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A CHARACTERIZATION OF ROBERTS' INEQUALITY FOR BOXICITY

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F.S. Roberts defined the boxicity of a graph G as the smallest positive integer n for which there exists a function F assigning to each vertex $x \in G$ a sequence $F(x)(1), F(x)(2), \ldots, F(x)(n)$ of closed intervals of $\mathbf{R}$ so that distinct vertices x and y are adjacent in G if and only if $F(x)(i) \cap F(y)(i) \neq \emptyset$ for $i = 1, 2, 3, \ldots, n$. Roberts then proved that if G is a graph having $2n+1$ vertices, then the boxicity of G is at most n. In this paper, we provide an explicit characterization of this inequality by determining for each $n \geq 1$ the minimum collection $\mathscr{C}_n$ of graphs so that a graph G having $2n+1$ vertices has boxicity n if and only if it contains a graph from $\mathscr{C}_n$ as an induced subgraph. We also discuss combinatorial connections with analogous characterization problems for rectangle graphs, circular arc graphs, and partially ordered sets.

1. Introduction

In this paper all graphs are finite and have no loops or multiple edges. For a graph G, we write $x \perp y$ in G when x and y are adjacent vertices in G and $x \pm y$ in G when x and y are nonadjacent. We denote the number of vertices in G by $|G|$. A graph H is called an *induced subgraph* of a graph G when the vertex set of H is a subset of the vertex set of G and distinct vertices of H are adjacent in H if and only if they are adjacent in G. When H is an induced subgraph of G, we will also say G *contains* H. We do not distinguish between isomorphic graphs.

A graph G is an *interval graph* when there is a function f which assigns to each vertex $x \in G$ a closed interval $f(x)$ of the real line $\mathbf{R}$ so that $x \perp y$ in G if and only if $f(x) \cap f(y) \neq \emptyset$ and $x \neq y$. Alternatively, an interval graph is the intersection graph of a family of closed intervals of the real line $\mathbf{R}$.

The concept of an interval graph extends very naturally to higher dimensions by considering the intersection graph of a family of "boxes" in n-dimensional Euclidean space $\mathbf{R}^n$. Roberts [3] defined the *boxicity* of a graph G, denoted $\operatorname{Box}(G)$, as the smallest positive integer n for which G is the intersection graph of a family of boxes in $\mathbf{R}^n$. Formally, $\operatorname{Box}(G)$ is the smallest positive integer n for which there exists a function F which assigns to each vertex $x \in G$, a sequence $F(x)(1), F(x)(2), \ldots, F(x)(n)$ of closed intervals of $\mathbf{R}$ so that $x \perp y$ in G if and only if $x \neq y$ and $F(x)(i) \cap F(y)(i) \neq \emptyset$ for $i = 1, 2, \ldots, n$. The function F is called an *interval coordinatization* of length n for G. By convention, we define

$\mathrm{Box}(G)=0$ when G is a complete graph. Therefore, a graph G is an interval graph if and only if $\mathrm{Box}(G)\leq 1$.

Roberts proved that if G is a graph having $2n+1$ vertices (where $n\geq 1$), then $\mathrm{Box}(G)\leq n$. The principal result of this paper will be an explicit characterization of this inequality. For each $n\geq 1$, we will determine the minimum collection $\mathscr{C}_n$ of graphs so that if G is a graph and $|G|=2n+1$, then $\mathrm{Box}(G)=n$ if and only if G contains a graph from $\mathscr{C}_n$ as an induced subgraph.

2. Some inequalities for boxicity

If A is a subset of the vertex set $V(G)$ of a graph G, we denote by $G-A$ the subgraph of G with vertex set $V(G)-A$. It is obvious that if H is an induced subgraph of G, then $\mathrm{Box}(H)\leq\mathrm{Box}(G)$.

We now state without proof two elementary lemmas due to Roberts [3].

Lemma 1. *If x is a vertex of G, then* $\mathrm{Box}(G)\leq 1+\mathrm{Box}(G-\{x\})$.

Lemma 2. *If $x\pm y$ in G, then* $\mathrm{Box}(G)\leq 1+\mathrm{Box}(G-\{x,y\})$.

It is easy to verify that every graph on three vertices is an interval graph and thus has boxicity at most one. The following inequality then follows from Lemma 2 by induction on n.

Theorem 1 (Roberts). *If $|G|=2n+1$ where $n\geq 1$, then* $\mathrm{Box}(G)\leq n$.

The *join* of two graphs G and H, denoted $G\oplus H$, is the graph formed by adding to disjoint copies of G and H all edges with one endpoint in G and the other in H. We illustrate this definition with the graphs shown in Fig. 1.

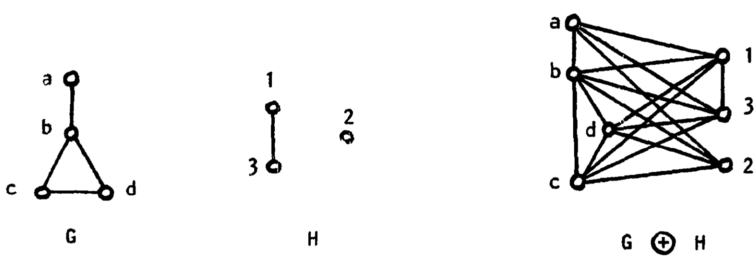

Fig. 1.

The following lemma shows that boxicity is additive with respect to the join operation on graphs.

Lemma 3. $\mathrm{Box}(G\oplus H)=\mathrm{Box}(G)+\mathrm{Box}(H)$ *for every pair of graphs G and H.*

Proof. Let $t_1=\mathrm{Box}(G)$, $t_2=\mathrm{Box}(H)$, and $t_3=\mathrm{Box}(G\oplus H)$. We further assume that $t_1 \geq 1$ and that $t_2 \geq 1$, i.e., neither G nor H is complete. The argument when $t_1 = 0$ or $t_2 = 0$ will follow with minor modifications.

We first show that $t_3 \leq t_1 + t_2$. Let F_1 be an interval coordinatization of length t_1 for G and let F_2 be an interval coordinatization of length t_2 for H. Then choose an interval $[a, b]$ of $\mathbf{R}$ so that $F_1(x)(i) \cup F_2(y)(j) \subseteq [a, b]$ for every $x \in G$, $y \in H$, $i \leq t_1$ and $j \leq t_2$. For each vertex $z \in G \oplus H$ and each positive integer $k \leq t_1 + t_2$ we then define a closed interval $F_3(z)(k)$ of $\mathbf{R}$ by the following rule.

$$F_3(z)(k) = \begin{cases} F_1(z)(k) & \text{if } z \in G \text{ and } 1 \leq k \leq t_1, \\ [a, b] & \text{if } z \in G \text{ and } t_1 + 1 \leq k \leq t_1 + t_2, \\ [a, b] & \text{if } z \in H \text{ and } 1 \leq k \leq t_1, \\ F_2(z)(k - t_1) & \text{if } z \in H \text{ and } t_1 + 1 \leq k \leq t_1 + t_2. \end{cases}$$

It follows immediately that F_3 is an interval coordinatization of length $t_1 + t_2$ for $G \oplus H$, and thus $t_3 \leq t_1 + t_2$.

We now show that $t_3 \geq t_1 + t_2$. Let F be an interval coordinatization of length t_3 of $G \oplus H$. Then let S_1 and S_2 be the subsets of $\{1, 2, 3, \ldots, t_3\}$ defined by

$$S_1 = \{i\colon \text{ There exist nonadjacent vertices } x_1, x_2 \in G \text{ so that } F(x)(i) \cap F(x_2)(i) = \emptyset\}$$

and

$$S_2 = \{i\colon \text{ There exist nonadjacent vertices } y_1, y_2 \in H \text{ so that } F(y_1)(i) \cap F(y_2)(i) = \emptyset\}.$$

We show that $S_1 \cap S_2 = \emptyset$, $|S_1| \geq t_1$ and $|S_2| \geq t_2$. This will allow us to conclude that $t_3 = |S| \geq |S_1| + |S_2| \geq t_1 + t_2$.

To see that $S_1 \cap S_2 = \emptyset$, we observe that if $i \in S_1$, x_1, $x_2 \in G$, and $F(x_1)(i) \cap F(x_2)(i) = \emptyset$, then

$$F(x_1)(i) \cap F(y)(i) \neq \emptyset \neq F(x)(i) \cap F(y)(i)$$

for every $y \in H$, i.e., the interval $F(y)(i)$ contains the open interval of $\mathbf{R}$ lying between the disjoint closed intervals $F(x_1)(i)$ and $F(x_2)(i)$. Hence $F(y_1)(i) \cap F(y_2)(i) \neq \emptyset$ for every $y_1, y_2 \in H$ and thus $i \notin S_2$.

To see that $|S_1| \geq t_1$, let $|S_1| = m$ and $S_1 = \{i_1, i_2, \ldots, i_m\}$. Then the function F' defined by $F'(x)(j) = F(x)(i_j)$ for every $x \in G$ and every $j \leq m$ is an interval coordinatization of length m for G; hence, $m \geq t_1$. The same argument shows that $|S_2| \geq t_2$, so that our argument is complete in the case when $t_1 \geq 1$ and $t_2 \geq 1$.

We now consider the case when $t_1 = 0$ or $t_2 = 0$. First, if both t_1 and t_2 are zero, then G, H, and $G \oplus H$ are complete graphs so that $\operatorname{Box}(G \oplus H) = 0 = \operatorname{Box}(G) + \operatorname{Box}(H)$. By symmetry, it remains only to consider the case where $t_1 = 0$ and $t_2 > 0$. Then $\operatorname{Box}(G \oplus H) \geq \operatorname{Box}(H)$ since H is a subgraph of $G \oplus H$. To show that $\operatorname{Box}(G \oplus H) \leq \operatorname{Box}(H)$, we choose an arbitrary interval representation F of length t_2 for H. We then select an interval $[a, b]$ of R so that $F(y)(i) \subseteq [a, b]$ for every $y \in H$ and $i \leq t_2$. Finally, we extend F to $G \oplus H$ by defining $F(x)(i) = [a, b]$

for every $x \in G$ and $i \leqslant t_2$. It is clear that we have obtained an interval representation of $G \oplus H$ of length t_2 so that $\mathrm{Box}(G \oplus H) \leqslant \mathrm{Box}(H)$, and with this observation, the proof of the lemma is complete.

Let G_1 be the graph consisting of two nonadjacent vertices. For $n \geqslant 1$, we then define G_n inductively by $G_{k+1} = G_k \oplus G_1$. It follows immediately that $|G_n| = 2n$ and $\mathrm{Box}(G_n) = n$. Therefore, Roberts' inequality (Theorem 1) is best possible. We now use the graph G_n for $n \geqslant 1$ to examine the sharpness of the following inequalities which follow easily from Lemmas 1 and 2.

Lemma 4. *If K is a complete subgraph of G with $|K| = k$, then* $\mathrm{Box}(G) \leqslant k + \mathrm{Box}(G - K)$.

Lemma 5. *If I is an independent induced subgraph of G with $|I| = i$, then* $\mathrm{Box}(G) \leqslant \{i/2\} + \mathrm{Box}(G - 1)$.

Label the vertices of G_n with the symbols $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ so that the subgraphs $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ are complete and a_i and b_j are adjacent if and only if $i \neq j$. Now suppose $1 \leqslant k \leqslant n$ and let K be the k-element complete subgraph $\{a_1, a_2, \ldots, a_k\}$. It follows immediately from Lemma 3 that $\mathrm{Box}(G) = k + \mathrm{Box}(G - K)$ so that Lemma 4 is also best possible.

To test the accuracy of Lemma 5, it is first necessary to modify the graph G_n. For each $n \geqslant 1$, let G_n^* be the graph obtained from G_n by removing all edges between distinct vertices of A.

Theorem 2. $\mathrm{Box}(G_n^*) = \{n/2\}$ *for all* $n \geqslant 1$.

Proof. Suppose first that $n = 2m$. For each $i \leqslant m$, let $F(a_{2i-1})(i) = [0, 1]$, $F[a_{2i})(i) = [4, 5]$, $F(b_{2i-1})(i) = [2, 4]$, $F(b_{2i})(i) = [1, 3]$, and if $j \neq 2i-1$, $j \neq 2i$, then $F(b_j)(i) = [0, 5]$, and $F(a_j)(i) = [2, 3]$. Clearly the function F is an interval coordinatization of length m for G_{2m}^*. Therefore, $\mathrm{Box}(G_{2m}^*) \leqslant m$ for all $m \geqslant 1$. The general result $\mathrm{Box}(G_n^*) \leqslant \{n/2\}$ now follows from Lemma 2.

On the other hand, suppose $\mathrm{Box}(G_n^*) = s$, and let F be an interval coordinatization of length s for G_n^*. For each $i \leqslant 2$, let

$$M(i) = \{j : F(a_j)(i) \cap F(b_j)(i) = \emptyset\}.$$

We first observe that $|M(i)| \leqslant 2$ for each $i \leqslant s$, for if j_1, j_2, and j_3 are distinct elements of $M(i)$, we may assume by symmetry that $F(b_{j_1})(i)$ lies entirely to the left of $F(a_{j_1})(i)$, $F(b_{j_2})(i)$ lies entirely to the left of $F(a_{j_2})(i)$, and that the right endpoint of $F(b_{j_2})(i)$ is at least as large as the right endpoint $F(b_{j_1})(i)$. But this would imply that $F(a_{j_2})(i) \cap F(b_{j_1})(i) = \emptyset$. Therefore, $|M(i)| \leqslant 2$.

Since we must clearly have $\sum_{i=1}^{s} |M(i)| \geqslant n$ it follows that $s \geqslant \{n/2\}$ and the argument is complete.

Now suppose $1 \leqslant k \leqslant n$ and I is the independent induced subgraph $\{a_1, a_2, \ldots, a_k\}$ in G_n^*. It follows from Theorem 2 and Lemma 3 that $\operatorname{Box}(G_n^* - 1) = \{(n-i)/2\}$. Therefore,

$$\operatorname{Box}(G_n^*) = \{i/2\} + \operatorname{Box}(G_n^* - 1),$$

and the inequality in Lemma 5 is also best possible.

3. A characterization of Roberts' inequality

Let H_2 be the 5 element cycle $\{c_1, c_2, c_3, c_4, c_5\}$ with $c_i \perp c_i + 1$ for $i = 1, 2, 3, 4$ and $c_5 \perp c_1$. For $n \geqslant 2$, we then define H_n inductively by $H_{k+1} = H_k \oplus G_1$. Since an interval graph does not contain a cycle of 4 or more vertices as an induced subgraph, we note that $\operatorname{Box}(H_2) = 2$. By Lemma 3, we then conclude that $\operatorname{Box}(H_n) = n$ for every $n \geqslant 2$.

Consider the graph W_3 shown in Fig. 2.

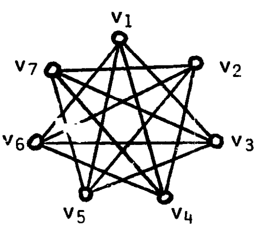

Fig. 2.

We will now show that this graph has boxicity 3. First, note that $\operatorname{Box}(W_3) \leqslant 3$ since $|W_3| = 7$. Now suppose that $\operatorname{Box}(W_3) < 3$ and let F be an interval coordinatization of length two for W_3. For $i = 1, 2$, let

$$E_i = \{\{v_j, v_k\}\colon\ 1 \leqslant j < k \leqslant 7,\ F(v_j)(i) \cap F(v_k)(i) = \emptyset\}.$$

It is easy to see that $|E_1 \cup E_2| \geqslant 7$ but that $|E_1| \leqslant 3$ and $|E_2| \leqslant 3$. The contradiction completes the argument.

We then define W_n for $n \geqslant 3$ by $W_{k+1} = W_k \oplus G_1$; by Lemma 3, we conclude that $\operatorname{Box}(W_n) = n$ for every $n \geqslant 3$.

The remainder of this section will be devoted to proving that the graphs G_n, H_n, and W_n provide an explicit characterization of Roberts' inequality for boxicity. In order to simplify the argument, we develop several preliminary lemmas. These lemmas will require the following result which follows from Lekkerkerker and Boland's characterization of interval graphs [2].

Lemma 6. *If* $|G| \leqslant 5$, *then* $\operatorname{Box}(G) = 2$ *if and only if* G *contains* G_2 *or* H_2.

Lemma 7. *If* $n \geqslant 1$ *and* $|G| = 2n$, *then* $\operatorname{Box}(G) = n$ *if and only if* $G = G_n$.

Proof. For $n = 1$, we note that the complete graph on two vertices has boxicity zero while the independent graph on two vertices G_1 has boxicity one. The result follows from Lemma 6 when $n = 2$.

Now assume validity for all values of $n \leqslant m$ where $m \geqslant 2$ and let G be a graph with $|G| = 2m + 2$ and $\mathrm{Box}(G) = m + 1$. If G is complete, then $\mathrm{Box}(G) = 0$, so G has vertices x, y with $x \pm y$. Then $G - \{x, y\}$ has $2m$ vertices and boxicity m and is therefore G_m. Label $G - \{x, y\} = G_m$ with the symbols $a_1, a_2, \ldots, a_m$, $b_1, b_2, \ldots, b_m$ so that $a_i \perp a_j$, $b_i \perp b_j$, and $a_i \perp b_j$ if and only if $i \neq j$ for $i, j = 1, 2, \ldots, m$. Now the graphs $G - \{a_1, b_1\}$ and $G - \{a_2, b_2\}$ each have $2m$ points and boxicity m and must also be copies of G_m. It follows that x and y are adjacent to every vertex of $G - \{x, y\}$ and therefore $G = \{x, y\} \oplus G_m = G_{m+1}$ and our proof is complete.

Suppose for some $n \geqslant 1$, G is a graph with $2n + 1$ vertices. If G has a vertex x of degree $2n$, then $G = \{x\} \oplus (G - \{x\})$ so that $\mathrm{Box}(G) = \mathrm{Box}(G - \{x\})$ and thus $\mathrm{Box}(G) = n$ if and only if G contains G_n.

Lemma 8. *Let G be a graph with $|G| = 7$. If G has a vertex of degree 5, then* $\mathrm{Box}(G) = 3$ *if and only if G contains G_3 or H_3.*

Proof. Choose a vertex x of degree 5 and then choose y with $x \pm y$. Now $G - \{x, y\}$ has 5 vertices and boxicity 2 and therefore contains G_2 or H_2. If y is adjacent to each vertex of $G - \{x, y\}$, then $G = \{x, y\} \oplus (G - \{x, y\})$ so that G contains G_2 or H_3.

Therefore, we may assume that there exists a vertex $z \in G - \{x, y\}$ with $z \pm y$. Therefore, $G - \{z, y\}$ has boxicity 2 and

$$G - \{z, y\} = \{x\} \oplus (G - \{x, y, z\}):$$

thus, $G - \{x, y, z\} = G_2$. Label $G - \{x, y, z\} = G_2$ with the symbols a_1, a_2, b_1, b_2 so that $a_1 \perp a_2$, $a_1 \perp b_2$, $b_1 \perp a_2$, and $b_1 \perp b_2$. If y is adjacent to each vertex in $G - \{x, y, z\}$, then G contains G_3 so we may assume without loss of generality that $y \pm a_1$. Then $G - \{a_2, b_2\}$ has boxicity 2 and thus contains G_2 or H_2, but this is not possible since y has degree at most one and x has degree 3 in $G - \{a_2, b_2\}$. The contradiction completes the proof.

Lemma 9. *Let G be a graph with $|G| = 7$. Then* $\mathrm{Box}(G) = 3$ *if and only if G contains G_3, H_3, or W_3.*

Proof. Let G be a graph with $|G| = 7$ and $\mathrm{Box}(G) = 3$. If G contains a vertex of degree 5 or 6, then G must contain G_3 or H_3, so we may assume without loss of generality that each vertex of G has degree at most 4.

Suppose that there exist a nonadjacent pair of vertices x, y so that $G - \{x, y\} = H_2$. Label the vertices of $G - \{x, y\}$ with the symbols c_1, c_2, c_3, c_4, c_5 so that $c_i \perp c_{i+1}$ for $i = 1, 2, 3, 4$ and $c_5 \perp c_1$. Since x has degree at most 4, we may assume that $x \pm c_2$. Then $G-\{c_1, c_3\}$ has boxicity 2 but does not contain G_2 or H_2. The contradiction allows us to conclude that for every nonadjacent pair of vertices x, y in G, $G-\{x, y\}$ contains G_2 but not H_2.

Now choose nonadjacent vertices x, y in G and a vertex z of $G-\{x, y\}$ so that $G-\{x, y, z\}=G_2$. Label the vertices of $G-\{x, y, z\}$ with the symbols a_1, a_2, b_1, b_2 as in the proof of Lemma 8. Suppose that $x \pm z$. If

$$G-y=\{x, y\} \oplus (G-\{x, y, z\})$$

or

$$G-z=\{x, y\} \oplus (G-\{x, y, z\}),$$

then G contains G_3 so we may assume without loss of generality that $y \pm a_1$. Then $G-\{a_2, b_2, y\}=G_2$ so that $x \perp a_1$, $x \perp b_1$, $z \perp a_1$, and $z \perp b_1$. And since b_1 has degree at most 4, we see that $y \pm b_1$. Therefore, $G-\{y, b_1, a_1\}=G_2$, $x \perp a_2$, $x \perp b_2$, $z \perp a_2$, $z \perp b_2$, and we conclude that

$$G-y=\{x, z\} \oplus (G-\{x, y, z\})=G_3.$$

The contradiction allows us to conclude that $x \perp z$ and $y \perp z$.

Since y has degree at most 4, we may assume that $y \pm a_1$. Then $G-\{a_2, b_2, a_1\}=G_2$ and thus $x \perp b_1$, $y \perp b_1$, and $z \pm b_1$. If $y \pm a_2$, then $G-\{b_1, z, y\}=G_2$ and thus $x \perp a_2$, $x \perp b_2$, and $x \pm a_1$. Therefore, $G-\{a_1, b_1, a_2\}=G_2$ and thus $y \perp b_2$ and $z \pm b_2$. But $G-\{z, b_2\}$ does not contain G_2. Therefore, we may assume that $y \perp a_2$. By symmetry, we may also assume $y \perp b_2$.

If $x \pm a_1$, then $G-\{x, a_1, y\}=G_2$ so $z \perp a_2$ and $z \perp b_2$. Therefore, $x \pm a_2$ and $x \pm b_2$. But $G-\{x, b_2\}$ does not contain G_2. The contradiction allows us to conclude that $x \perp a_1$. Since x has degree at most 4, we may then assume that $x \pm a_2$.

It follows that $G-\{x, a_2, b_1\}=G_2$ and thus $z \perp a_1$, $z \pm b_2$. Also we see that $G-\{y, a_1, b_2\}=G_2$ and thus $z \perp a_2$. Finally we note that if $x \pm b_2$, then $G-\{x, b_2\}$ does not contain G_2 so that we must have $x \perp b_2$. It follows that all adjacencies of G have been determined and that $G=W_3$.

We are now ready to establish our characterization of Roberts' inequality for boxicity. Theorem 3 will provide for each $n \geq 1$ the minimum collection $\mathscr{C}_n$ of graphs so that if $|G|=2n+1$, then $\mathrm{Box}(G)=n$ if and only if G contains a graph from $\mathscr{C}_n$ as an induced subgraph.

Theorem 3. *Let $n \geq 1$ and let G be a graph with $|G|=2n+1$.*

(i) *If $n=1$, then* $\mathrm{Box}(G)=n$ *if and only if G contains G_1.*

(ii) *If $n=2$, then* $\mathrm{Box}(G)=n$ *if and only if G contains G_2 or H_2.*

(iii) *If $n \geq 3$, then* $\mathrm{Box}(G)=n$ *if and only if G contains G_n, H_n or W_n.*

Proof. Part (i) is trivial since a complete graph has boxicity zero; part (ii) is

Lemma 6. We now proceed to prove part (iii) by induction on n. We first note that part (iii) is valid for $n=3$ in view of Lemma 9. We then assume validity for all $n \leq m$ where m is some integer with $m \geq 3$. Then let G be a graph with $|G|=2m+3$ and $\mathrm{Box}(G)=m+1$. We will now show that G contains G_{m+1}, H_{m+1}, or W_{m+1}.

Let x, y be any pair of nonadjacent vertices in G. Then $G-\{x, y\}$ has $2m+1$ vertices and boxicity $m+1$ and therefore must contain G_m, H_m, or W_m. Suppose first that there exists a nonadjacent pair of x, y of vertices of G so that $G-\{x, y\}=H_m$. Label the vertices of $G-\{x, y\}$ with the symbols $a_1, a_2, \ldots, a_{m-2}, b_1, b_2, \ldots, b_{m-2}, c_1, c_2, c_3, c_4, c_5$ in the obvious fashion.

As in the proof of Lemma 9, if $x \pm c_2$, then $G-\{c_1, c_3\}$ has boxicity m but does not contain G_m, H_m, or W_m since c_2 has degree at most $2m-3$ and x has degree at most $2m-2$ in $G-\{c_1, c_3\}$. We may therefore conclude that x and y are both adjacent to c_1, c_2, c_3, c_4, and c_5.

Now consider the graph $G-\{c_1, c_3\}$ which has boxicity m. Since c_4 and c_5 have degree $2m-1$ in $G-\{c_1, c_3\}$, $c_4 \pm c_2$, and $c_5 \pm c_2$, we see that $G-\{c_1, c_3\}$ is not W_m or H_m. And therefore, $G-\{c_1, c_3\}$ must contain G_m. It is easy to see that we must have either $G-\{c_1, c_3, c_4\}=G_m$ or $G-\{c_1, c_3, c_5\}=G_m$. In either case, x and y are both adjacent to $a_1, a_2, \ldots, a_{m-2}, b_1, b_2, \ldots, b_{m-2}$ so that

$$G=\{x, y\} \oplus (G-\{x, y\})=H_{m+1}.$$

Now suppose that x and y are nonadjacent vertices of G and that $G-\{x, y\}=W_m$. Suppose first that $m=3$ and label $G-\{x, y\}$ with the symbols $v_1, v_2, \ldots, v_7$ as shown in Fig. 2. Suppose further that $x \pm v_1$. Then $G-\{v_3, v_4\}$ has boxicity 3 but v_1 has degree at most 3 so $G-\{v_3, v_4\}$ is neither W_3 or H_3. But it is easy to see that $G-\{v_3, v_4\}$ does not contain G_3 either. We may therefore conclude that x and y are adjacent to $v_1, v_2, \ldots, v_7$ and therefore, $G=W_4$.

Now suppose that $m \geq 4$ and label the vertices of $G-\{x, y\}$ with the symbols $a_1, a_2, \ldots, a_{m-3}, b_1, b_2, \ldots, b_{m-3}, v_1, v_2, \ldots, v_7$ in the obvious fashion. As in the preceding paragraph, we may conclude that x and y are adjacent to $v_1, v_2, \ldots, v_7$. Now consider the graph $G-\{v_1, v_2\}$ which has boxicity m.

Now v_3 has degree $2m-1$ and $v_3 \pm v_4$ and $v_5 \pm v_4$ in $G-\{v_1, v_2\}$ so $G-\{v_1, v_2\}$ is not W_m or H_m. Therefore, $G-\{v_1, v_2\}$ must contain G_m. Clearly this requires $G-\{v_1, v_2, v_5\}=G_m$ and thus, x and y are adjacent to $a_1, a_2, \ldots, a_{m-3}, b_1, b_2, \ldots, b_{m-3}$. Therefore,

$$G=\{x, y\} \oplus (G-\{x, y\})=W_{m+1}.$$

We may now assume that whenever x and y are nonadjacent vertices of G, the graph $G-\{x, y\}$ contains G_m, but not H_m or W_m. Choose a nonadjacent pair of vertices x, y and a vertex z so that $G-\{x, y, z\}=G_m$. Label the vertices of $G-\{x, y, z\}$ with the symbols $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m$ in the usual fashion. Now suppose that $x \pm z$. If $y \pm a_1$, then $G-\{a_2, b_2\}$ does not contain G_m so we may assume that y is adjacent to $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m$. Similarly, if $x \pm a_1$, then $G-\{a_2, b_2\}$ does not contain G_m, so we may assume that x is also adjacent to $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m$. But this implies that $G-z=G_{m+1}$. We may therefore assume that $x \perp z$. By symmetry, we may also assume $y \perp z$.

Now suppose that $x \pm a_1$. Then we must have $G-\{a_2, b_2, a_1\}=G-\{a_3, b_3, a_1\}=G_m$ and thus, $G-a_1=G_{m+1}$. We may therefore assume by symmetry that x and y are both adjacent to $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m$ and, therefore, $G-z=G_{m+1}$. With this case, the argument is complete.

4. The characterization of rectangle graphs

A graph G with $\mathrm{Box}(G) \leqslant 2$ is the intersection graph of a family of rectangles (with sides parallel to the x and y axes) in the plane, so it is natural to refer to a graph with boxicity at most 2 as a *rectangle graph.* In this section, we discuss the problem of providing a forbidden subgraph characterization of rectangle graphs. While this is a very difficult unsolved combinatorial problem, we will solve the subproblem of determining a forbidden subgraph characterization for rectangle graphs with clique covering number two. We accomplish this by establishing combinatorial connections between this problem and characterization problems for partially ordered sets and circular arc graphs as discussed in [6]. In the interests of brevity, we provide only the key definitions here and refer the reader to [6] for details. If $[a, b]$ and $[c, d]$ are closed intervals of the real line $\mathbf{R}$, we write $[a, b] \lhd [c, d]$ when $b<c$ in $\mathbf{R}$. The *interval dimension* of a partially ordered set X, denoted $\mathrm{I\,Dim}(X)$, is then the smallest positive integer n for which there exists a function F assigning to each point $x \in X$ a sequence $F(x)(1), F(x)(2), \ldots, F(x)(n)$ of closed intervals of $\mathbf{R}$ so that $x<y$ in X if and only if $F(x)(i) \lhd F(y)(i)$ for $i=1,2,3, \ldots, n$.

A partially ordered set X is said to be *t-interval irreducible* when $\mathrm{I\,Dim}(X)=t$ and $\mathrm{I\,Dim}(X-x)=t-1$ for every $x \in X$. Let $\mathcal{P}_2$ denote the collection of all 3-interval irreducible partially ordered sets of height 1.

A graph G is called a *circular arc* graph when it is the intersection graph of a family of arcs of a circle. Let $\mathcal{A}_2$ denote the collection of all graphs with clique covering number two which are not circular arc graphs but have the property that the removal of any vertex leaves a circular arc graph. Also, let $\mathcal{B}_2$ denote the collection of all graphs with clique covering number two which have boxicity 3, but have the property that the removal of any vertex leaves a subgraph with boxicity 2.

For a graph G, we denote by $\bar{G}$, the *complement* of G, i.e., $x \perp y$ in $\bar{G}$ if and only if $x \pm y$ in G. Now let X be a partially ordered set of height one with maximal elements $a_1, a_2, \ldots, a_m$ and minimal elements $b_1, b_2, \ldots, b_n$. We associate with X, graphs G_X and G_X^*, each having

$$\{a_1, a_2, \ldots, a_m\} \cup \{b_1, b_2, \ldots, b_n\}$$

as vertex sets. In G_X and G^*_X, the subgraphs induced by $\{a_1, a_2, \ldots, a_m\}$ and $\{b_1, b_2, \ldots, b_n\}$ are complete. In G_X we define $a_i \perp b_j$ if and only if $b_j < a_i$ while in G^*_X we define $a_i \pm b_j$ if and only if $b_j < a_i$.

Dually, for a graph G with vertex set $\{a_1, a_2, \ldots, a_m\} \cup \{b_1, b_2, \ldots, b_n\}$ for which the subgraphs induced by $\{a_1, a_2, \ldots, a_m\}$ and $\{b_1, b_2, \ldots, b_n\}$ are complete, we denote by X_G the partially ordered set of height one for which $G = G_{X_G}$. Among the results established in [6] is the following theorem relating circular arc graphs to partially ordered sets.

Theorem 4. *Let X be a partially ordered set of height one. Then $X \in \mathcal{P}_2$ if and only if $G^*_X \in \mathcal{A}_2$.*

Now let G be a graph with vertex set $\{a_1, a_2, \ldots, a_m\} \cup \{b_1, b_2, \ldots, b_n\}$ for which the subgraphs induced by $\{a_1, a_2, \ldots, a_m\}$ and $\{b_1, b_2, \ldots, b_n\}$ are complete. Suppose that $\operatorname{Box}(G) = 2$ and let F be an interval coordinatization of length two for G. Since $\operatorname{Box}(G) = 2$, assume by symmetry, that for $k = 1, 2$ there exist i_k, j_k with $1 \leq i_k \leq m$ and $1 \leq j_k \leq n$ so that $F(a_{i_k})(k) \lhd F(b_{i_k})(k)$. Clearly, we may further assume that $F(x)(k)$ is a subset of the open interval $(0, 1)$ for each vertex x of G and $k = 1, 2$.

Now consider the function F' which assigns to each vertex x of G a pair $F'(x)(1)$, $F'(x)(2)$ of closed intervals of R defined as follows; For $i = 1, 2, \ldots, m$ and $k = 1, 2$, let $F'(a_i)(k) = [r, 1]$ where r is the right end point of $F(a_i)(k)$; for $j = 1, 2, \ldots, n$ and $k = 1, 2$, let $F'(b_j)(k) = [0, l]$ where l is the left end point of $F(b_j)(k)$. It follows that for $k = 1, 2$, $i = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, n$, we have $F(a_i)(k) \cap F(b_j)(k) \neq \emptyset$ if and only if $F'(b_j)(k) \lhd F'(a_i)(k)$ and therefore, F' is an interval representation of length two for the partially ordered set X_G. This process is easily seen to be reversible and we have thus established the following theorem relating rectangle graphs to circular arc graphs and partially ordered sets.

Theorem 5. *Let X be a partially ordered set of height one. Then the following statements are equivalent.*

(i) $X \in \mathcal{P}_2$,
(ii) $G_X \in \mathcal{R}_2$,
(iii) $G^*_X \in \mathcal{A}_2$.

The reader is referred to [6] where a complete determination of $\mathcal{P}_2$ is given.

5. Some related topics

We conclude this paper with some references to related papers. First, we note that the author and K.P. Bogart [4] have proved a characterization theorem for interval dimension which is analogous to Theorem 3. For an integer $n \geq 2$, let S^n_n

denote the partially ordered set Y of height one for which $G_n = G_Y$. Then it follows that for each $n \geq 2$, if X is a partially ordered set having $2n+1$ points, then $\mathrm{I\,Dim}(X) = n$ if and only if X contains S_n^0.

We also refer the reader to [1] where Feinberg has extended the concept of boxicity to arcs on a circle by defining the circular dimension of a graph, $D(G)$, as the smallest positive integer n for which there exists a function F assigning to each vertex x of G a sequence $F(x)(1), F(x)(2), \ldots, F(x)(n)$ of arcs on a circle so that $x \perp y$ in G if and only if $x \neq y$ and $F(x)(i) \cap F(y)(i) \neq \emptyset$ for $i = 1, 2, \ldots, n$. Since $D(G) \leq \mathrm{Box}(G)$, we have the analogous inequality $D(G) \leq [|G|/2]$. However, Feinberg observed that $D(G_n) = 1$ for all $n \geq 1$. Feinberg constructed for each $n \geq 1$ a graph with $2^n + n - 1$ vertices and circular dimension n and conjectured that this family characterized graphs with maximum circular dimension. This conjecture is incorrect, since it is straightforward to prove, using Erdos' probabilistic methods, that for large n, there exists a graph with n vertices whose circular dimension exceeds $n/(4 \log n)$. However, the general question of the relative accuracy of $D(G) \leq [|G|/2]$ is unanswered.

Finally, we mention the paper by Trotter and Harary [5], who defined the *interval number* of a graph G, denoted $i(G)$, as the smallest positive integer n for which there exists a function F assigning to each vertex x of G a sequence $F(x)(1), F(x)(2), \ldots, F(x)(n)$ of closed intervals of $\mathbf{R}$ so that distinct vertices x, y of G are adjacent in G if and only if $F(x)(i) \cap F(y)(j) \neq \emptyset$ for some i, j with $1 \leq i \leq n$ and $1 \leq j \leq n$. Alternately, $i(G)$ is the smallest n for which G is the intersection graph of a family of sets where each set is the union of n intervals of $\mathbf{R}$. Trotter and Harary showed that the complete bipartite graph $K_{m,n}$ has interval number $\{(mn+1)/(m+n)\}$.

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