Journal of Pure and Applied Algebra 216 (2012) 593–597

Contents lists available at SciVerse ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Non-canonical isomorphisms

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ARTICLE INFO

Article history:
Received 15 November 2010
Received in revised form 20 June 2011
Available online 8 September 2011
Communicated by G. Rosolini

ABSTRACT

Many categorical axioms assert that a particular canonically defined natural transformation between certain functors is invertible. We give two examples of such axioms where the existence of any natural isomorphism between the functors implies the invertibility of the canonical natural transformation. The first example is distributive categories, the second (semi-)additive ones. We show that each follows from a general result about monoidal functors.

1. Introduction

In any category $\mathscr{D}$ with finite products and coproducts there is a natural family of maps

$$(X \times Y) + (X \times Z) \xrightarrow{\delta_{X,Y,Z}} X \times (Y + Z)$$

induced, via the universal property of the coproduct $X \times Y + X \times Z$, by the morphisms $X \times i$ and $X \times j$, where i and j denote the coproduct injections of $Y + Z$. Such a $\mathscr{D}$ is said to be *distributive* [2,3] if the canonical maps are invertible; in other words, if the functor $X \times - \colon \mathscr{D} \to \mathscr{D}$ preserves binary coproducts, for all objects X. As observed by Cockett [3], it follows that $X \times 0 \cong 0$, so $X \times -$ in fact preserves finite coproducts. Examples of distributive categories include the category of sets, the category of topological spaces, the opposite of the category of commutative rings, any topos, or any distributive lattice, seen as a preorder.

Claudio Pisani has asked [7] whether the existence of *any* natural family of isomorphisms

$$(X \times Y) + (X \times Z) \xrightarrow{\psi_{X,Y,Z}} X \times (Y + Z)$$

implies that $\mathscr{D}$ is distributive. Such ψ are the non-canonical isomorphisms of the title. Rather to my surprise, this turns out to be true, and is the first result of the paper.

The second result is an analogue for semi-additive categories. Recall that a category is pointed when it has an initial object which is also terminal ($1 = 0$), and that for any two objects Y and Z in a pointed category there is a unique morphism from Y to Z which factorizes through the zero object; this morphism is called $0_{Y,Z}$ or just 0. If the category has finite products and coproducts, then there is a natural family of morphisms

$$Y + Z \xrightarrow{\alpha_{Y,Z}} Y \times Z$$

induced by the identities on Y and Z and the zero morphisms $0 \colon Y \to Z$ and $0 \colon Z \to Y$. The category is semi-additive when these $\alpha_{Y,Z}$ are invertible [6, VII.2]. A semi-additive category admits a canonical enrichment over commutative monoids;

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doi:10.1016/j.jpaa.2011.07.012

conversely, any category enriched over commutative monoids which has either finite products or finite coproducts is semi-additive. Examples of semi-additive categories include the category of abelian groups, the category of commutative monoids, the category of R-modules for a ring R, or any abelian category. Our result for semi-additive categories asserts once again that the existence of any natural isomorphism $Y + Z \cong Y \times Z$ implies that the category is semi-additive.

We prove the result about distributive categories in Section 2, and the result about semi-additive categories in Section 3. Finally in Section 4, we show how the common part of the two arguments follows from a more general result about monoidal functors.

2. Non-canonical distributivity isomorphisms

This section involves, as in the introduction, a category $\mathscr{D}$ with finite products and coproducts and a natural family of isomorphisms $(X \times Y) + (X \times Z) \cong X \times (Y + Z)$. First we show, in the following lemma, that such a $\mathscr{D}$ will be distributive if $X \times 0 \cong 0$. Later on, we shall see that this lemma follows from a more general result about coproduct-preserving functors due to Caccamo and Winskel; and that this in turn is a special case of a still more general result about monoidal functors: this is our Theorem 6 below.

Lemma 1. *Suppose that we have, in a category $\mathscr{D}$ with finite products and coproducts, natural isomorphisms*

$$(X \times Y) + (X \times Z) \xrightarrow{\psi} X \times (Y + Z)$$

and that $X \times 0 \cong 0$. Then the category $\mathscr{D}$ is distributive.

Proof. Since an isomorphism $X \times 0 \cong 0$ is inverse to the unique map $0 \to X \times 0$ it is unique, and so we may identify $X \times 0$ with 0. We also identify $X + 0$ with X for all objects X. Thus the isomorphism $\varphi_{X,Y,0} \colon (X \times Y) + (X \times 0) \cong X \times (Y + 0)$ takes the form $X \times Y \cong X \times Y$. By naturality with respect to the unique map $0 \to Z$, the diagram

$$\begin{tikzcd} X \times Y \arrow[r, "\psi_{X,Y,0}"] \arrow[d, "i"'] & X \times Y \arrow[d, "X \times i"] \\ (X \times Y) + (X \times Z) \arrow[r, "\psi_{X,Y,Z}"'] & X \times (Y + Z) \end{tikzcd}$$

commutes, and similarly we have a commutative diagram

$$\begin{tikzcd} X \times Z \arrow[r, "\psi_{X,0,Z}"] \arrow[d, "j"'] & X \times Z \arrow[d, "X \times j"] \\ (X \times Y) + (X \times Z) \arrow[r, "\psi_{X,Y,Z}"'] & X \times (Y + Z) \end{tikzcd}$$

and now combining these we get a commutative diagram

$$\begin{tikzcd} (X \times Y) + (X \times Z) \arrow[r, "\psi_{X,Y,0} + \psi_{X,0,Z}"] \arrow[d, equal] & (X \times Y) + (X \times Z) \arrow[d, "\delta_{X,Y,Z}"] \\ (X \times Y) + (X \times Z) \arrow[r, "\psi_{X,Y,Z}"'] & X \times (Y + Z). \end{tikzcd}$$

In this last diagram, the ψ's are all invertible, and hence so is δ. □

Thus to prove our result about non-canonical distributivity isomorphisms, it will suffice to show that $X \times 0 \cong 0$ whenever we have the natural isomorphisms ψ; a little surprisingly, perhaps, it is this part which requires more work. First we prove a different result involving non-canonical isomorphisms.

Proposition 2. *Let $\mathscr{C}$ be a category with binary coproducts and isomorphisms $\theta_C \colon C + C \cong C$ natural in C. Then the codiagonal $\nabla_C \colon C + C \to C$ is also invertible for each C, and so $\mathscr{C}$ is a preorder.*

Proof. For each C we have the injections $i_C, j_C \colon C \to C + C$ and these are natural in C. By naturality of θ with respect to the morphism θ_C, the diagram

$$\begin{tikzcd} C + C + C + C \arrow[r, "\theta_C + \theta_C"] \arrow[d, "\theta_{C+C}"'] & C + C \arrow[d, "\theta_C"] \\ C + C \arrow[r, "\theta_C"'] & C \end{tikzcd}$$

commutes, and now since θ_C is invertible, $\theta_C + \theta_C = \theta_{C+C}$. In the diagram

$$\begin{tikzcd}
C+C \arrow[rrr, "\theta_C"] \arrow[dr, "i_{C+C}"] \arrow[dd, "\nabla_C"'] & & & C \arrow[dr, "i_C"] & \\
& C+C+C+C \arrow[rrr, "\theta_C+\theta_C", "\theta_{C+C}"'] \arrow[dd, "\nabla_C+\nabla_C"'] & & & C+C \arrow[dd, "\nabla_C"'] \\
C \arrow[dr, "i_C"'] & & & & \\
& C+C \arrow[rrr, "\theta_C"'] & & & C
\end{tikzcd}$$

the top right region commutes by definition of $\theta_C + \theta_C$, the bottom left commutes by definition of $\nabla_C + \nabla_C$, and the bottom right region commutes by naturality of θ with respect to the codiagonal ∇_C. Now $\nabla_C i_C = 1$ by definition of the codiagonal ∇_C, and so $\theta_C = \nabla_C i_C \theta_C = \theta_C i_C \nabla_C$; but θ_C is invertible, so $i_C \nabla_C = 1$. This proves that ∇_C is invertible with inverse i_C.

Finally, since $\nabla_C i_C = 1 = \nabla_C j_J$ and ∇_C is invertible, the coproduct injections i_C and j_C are equal. It follows that any two maps $f, g: C \to D$ are equal. This proves that $\mathscr{C}$ is a preorder. □

Recall that an object T is called *subterminal* if for any object X there is at most one morphism from X to T. If, as here, a terminal object exists, this is equivalent to saying that the unique map $T \to 1$ is a monomorphism. It is also equivalent to saying that the product $T \times T$ exists and the diagonal map $T \to T \times T$ is invertible.

Proposition 3. *The product* 0×0 *in* $\mathscr{D}$ *is initial, and so* 0 *is subterminal.*

Proof. For the first part, observe that $\psi_{0,0,1}$ gives an isomorphism $0 \times 0 + 0 \times 1 \cong 0 \times (0 + 1)$, and that $0 + 1 \cong 1$ and $0 \times 1 \cong 0$. Thus $0 \times 0 + 0 \cong 0$ and so 0×0 is initial as claimed. For the second, we have an isomorphism $0 \cong 0 \times 0$, and since 0 is initial, this can only be the diagonal $\Delta: 0 \to 0 \times 0$. Thus any morphism $X \to 0 \times 0$ factorizes through the diagonal, and so any two morphisms $X \to 0$ are equal. □

Theorem 4. *If* $\mathscr{D}$ *is a category with finite products and coproducts, and with a natural family*

$$\psi_{X,Y,Z}: (X \times Y) + (X \times Z) \cong X \times (Y + Z)$$

of isomorphisms, then $\mathscr{D}$ *is distributive.*

Proof. By Lemma 1, it will suffice to show that $X \times 0 \cong 0$. Let $\mathscr{C}$ be the slice category $\mathscr{D}/0$ consisting of all objects X equipped with a map $X \to 0$. Since 0 is subterminal, such a map is unique if it exists, and compatibility with these maps is automatic; thus the projection $\mathscr{D}/0 \to \mathscr{D}$ is fully faithful and injective on objects. Since $\mathscr{D}$ has finite coproducts, so does $\mathscr{D}/0$ and the projection preserves them. In general, products in slice categories are pullbacks in the base category, but in the case of $\mathscr{D}/0$, since 0 is subterminal, these pullbacks are just products, so $\mathscr{D}/0$ has products and the projection preserves them. Summing up, $\mathscr{D}/0$ is a full subcategory of $\mathscr{D}$ closed under finite products and coproducts.

The isomorphisms ψ for $\mathscr{D}$ therefore restrict to $\mathscr{C} = \mathscr{D}/0$, but now $\mathscr{C}$ has the further property that it is pointed, so $1 \cong 0$. Thus for each object $C \in \mathscr{C}$ we have isomorphisms

$$C + C \cong C \times 1 + C \times 1 \cong C \times 0 + C \times 0 \xrightarrow{\psi} C \times (0 + 0) \cong C \times 0 \cong C \times 1 \cong C$$

which are clearly natural in C. It follows, by Proposition 2, that $\mathscr{C}$ is a preorder. For any $C \in \mathscr{C}$ we have morphisms $0 \to C$ and $C \to 1 \to 0$ and so $C \cong 0$. In particular, for each $X \in \mathscr{D}$ the projection $X \times 0 \to 0$ gives an object of $\mathscr{D}/0$, and so $X \times 0 \cong 0$. □

3. Non-canonical semi-additivity isomorphisms

We now give an analogous result for semi-additivity. An interesting feature is that this does not require us to assume that the category is pointed, although that will of course be a consequence.

Theorem 5. *If* $\mathscr{A}$ *is a category with finite products and coproducts and with a natural family*

$$\psi_{Y,Z}: Y + Z \cong Y \times Z$$

of isomorphisms, then $\mathscr{A}$ *is semi-additive.*

Proof. Taking $Y = 1$ and $Z = 0$ gives an isomorphism $\psi_{1,0}\colon 1 \cong 1 \times 0$; composing with the projection $1 \times 0 \to 0$ gives a morphism $1 \to 0$. By uniqueness of morphisms into 1 and out of 0, this is inverse to the unique map $0 \to 1$, and so $\mathscr{A}$ is pointed.

Taking one of Y and Z to be 0 gives natural isomorphisms $\psi_{Y,0}\colon Y \cong Y$ and $\psi_{0,Z}\colon Z \cong Z$. By naturality of the $\psi_{Y,Z}$, the diagrams

$$\begin{tikzcd} Y \arrow[r, "\psi_{Y,0}"] \arrow[d, "i"'] & Y \arrow[d, "\binom{Y}{0}"] \\ Y+Z \arrow[r, "\psi_{Y,Z}"'] & Y \times Z \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r, "\psi_{0,Z}"] \arrow[d, "j"'] & Z \arrow[d, "\binom{0}{Z}"] \\ Y+Z \arrow[r, "\psi_{Y,Z}"'] & Y \times Z \end{tikzcd}$$

commute, and so also

$$\begin{tikzcd} Y+Z \arrow[r, "\psi_{Y,0}+\psi_{0,Z}"] \arrow[d, equal] & Y+Z \arrow[d, "\alpha_{Y,Z}"] \\ Y+Z \arrow[r, "\psi_{Y,Z}"'] & Y \times Z \end{tikzcd}$$

commutes. Just as in the proof of Lemma 1, $\psi_{Y,0} + \psi_{0,Z}$ and $\psi_{Y,Z}$ are invertible, and hence so is $\alpha_{Y,Z}$. $\square$

4. Non-canonical isomorphisms for monoidal functors

In this section we prove a general result on monoidal functors, which could be used in the proof of both of the other theorems. We write as if the monoidal categories in question were strict; thanks to the coherence theorem for monoidal categories there is no danger in doing this. Recall that if $\mathscr{A}$ and $\mathscr{B}$ are monoidal categories, a monoidal functor $F\colon \mathscr{A} \to \mathscr{B}$ consists of a functor (also called F) equipped with maps $\varphi_{Y,Z}\colon FY \otimes FZ \to F(Y \otimes Z)$ and $\varphi_0\colon I \to FI$ which need not be invertible, but which are natural and coherent [4]. The monoidal functor is said to be *strong* if $\varphi_{Y,Z}$ and φ_0 are invertible, and *normal* if φ_0 is invertible. Given monoidal functors $(F, \varphi), (G, \psi)\colon \mathscr{A} \to \mathscr{B}$ a natural transformation $\alpha\colon F \to G$ is *monoidal* if the diagrams

$$\begin{tikzcd} FY \otimes FZ \arrow[r, "\alpha_X \otimes \alpha_Y"] \arrow[d, "\varphi_{X,Y}"'] & GY \otimes GZ \arrow[d, "\psi_{X,Y}"] \\ F(Y \otimes Z) \arrow[r, "\alpha_{X \otimes Y}"'] & G(Y \otimes Z) \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "\varphi_0"] \arrow[dr, "\psi_0"'] & FI \arrow[d, "\alpha I"] \\ & GI \end{tikzcd}$$

commute. Recall further [5] that if $\mathscr{C}$ is braided monoidal, then the functor $\otimes\colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is strong monoidal, with structure maps

$$W \otimes X \otimes Y \otimes Z \xrightarrow{W \otimes \gamma \otimes D} W \otimes Y \otimes X \otimes Z \qquad I \xrightarrow{\lambda} I \otimes I$$

where γ denotes the braiding and λ the canonical isomorphism.

Theorem 6. *Let $\mathscr{A}$ and $\mathscr{B}$ be braided monoidal categories, and $F = (F, \varphi, \varphi_0) : \mathscr{A} \to \mathscr{B}$ a normal monoidal functor (so φ_0 is invertible). Suppose further that we have a monoidal isomorphism*

$$\begin{tikzcd} \mathscr{A} \times \mathscr{A} \arrow[r, "F \times F"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\Downarrow\psi"] & \mathscr{B} \times \mathscr{B} \arrow[d, "\otimes"] \\ \mathscr{A} \arrow[r, "F"'] & \mathscr{B} \end{tikzcd}$$

where $F \times F$ is made monoidal via the monoidal structure on F, and the vertical arrows are made monoidal using the braidings on $\mathscr{A}$ and $\mathscr{B}$. Then φ is invertible, and so F is strong monoidal.

Proof. The fact that ψ is monoidal means in particular that the diagram

$$\begin{tikzcd}
FW \otimes FX \otimes FY \otimes FZ \arrow[r, "\psi_{W,X} \otimes \psi_{Y,Z}"] \arrow[d, "1 \otimes \gamma \otimes 1"'] & F(W \otimes X) \otimes F(Y \otimes Z) \arrow[d, "\varphi_{W \otimes X, Y \otimes Z}"] \\
FW \otimes FY \otimes FX \otimes FZ \arrow[d, "\varphi_{W,Y} \otimes \varphi_{X,Z}"'] & F(W \otimes X \otimes Y \otimes Z) \arrow[d, "F(1 \otimes \gamma \otimes 1)"] \\
F(W \otimes Y) \otimes F(X \otimes Z) \arrow[r, "\psi_{W \otimes Y, X \otimes Z}"'] & F(W \otimes Y \otimes X \otimes Z)
\end{tikzcd}$$

commutes. Taking $X = Y = I$ and twice using the isomorphism φ_0 gives commutativity of

$$\begin{tikzcd}
FW \otimes FZ \arrow[r, "1 \otimes \varphi_0 \otimes \varphi_0 \otimes 1"] \arrow[dr, equal] & FW \otimes FI \otimes FI \otimes FZ \arrow[r, "\psi_{W,I} \otimes \psi_{I,Z}"] \arrow[d, "\varphi_{W,I} \otimes \varphi_{I,Z}"] & FW \otimes FZ \arrow[d, "\varphi_{W,Z}"] \\
& FW \otimes FZ \arrow[r, "\psi_{W,Z}"'] & F(W \otimes Z)
\end{tikzcd}$$

in which all arrows except the φ's are invertible. Since the exterior commutes, it follows that $\varphi_{W,Z}$ too is invertible. □

Remark 7. In general, for a functor between monoidal categories there is no canonical way to make it monoidal. But for a monoidal functor (F, φ, φ_0), we do have a canonical monoidal natural transformation $\otimes(F \times F) \to F\times$, namely φ itself. So the theorem asserts that the existence of a non-canonical monoidal natural isomorphism ψ implies the invertibility of the canonical φ. In the proof of the theorem we have used rather less than was assumed in the statement. For example, we do not use the nullary part of the assumption that the natural transformation is monoidal.

The following corollary appeared (in dual form) as [1, Theorem 3.3]:

Corollary 8 (*Caccamo–Winskel*)**.** *Let $\mathscr{A}$ and $\mathscr{B}$ be categories with finite coproducts, and $F : \mathscr{A} \to \mathscr{B}$ a functor which preserves the initial object. If there is a natural family of isomorphisms*

$$FX + FY \xrightarrow{\psi_{X,Y}} F(X + Y)$$

then F preserves finite coproducts.

Proof. In this case F has a unique structure of normal monoidal functor $(\mathscr{A}, +, 0) \to (\mathscr{B}, +, 0)$, and ψ is always monoidal. □

In particular if $\mathscr{D}$ has finite products and coproducts, we may recover Lemma 1 by applying the corollary to the functor $X \times - : \mathscr{D} \to \mathscr{D}$.

Section 3 involves the case of the theorem where the categories $\mathscr{A}$ and $\mathscr{B}$ are the same, but the monoidal structure on $\mathscr{A}$ is cartesian and that on $\mathscr{B}$ is cocartesian. The functor F is the identity. One proves that $0 \to 1$ is invertible, as in the proof of Theorem 5; and then the identity functor $1_{\mathscr{A}}$ has a unique structure of normal monoidal functor $(\mathscr{A}, \times, 1) \to (\mathscr{A}, +, 0)$, with binary part precisely the canonical morphism $\alpha : Y + Z \to Y \times Z$. Furthermore, any natural isomorphism $\psi_{Y,Z} : Y + Z \cong Y \times Z$ is monoidal.

Acknowledgement

The support of the Australian Research Council and DETYA is gratefully acknowledged.

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