $\#E(\mathscr{A}_0) = 0$, then $e(\mathscr{A}_0, \mathscr{A}) = r_m(R[\mathscr{A}_0]) = 0$. Let $\#E(\mathscr{A}_0) > 0$. By Proposition 4.6 and Lemma 4.9, we may assume that $\mathscr{A}_0$ is s.c. By Corollary 4.2, and definition of $e(\mathscr{A}_0, \mathscr{A})$, it will suffice to show that for each $t \subseteq Q$, $r_m(R[\mathscr{A}_0] - \Sigma^*\Sigma^2(t)\Sigma^*) = e(\mathscr{A}_0 - E_0, \mathscr{A})$, where $E_0 = \{(q, a, q') \mid q \in t, a \in \Sigma, q' \in M(q, a)$, and $q_d \notin M_d(t, a)\} \cap E(\mathscr{A}_0)$. We may assume that $R[\mathscr{A}_0] \neq R[\mathscr{A}_0] - \Sigma^*\Sigma^2(t)\Sigma^*$. We show that $R[\mathscr{A}_0) - \Sigma^*\Sigma^2(t)\Sigma^* = R[\mathscr{A}_0 - E_0]$. Let $w \in R[\mathscr{A}_0 - E_0]$, and assume that $w \notin R[\mathscr{A}_0] - \Sigma^*\Sigma^2(t)\Sigma^*$. Since $w \in R[\mathscr{A}_0]$, it follows that $w \in \Sigma^*\Sigma^2(t)\Sigma^*$. Then for some $v_1, v_2 \in \Sigma^*$, and $a, b \in \Sigma$, $ab \in \Sigma^2(t)$, and $w = v_1 ab v_2$. By definition of $\Sigma^2(t)$, for some $q, q'' \in Q$, and $q' \in t$, $M(q, a) = q'$, $M(q', b) = q''$, and $q_d \notin M_d(t, b)$. Then $(q', b, q'') \in E_0$, and $w \notin R[\mathscr{A}_0 - E_0]$, which is a contradiction. Thus $R[\mathscr{A}_0 - E_0] \subseteq R[\mathscr{A}_0] - \Sigma^*\Sigma^2(t)\Sigma^*$. Similarly $R[\mathscr{A}_0] - \Sigma^*\Sigma^2(t)\Sigma^* \subseteq R[\mathscr{A}_0 - E_0]$. By the inductive hypothesis, $e(\mathscr{A}_0 - E_0, \mathscr{A}) = r_m(R[\mathscr{A}_0 - E_0]) = r_m(R[\mathscr{A}_0] - \Sigma^*\Sigma^2(t)\Sigma^*)$. This completes the proof.

COROLLARY 4.4. *If for any* $t \in \{t' \subseteq Q \mid \#t' = 2\}$, *the set* $\{(q, a, q') \mid q \in t, a \in \Sigma, q' = M(q, a)$ *and* $q_d \notin M_d(t, a)\}$ *is empty, then* $h(R) = r(\mathscr{A})$.

EXAMPLE 4.1. There exists a 1-reset automaton $\mathscr{A}$ such that $e(\mathscr{A}) < r(\mathscr{A})$. Let $\mathscr{A} = \langle \Sigma, Q, M \rangle$, $\Sigma = \{0, 1, 2, 3, 4, 5\}$, $Q = \{q_0, q_1, q_2, q_3, q_4, q_5\}$, and for all $q_i, q_j \in Q$, and $k \in \Sigma$, $q_j = M(q_i, k)$ iff $j = k$, and one of the following holds: (1) $i, j \in \{0, 1, 2\}$, (2) $i \in \{0, 1, 2\}$, $j \in \{3, 4, 5\}$, and $j = i + 3$, (3) $i \in \{3, 4, 5\}$, $j \in \{0, 1, 2\}$, and $i = j + 3$, or (4) $i, j \in \{3, 4, 5\}$, and $i \equiv j + 1 \pmod 3$. (As a graph representation, $\mathscr{A}$ consists of two main circles, the inner circle containing q_0, q_1, q_2, and the outer circle containing q_3, q_4, q_5, and the set of edges defined as above).

One can see with some efforts that $r(\mathscr{A}) = 4$, and $e(\mathscr{A}) = e(\mathscr{A}, \mathscr{A} - E_0) + 1 = 3$, where $E_0 = \{(q, a, q') \mid q \in \{q_0, q_1, q_2\}, a \in \Sigma, q' = M(q, a)$, and $q_d \notin M_d(\{q_0, q_1, q_2\}, a)\}$.

EXAMPLE 4.2. There exists a star event R such that $h(R) = h(\text{root}(R))$. Let $\mathscr{A}$ be as in Example 4.1. Let $R = \mathscr{A}(q_0, q_0)$. Then $h(R) = e(\mathscr{A}) = 3$. Moreover one can see that $h(\text{root}(R)) = h(R[\mathscr{A} - [q_0]]) = e(\mathscr{A} - [q_0]) = 3$.

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